Z Transform (H.1)
Definition

20170201

Based on
Complex Analysis for Mathematics and Engineering
J. Mathews

z - Transform

$$X(z) = \sum_{k=-\infty}^{+\infty} x[k]\, z^{-k}$$

$$z = |r|\, e^{j2\pi F} = |r|\, e^{j\Omega}$$

$$x[n] \longleftrightarrow X(z)$$

One Sided z-transform

$$X(z) = \sum_{k=0}^{+\infty} x[k]\, z^{-k}$$

Inverse z-Transform

$$X(z) = \mathcal{Z}\left[\{x_n\}_{n=0}^{\infty}\right] = \sum_{n=0}^{\infty} x_n z^{-n} = \sum_{n=0}^{\infty} x[n] z^{-n}$$

$x[n] \rightarrow X(z)$

$$x_n = x[n] = \mathcal{Z}^{-1}[X(z)] = \frac{1}{2\pi i}\int_C X(z) z^{n-1} dz$$

$x[n] \leftarrow X(z)$

Admissible Form of z-transform

$$X(z) = \sum_{n=0}^{\infty} x[n] z^{-n}$$

$X(z)$: admissible z-transform

if $X(z)$ is a rational function

$$\boxed{X(z) = \frac{P(z)}{Q(z)}} = \frac{b_0 + b_1 z^1 + b_2 z^2 + \cdots + b_{p-1} z^{p-1} + b_p z^p}{a_0 + a_1 z^1 + a_2 z^2 + \cdots + a_{q-1} z^{q-1} + a_q z^q}$$

$P(z)$: a polynomial of degree p

$Q(z)$: a polynomial of degree q

Residue Theorem

D: simply connected domain
C: simple closed contour (CCW) in D

if $f(z)$ is analytic inside C and on C
except at the points $\boxed{z_1, z_2, \cdots, z_k}$ in C

then
$$\frac{1}{2\pi i}\int_C f(z)\,dz = \sum_{j=1}^{k} \operatorname{Res}(f(z), z_j)$$

singular points of $f(z)$: $z_1, z_2, \cdots, z_k$

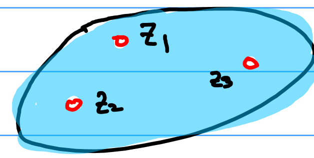
Integration of a function of a complex var.

$$\boxed{\oint_C f(z)\,dz} = 2\pi i \sum_{k=1}^{n} \mathrm{Res}(f(z), z_k)$$

finite number k of

Singular points z_k

residue theorem



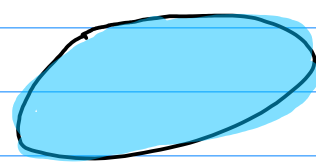
$$\boxed{\oint_C f(z)\,dz} = 0 \quad \text{if } \boxed{f(z) = F'(z)} \text{ on } C$$

: $F(z)$ is an antiderivative of $f(z)$

fundamental theorem of calculus

$$\boxed{\oint_C f(z)\,dz} = 0 \quad \text{if } f(z) \text{ is analytic within and on } C$$

no singularity



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Digital Signal Processing, Wiley, 2000

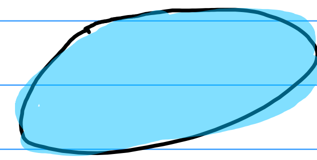
$\boxed{\oint_C f(z)\,dz} = 0$ if $f(z)$ is continuous in D and

$\boxed{f(z) = F'(z)}$: $F(z)$ is an antiderivative of $f(z)$

fundamental theorem of calculus

$\boxed{\oint_C f(z)\,dz} = 0$ if $f(z)$ is analytic within and on C

no singularity



can expand $f(z)$ about any point z_m
over powers of $(z - z_m)$

whether or not $f(z)$ is singular at z_m
or at other points between z and z_m

$$\boxed{f(z) = \sum_{n=n_1}^{\infty} a_n^{\{m\}} (z - z_m)^n}$$

① Laurent Series Expansion of $f(z)$ at z_m
general n_1 — depend on $f(z)$ and z_m

② z-transform of $a_n^{\{m\}}$
general n_1 — depend on $f(z)$
$\boxed{z_m = 0}$

③ Taylor Series Expansion of $f(z)$ at z_m
positive n_1 — depend on $f(z)$ and z_m $(n_1 > 0)$

④ MacLaurin Series Expansion of $f(z)$ at z_m
positive n_1 — depend on $f(z)$ $(n_1 > 0)$
$\boxed{z_m = 0}$

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* Expansion of $f(z)$ about any point z_m over powers of $(z - z_m)$

$$\boxed{f(z) = \sum_{n=n_1}^{\infty} a_n^{\{m\}} (z - z_m)^n}$$

$$a_n^{\{m\}} = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_m)^{n+1}} \, dz$$ for general $f(z)$

$$a_n^{\{m\}} = \sum_k \text{Res}\left[\frac{f(z)}{(z - z_m)^{n+1}}, z_k \right]$$ for general $f(z)$

$$a_n^{\{m\}} = \frac{1}{n!} f^{(n)}(z_m) \qquad n_1 \geq 0$$ for analytic $f(z)$ within C

analytic $f(z) \longrightarrow \dfrac{f(z)}{(z - z_m)^{n+1}}$ has a pole at z_m order of $n+1$

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$$\boxed{f(z) = \sum_{n=n_1}^{\infty} a_n^{\{m\}} (z - z_m)^n}$$

$\boxed{z_m}$: possible poles of $f(z)$
not necessarily poles

$$a_n^{\{m\}} = \frac{1}{2\pi i} \oint_C \frac{f(z')}{(z'-z_m)^{n+1}} \, dz'$$

$$= \sum_k \mathrm{Res}\left(\frac{f(z)}{(z-z_m)^{n+1}}, z_k \right)$$

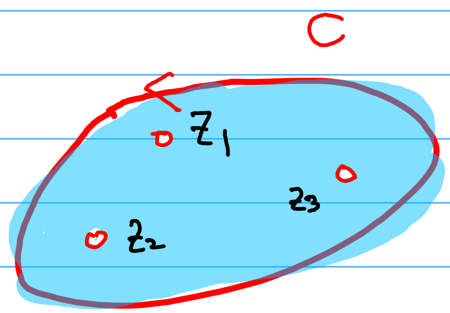
$\boxed{z_k}$: poles of $\dfrac{f(z)}{(z-z_m)^{n+1}}$
within C

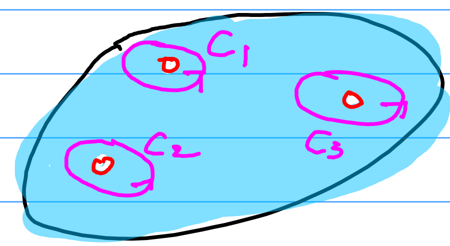
$$= \frac{1}{n!} f^{(n)}(z_m) \qquad n_1 \geq 0$$

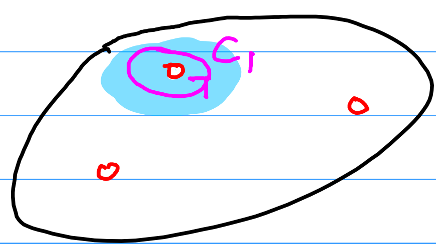
Residue Theorem

assumed there are (m) singularities (poles) of $f(z)$ in a region

C_m is taken to enclose only one pole z_m



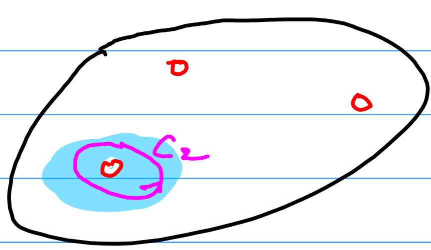




$a_n^{\{1\}}$ expanded at z_1

C_1 encloses z_1 only

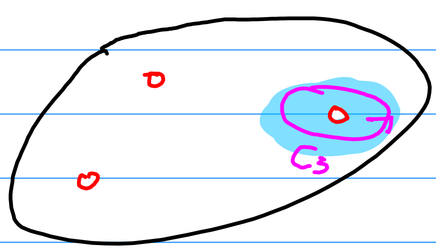
$\tilde{a}_{-1}^{\{1\}} = Res(f(z), z_1)$



$a_n^{\{2\}}$ expanded at z_2

C_2 encloses z_2 only

$\tilde{a}_{-1}^{\{2\}} = Res(f(z), z_2)$

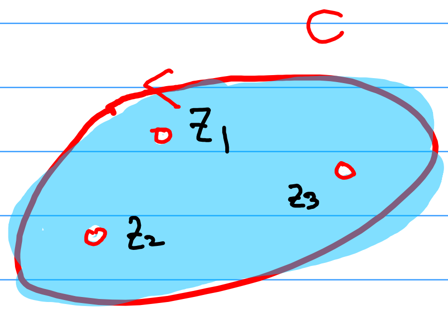


$a_n^{\{3\}}$ expanded at z_3

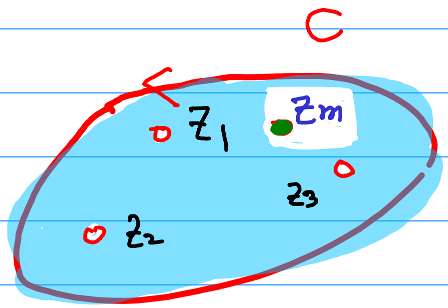
C_3 encloses z_3 only

$\tilde{a}_{-1}^{\{3\}} = Res(f(z), z_3)$

expand at z_m



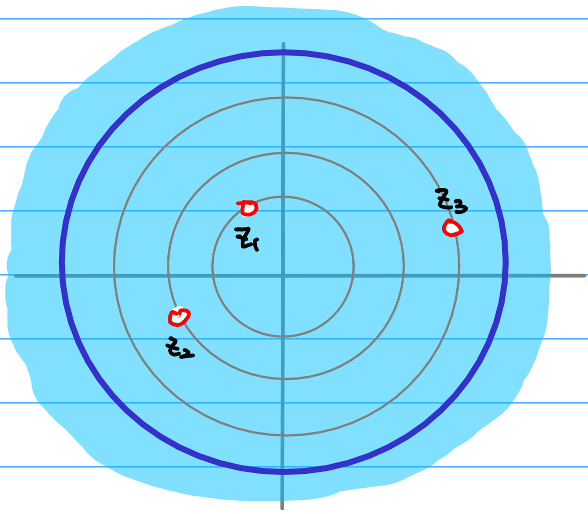
$$\boxed{\begin{aligned} a_n^{\{m\}} &= \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_m)^{n+1}}\,dz \\ &= \sum_k \mathrm{Res}\left(\frac{f(z)}{(z-z_m)^{n+1}},\, z_k\right) \end{aligned}}$$



$n \geq 0$ $\quad z_1, z_2, z_3, z_m$

$n < 0$ $\quad z_1, z_2, z_3$

expand at $z = 0$



$$a_n^{\{m\}} = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z^{n+1}}\, dz$$
$$= \sum_k \mathrm{Res}\left(\frac{f(z)}{z^{n+1}}, z_k\right)$$

Poles z_k

$n \geq 0$ $\quad z_1, z_2, z_3, 0$

$n < 0$ $\quad z_1, z_2, z_3$

expansion at z_m

$\boxed{n = -1}$ $\quad n+1=0 \quad (z-z_m)^{n+1} = 1$

$$\boxed{\begin{aligned} a_n^{\{m\}} &= \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_m)^{n+1}}\,dz \\ &= \sum_k \mathrm{Res}\left(\frac{f(z)}{(z-z_m)^{n+1}}, z_k\right) \end{aligned}}$$

$$\boxed{\begin{aligned} a_{-1}^{\{m\}} &= \frac{1}{2\pi i}\oint_C f(z)\,dz \\ &= \sum_k \mathrm{Res}\left(f(z), z_k\right) \end{aligned}}$$

$$a_{-1}^{\{m\}} = \frac{1}{2\pi i}\oint_C f(z)\,dz = \sum_k \mathrm{Res}(f(z), z_k)$$

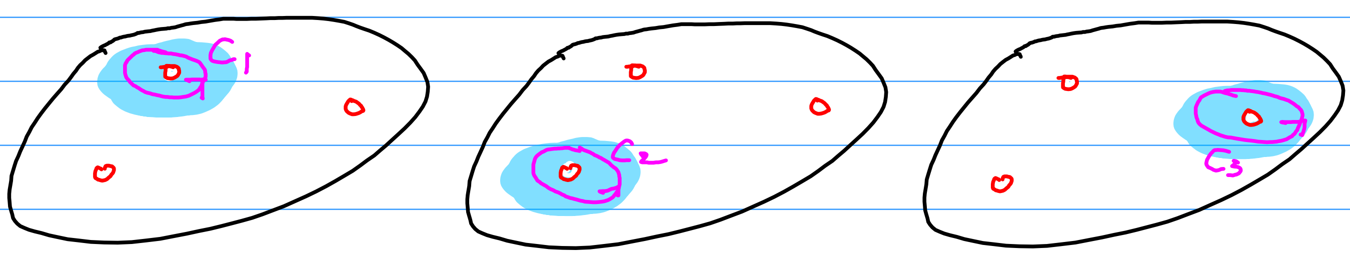
if C encloses only one pole, and expand at that pole

$$a_{-1}^{\{0\}} = \frac{1}{2\pi i}\oint_{C_0} f(z)\,dz = \mathrm{Res}(f(z), z_0)$$

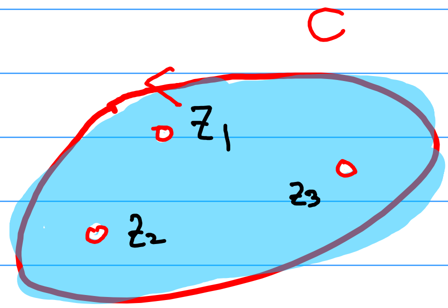
$$\boxed{\tilde{a}_{-1}^{\{m\}} = \mathrm{Res}(f(z), z_m)}$$

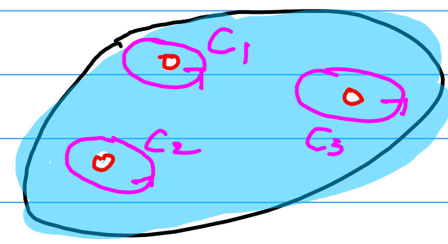
the residue of $f(z)$ at z_m

using C_m which is in the anulus ROC



$\tilde{a}_{-1}^{\{1\}} = \mathrm{Res}(f(z), z_1)$ $\quad$ $\tilde{a}_{-1}^{\{2\}} = \mathrm{Res}(f(z), z_2)$ $\quad$ $\tilde{a}_{-1}^{\{3\}} = \mathrm{Res}(f(z), z_3)$





$$\boxed{\oint_C f(z)\,dz = 2\pi j \sum_{k=1}^{M} \tilde{a}_{-1}^{\{k\}} = 2\pi j \sum_{k=1}^{M} \mathrm{Res}\left(f(z), z_k\right)}$$

residue theorem

$$\boxed{a_n = \sum_{k=1}^{M} \mathrm{Res}\left(\frac{f(z)}{(z-z_m)^{n+1}}, z_k\right)}$$

Laurent coefficient

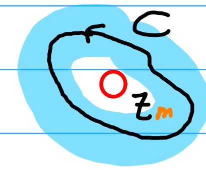
C encloses k poles

C_k encloses only the k-th pole

$\tilde{a}_{-1}^{\{k\}}$ the residue of the k-th pole enclosed by C, z_k

$$f(z) = \sum_{n=n_1}^{\infty} a_n^{\{m\}} (z - z_m)^n$$

$$a_n^{\{m\}} = \frac{1}{2\pi i} \oint_C \frac{f(z')}{(z'-z_m)^{n+1}} dz'$$



$$= \sum_k \mathrm{Res}\left(\frac{f(z)}{(z-z_m)^{n+1}}, z_k\right)$$

C is in the same region of analyticity of $f(z)$

typically a circle centered on z_m

z_k within C : singularities of $\frac{f(z)}{(z-z_m)^{n+1}}$

$n_1 = n_{f,m}$ depends on $f(z)$, z_m

$a_n^{\{m\}}$ depends on $f(z)$, z_m, region of analyticity

Whether $f(z)$ is singular at $z = z_m$ or not

or at other points between z and z_m

We can expand $f(z)$ about any point z_m

over powers of $(z - z_m)$.

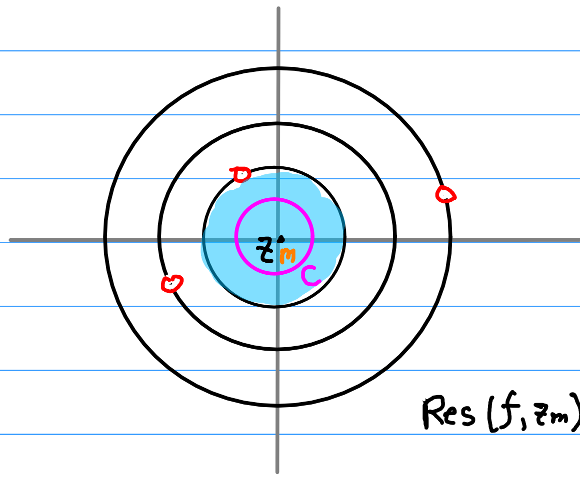
$$f(z) = \sum_{n=n_1}^{\infty} a_n^{\{m\}} (z - z_m)^n$$

$$a_n^{\{m\}} = \frac{1}{2\pi i} \oint_C \frac{f(z')}{(z'-z_m)^{n+1}} \, dz'$$

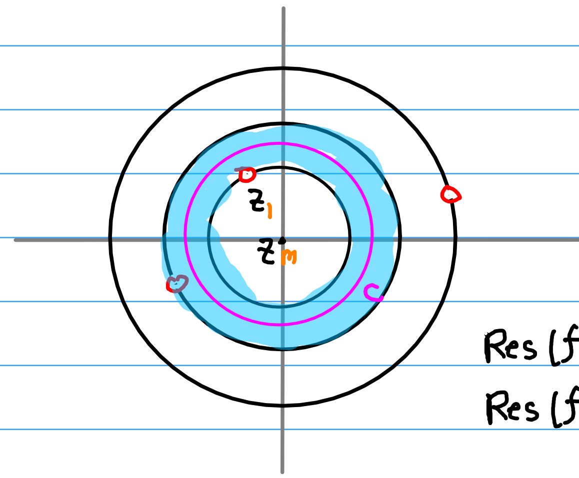
$$= \sum_k \text{Res}\left(\frac{f(z)}{(z-z_m)^{n+1}}, z_k \right)$$

z_k within C : singularities of $\dfrac{f(z)}{(z-z_m)^{n+1}}$

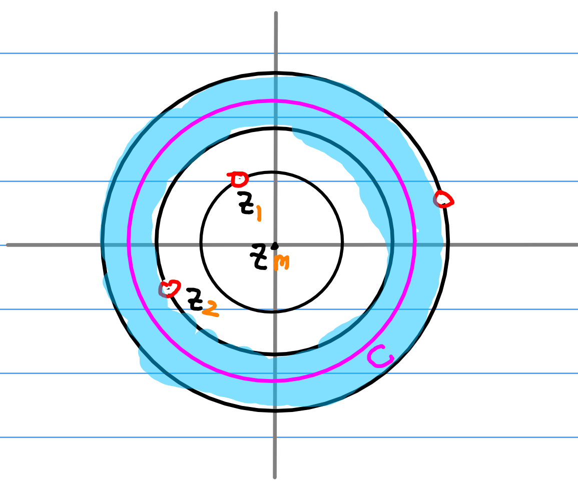
$$\begin{cases} \text{poles of } f(z) \cup z = z_m & n \geq 0 \\ \text{poles of } f(z) & n < 0 \end{cases}$$



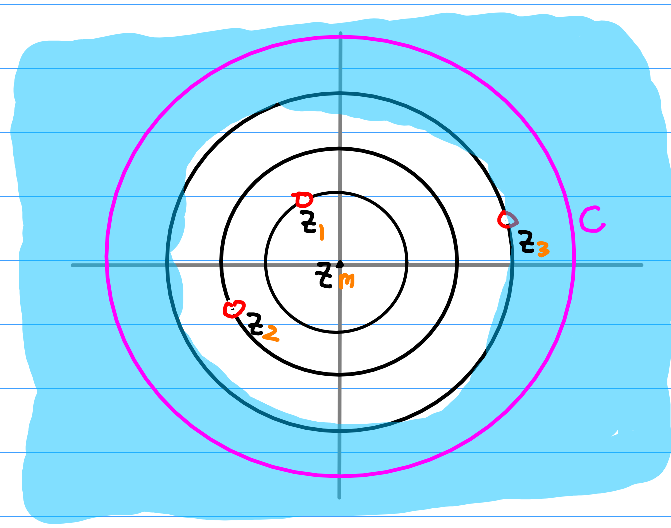
Res (f, z_m)



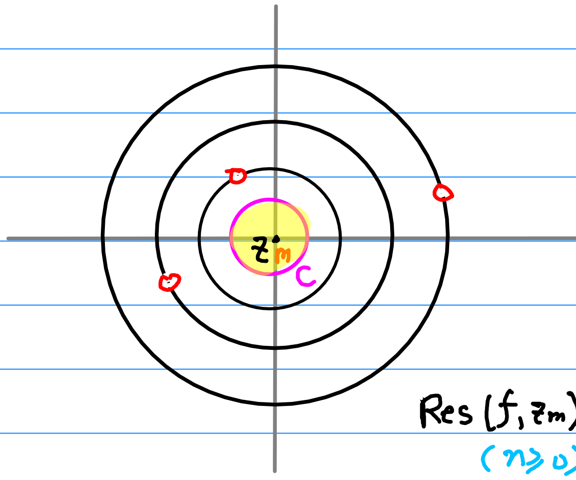
Res (f, z_1)
Res (f, z_m)



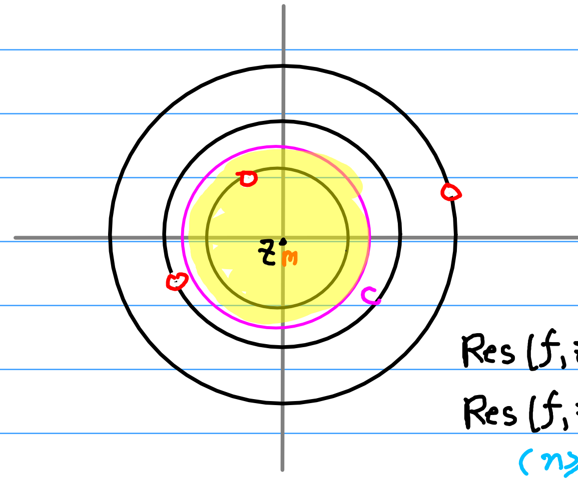
Res (f, z_1) + Res (f, z_1) + Res (f, z_m)



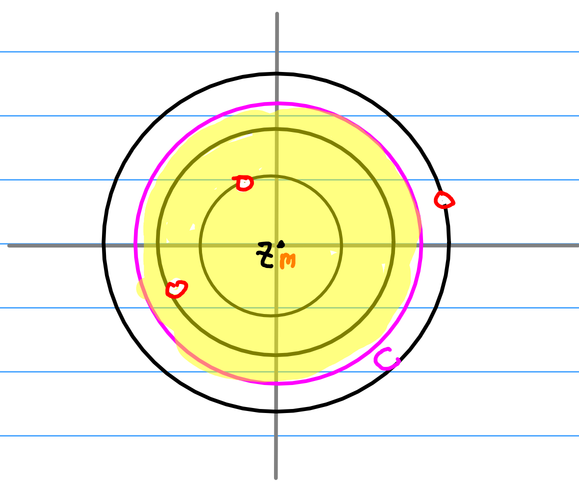
Res (f, z_1) + Res (f, z_1) + Res (f, z_3)
+ Res (f, z_m)



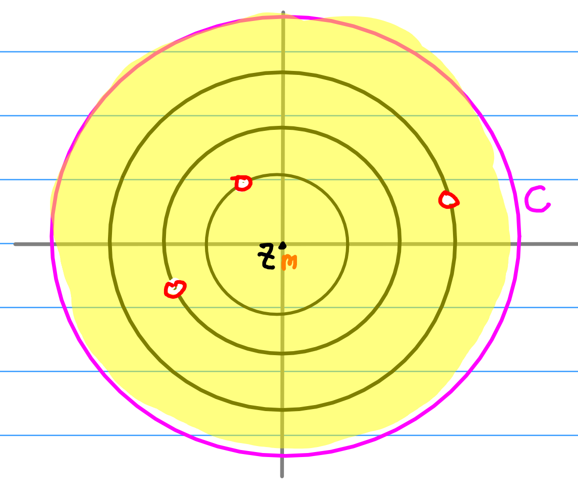
$\mathrm{Res}(f, z_m)$
$(n \geq 0)$



$\mathrm{Res}(f, z_1)$
$\mathrm{Res}(f, z_m)$
$(n \geq 0)$



$\mathrm{Res}(f, z_1) + \mathrm{Res}(f, z_1) + \mathrm{Res}(f, z_m)$
$(n \geq 0)$

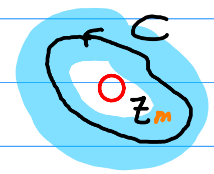


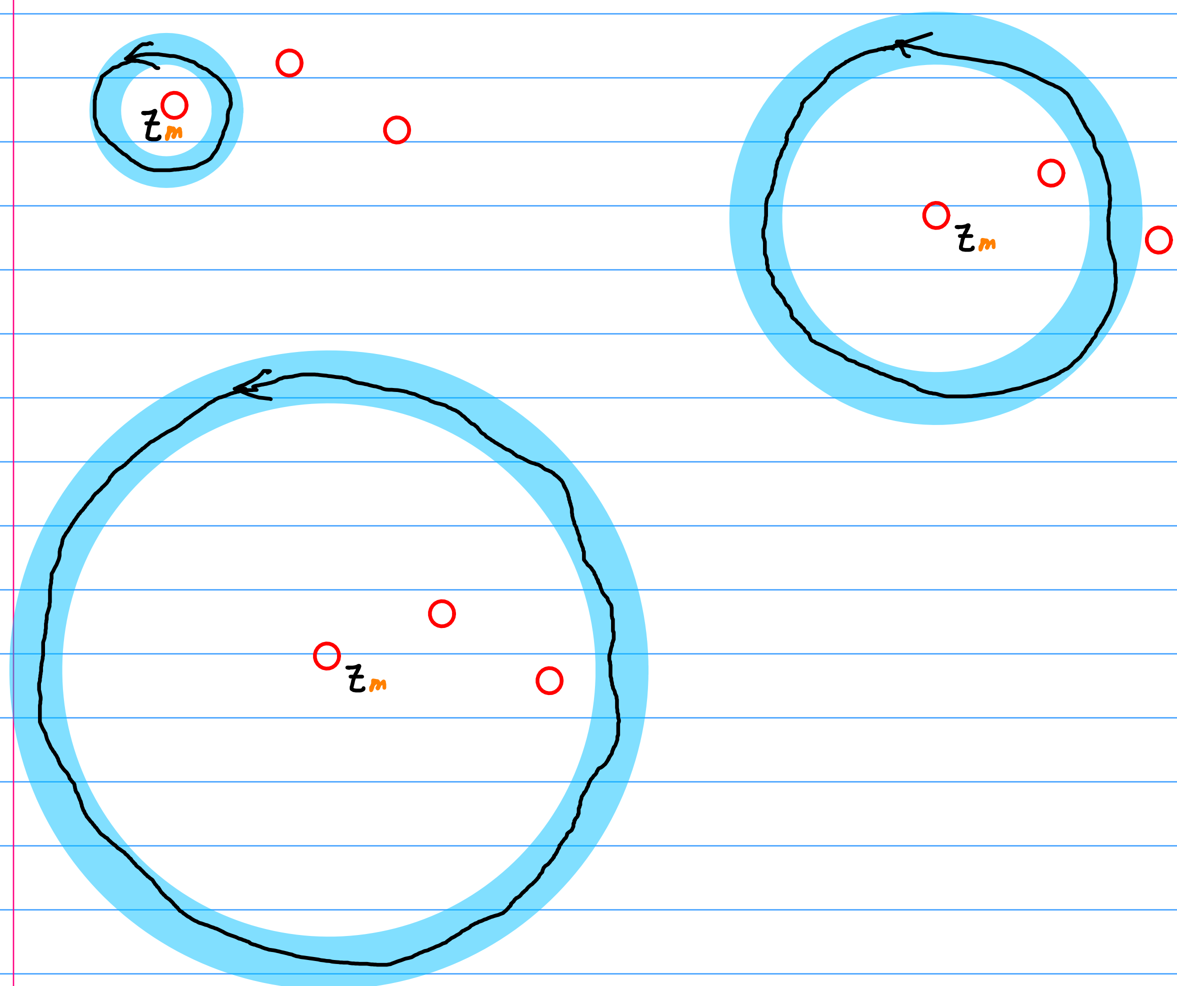
$\mathrm{Res}(f, z_1) + \mathrm{Res}(f, z_1) + \mathrm{Res}(f, z_3)$
$+ \mathrm{Res}(f, z_m)$
$(n \geq 0)$

$$f(z) = \sum_{n=n_1}^{\infty} a_n^{\{m\}} (z - z_m)^n$$

$$a_n^{\{m\}} = \frac{1}{2\pi i} \oint_C \frac{f(z')}{(z'-z_m)^{n+1}} \, dz'$$

$$= \sum_k \mathrm{Res}\left(\frac{f(z)}{(z-z_m)^{n+1}} , z_k \right)$$





Laurent's Theorem

f : analytic within the annular domain D

$$r < |z - z_0| < R$$

then

$$f(z) = \sum_{k=-\infty}^{+\infty} a_k (z - z_0)^k \quad , \text{ valid for } \quad r < |z - z_0| < R$$

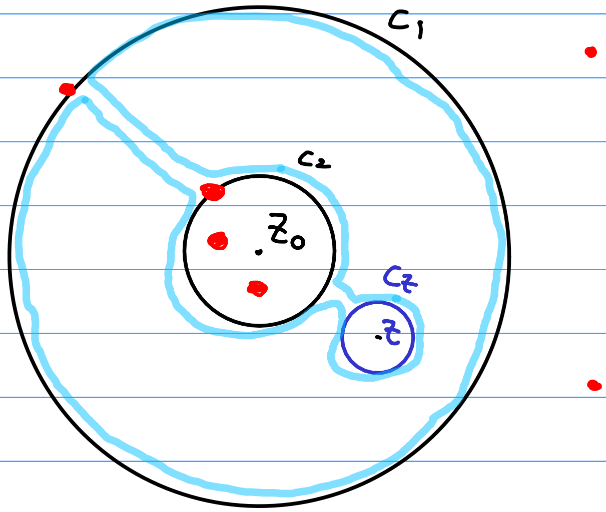
$$a_k = \frac{1}{2\pi i} \oint_C \frac{f(s)}{(s - z_0)^{k+1}} \, ds, \qquad k = 0, \pm 1, \pm 2, \cdots$$

C : a simple closed curve
that lies entirely within D
that encloses z_0

$$a_{-1} = \frac{1}{2\pi i} \oint_C f(s) \, ds \implies \oint_C f(s) \, ds = 2\pi i \cdot a_{-1}$$

$$a_{-1} = \frac{1}{2\pi i} \oint_C f(s) \, ds = Res(f(z), z_0)$$

$$= \begin{cases} \lim_{z \to z_0} (z - z_0) f(z) & (\text{simple}) \\ \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} (z - z_0)^n f(z) & (\text{order } n) \end{cases}$$



z_0 : expansion point

z : evaluation point

which poles of $f(z)$ lie between the point of evaluation z and the point z_0 about which the expansion is formed

$\dfrac{f(z')}{(z'-z_0)}$ is analytic between C_1 & C_2

deformation theorem $\qquad C_1 - C_2$ coincide

common contour C

Cauchy's Residue Theorem

$f(z)$: analytic on and within C

except a finite number of singular points

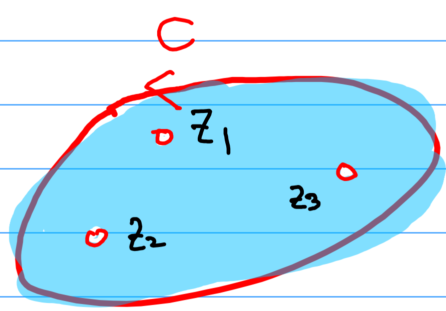
$z_1, z_2, \cdots, z_n$ within C

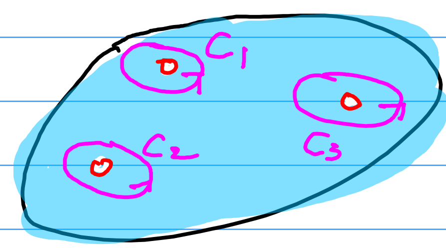
then

$$\int_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \mathrm{Res}(f(z), z_k)$$

D : a simply connected domain

C : a simple closed contour in D



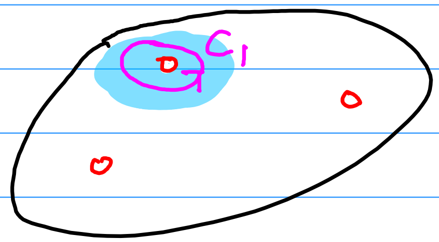


$\boxed{z_1}$ $\quad f(z) = \sum_{k=-\infty}^{+\infty} a_k (z - z_1)^k \qquad a_{-1}^{\{1\}} = \frac{1}{2\pi i} \oint_{C_1} f(s)\,ds = \mathrm{Res}(f(z), z_1)$

$\boxed{z_2}$ $\quad f(z) = \sum_{k=-\infty}^{+\infty} a_k (z - z_2)^k \qquad a_{-1}^{\{2\}} = \frac{1}{2\pi i} \oint_{C_2} f(s)\,ds = \mathrm{Res}(f(z), z_2)$

$\boxed{z_3}$ $\quad f(z) = \sum_{k=-\infty}^{+\infty} a_k (z - z_3)^k \qquad a_{-1}^{\{3\}} = \frac{1}{2\pi i} \oint_{C_3} f(s)\,ds = \mathrm{Res}(f(z), z_3)$

$\boxed{z_1}$

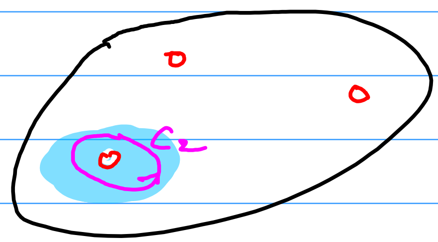


Laurent series expansion at z_1

$$f(z) = \sum_{k=-\infty}^{+\infty} a_k (z - z_1)^k$$

$$a_{-1}^{\{1\}} = \frac{1}{2\pi i} \oint_{C_1} f(s)\, ds = \text{Res}(f(z), z_1)$$

$\boxed{z_2}$

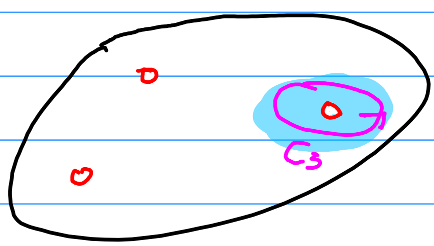


Laurent series expansion at z_2

$$f(z) = \sum_{k=-\infty}^{+\infty} a_k (z - z_2)^k$$

$$a_{-1}^{\{2\}} = \frac{1}{2\pi i} \oint_{C_2} f(s)\, ds = \text{Res}(f(z), z_2)$$

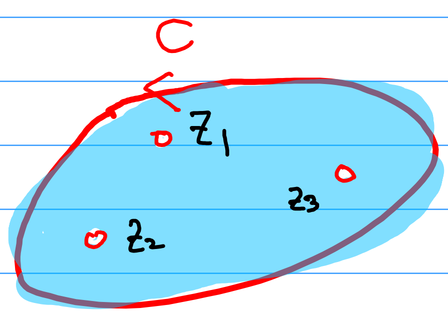
$\boxed{z_3}$



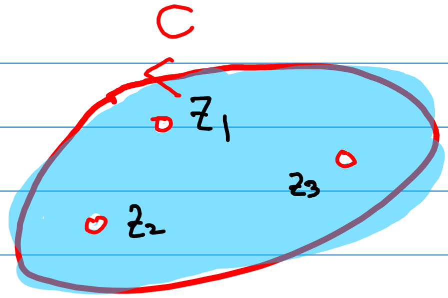
Laurent series expansion at z_3

$$f(z) = \sum_{k=-\infty}^{+\infty} a_k (z - z_3)^k$$

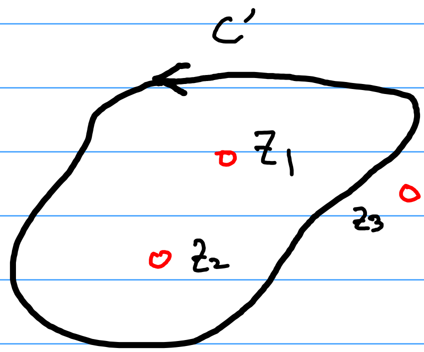
$$a_{-1}^{\{3\}} = \frac{1}{2\pi i} \oint_{C_3} f(s)\, ds = \text{Res}(f(z), z_3)$$



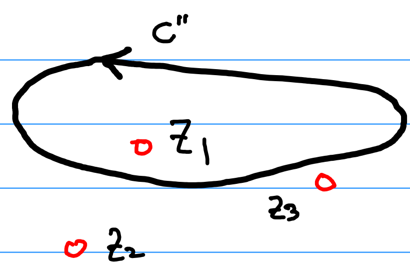
$$\int_C f(z)\, dz = 2\pi i \sum_{k=1}^{n} \text{Res}(f(z), z_k)$$



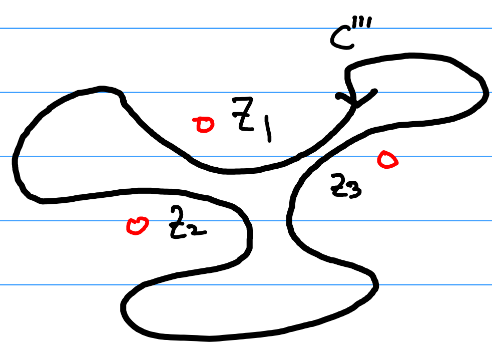
$$\int_C f(z)\,dz = 2\pi i \ \mathrm{Res}(f(z), z_1) + 2\pi i \ \mathrm{Res}(f(z), z_2) + 2\pi i \ \mathrm{Res}(f(z), z_3)$$



$$\int_{C'} f(z)\,dz = 2\pi i \ \mathrm{Res}(f(z), z_1) + 2\pi i \ \mathrm{Res}(f(z), z_2)$$

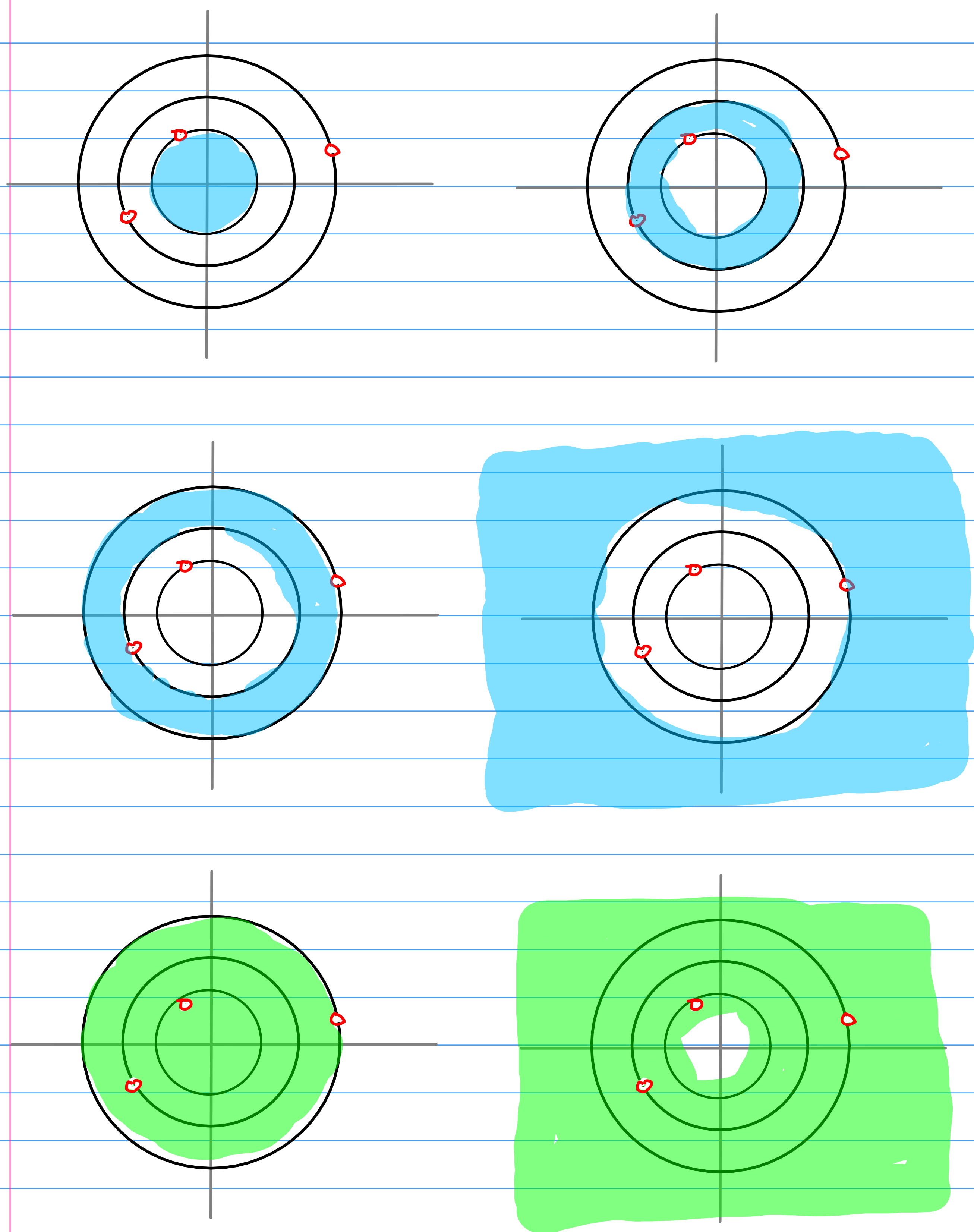


$$\int_{C''} f(z)\,dz = 2\pi i \ \mathrm{Res}(f(z), z_1)$$



$$\int_{C''} f(z)\,dz = 0$$

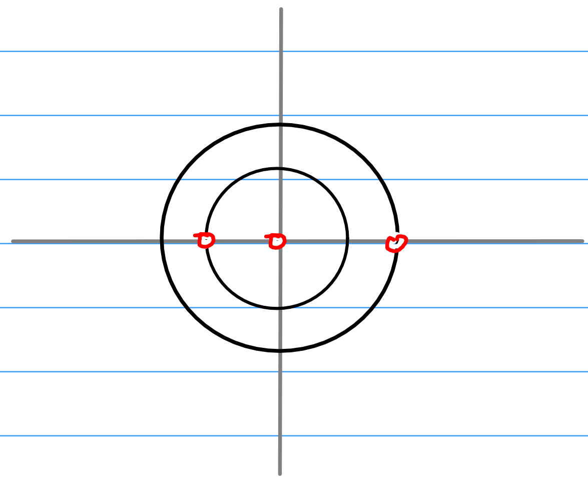
Different D, Different Laurent Series



z-transform

$\cdot f(z) = \dfrac{12}{z(2-z)(1+z)} = \dfrac{4}{z}\left(\dfrac{1}{1+z} + \dfrac{1}{2-z}\right)$

pole : $z=0$, $z=2$, $z=-1$



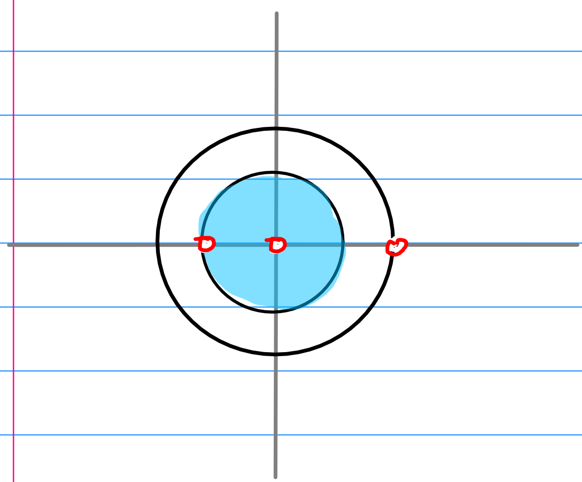
$0 < |z| < 1$

$$f(z) = -3 + 9z/2 - 15z^2/4 + 33z^3/8 + \cdots + 6/z$$

$|z| > 2$

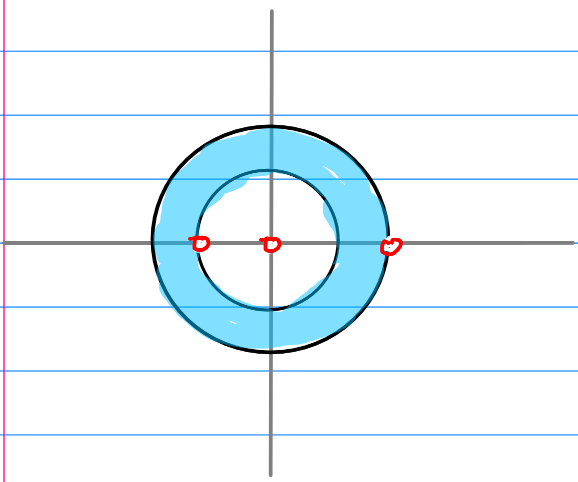
$$\frac{1}{1+z} = \frac{1}{z}\frac{1}{(1+z^{-1})} \qquad \frac{1}{2+z} = -\frac{1}{z}\frac{1}{1-2z^{-1}}$$

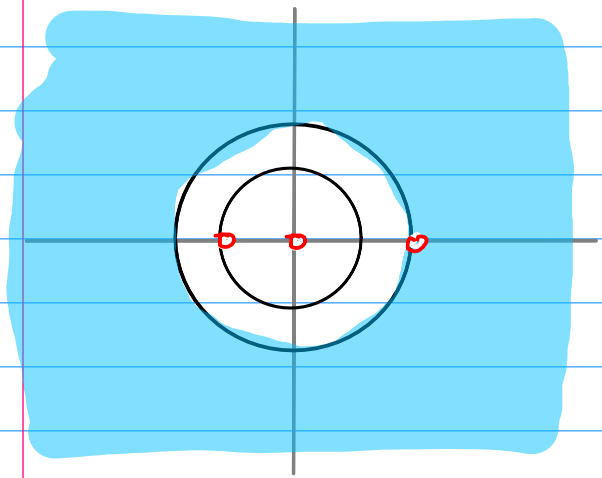
$$f(z) = -(12/z^3)\left(1 + 1/z + 3/z^2 + 5/z^3 + 11/z^4 + \cdots\right)$$



$0 < |z| < 1$

$$f(z) = -3 + 9z/2 - 15z^2/4 + 33z^3/8 + \cdots + 6/z$$





$|z| > 2$

$$\frac{1}{1+z} = \frac{1}{z}\frac{1}{(1+z^{-1})} \qquad \frac{1}{2+z} = -\frac{1}{z}\frac{1}{1-2z^{-1}}$$

$$f(z) = \frac{-1}{(z-1)(z-2)}$$

Complex Variables and Ap
Brown & Churchill

$$f(z) = \frac{-1}{(z-1)(z-2)} = \frac{1}{z-1} - \frac{1}{z-2}$$

D_1 : $|z| < 1$

D_2 : $1 < |z| < 2$

D_3 : $2 < |z|$

① D_1 $\quad |z| < 1, \quad \left|\frac{z}{2}\right| < 1$

$$f(z) = \frac{1}{z-1} - \frac{1}{z-2} = \frac{-1}{1-z} + \frac{1}{2}\frac{1}{1-\left(\frac{z}{2}\right)}$$

$$= -\sum_{n=0}^{\infty} z^n + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} = \sum_{n=0}^{\infty} \left(2^{-n-1} - 1\right) z^n \quad |z|<1$$

② D_2 $\quad 1 < |z| < 2 \Rightarrow \left|\frac{1}{z}\right| < 1, \quad \left|\frac{z}{2}\right| < 1$

$$f(z) = \frac{1}{z-1} - \frac{1}{z-2} = \frac{1}{z} \cdot \frac{1}{1-\left(\frac{1}{z}\right)} + \frac{1}{2}\frac{1}{1-\left(\frac{z}{2}\right)}$$

$$= \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}$$

$$= \sum_{n=1}^{\infty} \frac{1}{z^n} + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}$$

③ D_3 $\quad 2 < |z| \quad \left|\frac{2}{z}\right| < 1 \quad \left|\frac{1}{z}\right| < 1$

$$f(z) = \frac{1}{z-1} - \frac{1}{z-2} = \frac{1}{z}\frac{1}{1-\left(\frac{1}{z}\right)} - \frac{1}{z}\frac{1}{1-\left(\frac{2}{z}\right)}$$

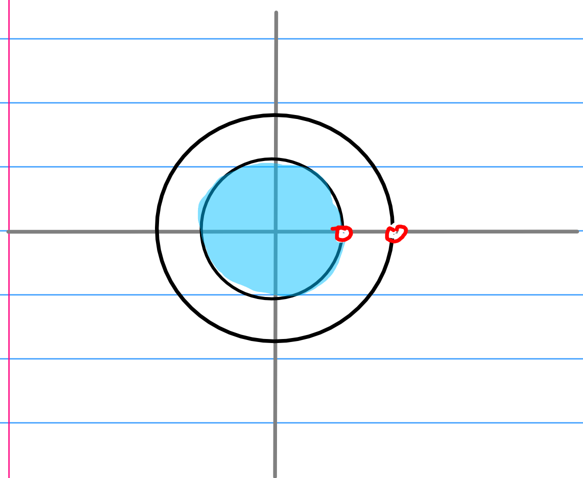
$$= \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} - \sum_{n=0}^{\infty} \frac{2^n}{z^{n+1}} = \sum_{n=0}^{\infty} \frac{1-2^n}{z^{n+1}}$$

$$= \sum_{n=1}^{\infty} \frac{1-2^{n-1}}{z^n}$$

$$f(z) = \frac{-1}{(z-1)(z-2)}$$

① D_1 $\quad |z|<1, \quad |\frac{z}{2}|<1$

$$\frac{f(z)}{z^{n+1}} = \frac{-1}{(z-1)(z-2)z^{n+1}}$$

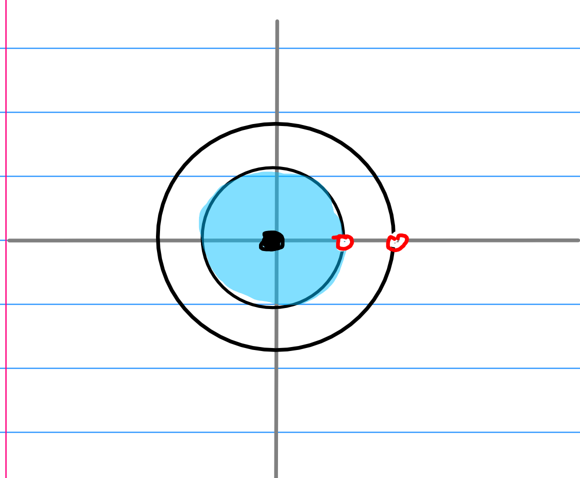


$$f(z) = \frac{1}{z-1} - \frac{1}{z-2} = \frac{-1}{1-z} + \frac{1}{2}\frac{1}{1-(\frac{z}{2})}$$

$$= -\sum_{n=0}^{\infty} z^n + \sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}} = \sum_{n=0}^{\infty}(2^{-n+1}-1)z^n \quad |z|<1$$

$$a_n = \quad \frac{f(z)}{z^{n+1}} = \frac{1}{(z-1)(z-2)z^{n+1}} \qquad \frac{1}{z-1} - \frac{1}{z-2}$$

$$a_n = \sum_{k=1}^{M} \mathrm{Res}\left(\frac{f(z)}{(z-z_k)^{n+1}}, z_k\right) = \mathrm{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 0\right)$$



$$a_n = \sum_{k=1}^{M} \mathrm{Res}\left(\frac{f(z)}{(z-z_k)^{n+1}}, z_k\right) = \mathrm{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 0\right)$$

$n \geq 0$ then the pole $z=0$

$$\frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} (z-z_0)^n f(z) \quad (\text{order } n)$$

$$\frac{d}{dz}\left((z-1)^{-1} - (z-2)^{-1}\right) = (-1)\left((z-1)^{-2} - (z-2)^{-2}\right)$$

$$\frac{d^2}{dz^2}\left((z-1)^{-1} - (z-2)^{-1}\right) = (-1)(-2)\left((z-1)^{-3} - (z-2)^{-3}\right)$$

$$\frac{d^3}{dz^3}\left((z-1)^{-1} - (z-2)^{-1}\right) = (-1)(-2)(-3)\left((z-1)^{-4} - (z-2)^{-4}\right)$$

$$\frac{d^n}{dz^n}\left((z-1)^{-1} - (z-2)^{-1}\right) = (-1)^n n! \left((z-1)^{-n-1} - (z-2)^{-n-1}\right)$$

$$\frac{1}{n!} \lim_{z \to 0} \frac{d^n}{dz^n}\left((z-1)^{-1} - (z-2)^{-1}\right) = (-1)^n \lim_{z \to 0} \left((z-1)^{-n-1} - (z-2)^{-n-1}\right)$$

$$= (-1)^n \left((-1)^{-n-1} - (-2)^{-n-1}\right)$$

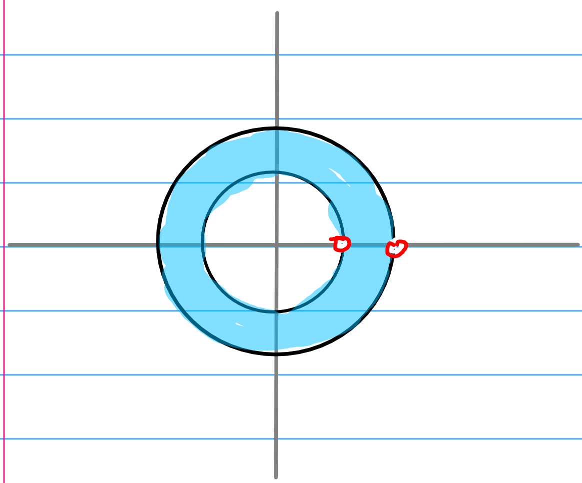
$$= -1 + 2^{-n-1}$$

$$a_n = -1 + 2^{-n-1} \qquad (n \geq 0)$$

$$= -\sum_{n=0}^{\infty} z^n + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} = \sum_{n=0}^{\infty} (2^{-n-1} - 1) z^n \qquad |z|<1$$

$$f(z) = \frac{-1}{(z-1)(z-2)}$$

② D_2 $\quad 1 < |z| < 2 \Rightarrow \left|\frac{1}{z}\right| < 1, \quad \left|\frac{z}{2}\right| < 1$



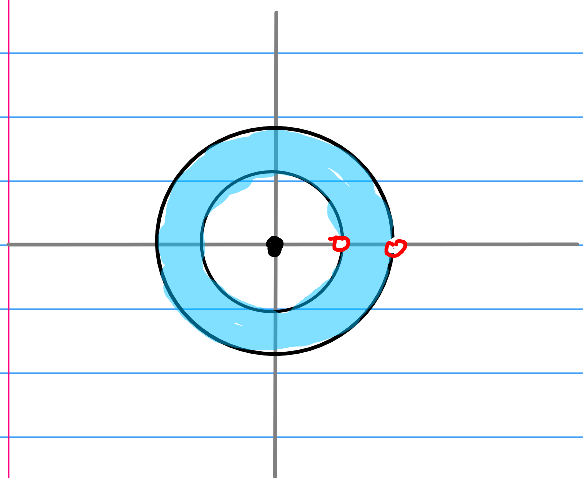
$$f(z) = \frac{1}{z-1} - \frac{1}{z-2} = \frac{1}{z} \cdot \frac{1}{1-\left(\frac{1}{z}\right)} + \frac{1}{2}\frac{1}{1-\left(\frac{z}{2}\right)}$$

$$= \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}$$

$$= \sum_{n=1}^{\infty} \frac{1}{z^n} + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}$$

$$a_n = \sum_{k=1}^{M} \mathrm{Res}\left(\frac{f(z)}{(z-z_n)^{n+1}}, z_k\right) = \mathrm{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 0\right)$$

$$+ \mathrm{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 1\right)$$



$$a_n = \sum_{k=1}^{M} \mathrm{Res}\left(\frac{f(z)}{(z-z_n)^{n+1}}, z_k\right) = \mathrm{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 0\right) + \mathrm{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 1\right)$$

$$\frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} (z-z_0)^n f(z) \quad (\text{order } n)$$

$$\frac{1}{n!} \lim_{z \to 0} \frac{d^n}{dz^n}\left((z-1)^{-1} - (z-2)^{-1}\right) = (-1)^n \lim_{z \to 0} \left((z-1)^{-n-1} - (z-2)^{-n-1}\right)$$

$$= (-1)^n \left((-1)^{-n-1} - (-2)^{-n-1}\right)$$

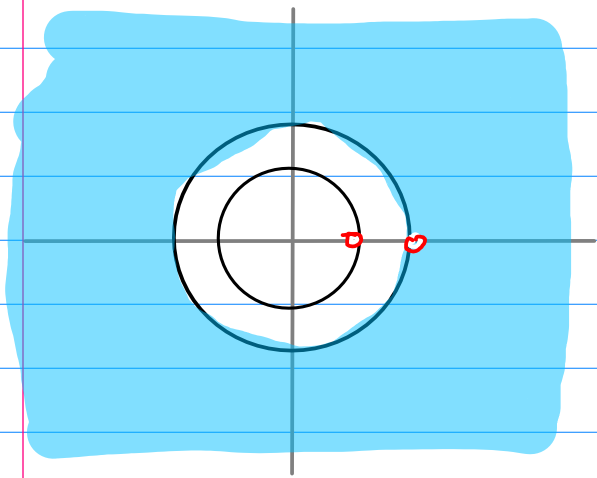
$$= -1 + 2^{-n-1}$$

$$\mathrm{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 0\right) = -1 + 2^{-n-1}$$

$$\mathrm{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 1\right) = \lim_{z \to 1} (z-1) \frac{-1}{(z-1)(z-2)z^{n+1}} = 1$$

$$f(z) = \frac{-1}{(z-1)(z-2)}$$

③ D_3 $\quad 2 < |z| \quad \left|\frac{2}{z}\right| < 1 \quad \left|\frac{1}{z}\right| < 1$

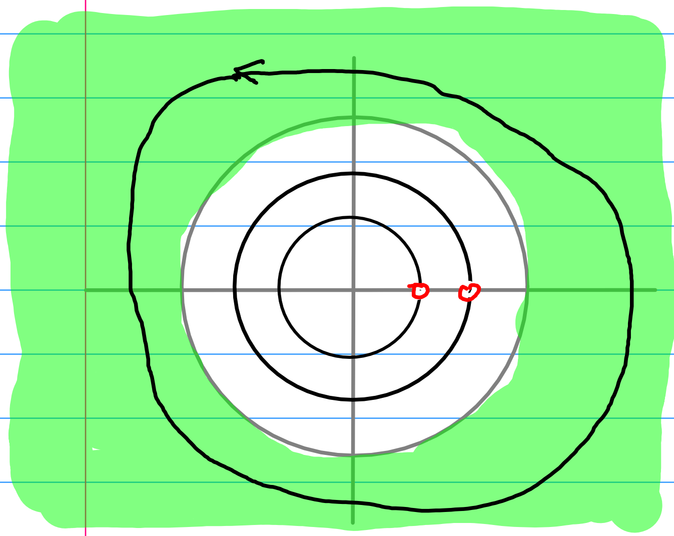


$$f(z) = \frac{1}{z-1} - \frac{1}{z-2} = \frac{1}{z}\frac{1}{1-\left(\frac{1}{z}\right)} - \frac{1}{z}\frac{1}{1-\left(\frac{2}{z}\right)}$$

$$= \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} - \sum_{n=0}^{\infty} \frac{2^n}{z^{n+1}} = \sum_{n=0}^{\infty} \frac{1-2^n}{z^{n+1}}$$

$$= \sum_{n=1}^{\infty} \frac{1-2^{n-1}}{z^n}$$

$$f(z) = \frac{-1}{(z-1)(z-2)}$$



$x[n]$

$$= \frac{1}{2\pi i} \int_C \boxed{X(z)\, z^{n-1}}\, dz$$

$$= \sum_{j=1}^{k} \mathrm{Res}\left(\boxed{X(z)\, z^{n-1}}, z_j\right)$$

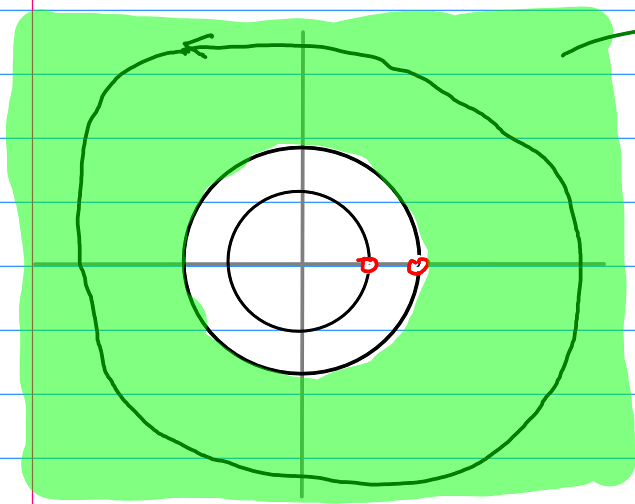
$$X(z) = \frac{-1}{(z-1)(z-2)}$$

$$X(z)\, z^{n-1} = \frac{-1}{(z-1)(z-2)}\, z^{n-1}$$

$$\mathrm{Res}\left(\boxed{X(z)\, z^{n-1}}, 1\right) = (z-1)\frac{-1}{(z-1)(z-2)}\, z^{n-1}\Big|_{z=1} = 1$$

$$\mathrm{Res}\left(\boxed{X(z)\, z^{n-1}}, 2\right) = (z-2)\frac{-1}{(z-1)(z-2)}\, z^{n-1}\Big|_{z=2} = -2^{n-1}$$

$$x[n] = 1 - 2^{n-1}$$



ROC (Region of Convergence)

$$|z|>2 \Rightarrow \frac{2}{|z|}<1$$

$$\left(\frac{2}{z}\right)^0+\left(\frac{2}{z}\right)^1+\left(\frac{2}{z}\right)^2+\cdots \xrightarrow[\text{converge}]{} \frac{1}{1-\frac{2}{z}}$$

$$|z|>2 \Rightarrow \frac{1}{|z|}<1$$

$$\left(\frac{1}{z}\right)^0+\left(\frac{1}{z}\right)^1+\left(\frac{1}{z}\right)^2+\cdots \xrightarrow[\text{converge}]{} \frac{1}{1-\frac{1}{z}}$$

$$f(z)=\frac{1}{z-1}-\frac{1}{z-2}=\frac{1}{z}\,\frac{1}{1-\left(\frac{1}{z}\right)}-\frac{1}{z}\,\frac{1}{1-\left(\frac{2}{z}\right)}$$

$$=\sum_{n=0}^{\infty}\frac{1}{z^{n+1}}-\sum_{n=0}^{\infty}\frac{2^n}{z^{n+1}}=\sum_{n=0}^{\infty}\frac{1-2^n}{z^{n+1}}$$

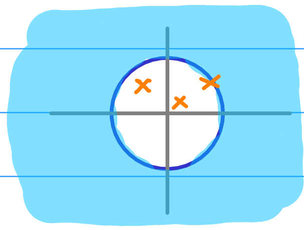
$$=\sum_{n=1}^{\infty}\frac{1-2^{n-1}}{z^n}$$

$$\left(\frac{1}{z}\right)+\left(\frac{1}{z}\right)^2+\left(\frac{1}{z}\right)^3+\cdots$$
$$+\frac{1}{2}\left\{\left(\frac{2}{z}\right)+\left(\frac{2}{z}\right)^2+\left(\frac{2}{z}\right)^3+\cdots\right\} \xrightarrow[\text{converge}]{} \frac{1}{z-1}-\frac{1}{z-2}=\frac{-1}{(z-1)(z-2)}$$

$$(1-2^0)z^{-1}+(1-2^1)z^{-2}+(1-2^2)z^{-3}+\cdots \xrightarrow[\text{converge}]{} \frac{-1}{(z-1)(z-2)} \quad (|z|>2)$$

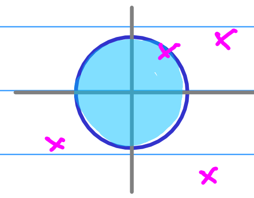
$$x[n]=1-2^{n-1} \longleftrightarrow X(z)=\frac{-1}{(z-1)(z-2)} \quad (|z|>2)$$

causal $x[n]=0 \quad (n<0)$



ROC: outside a circle

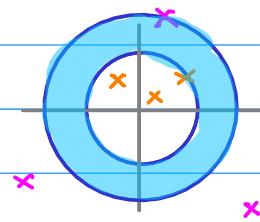
anti-causal $x[n]=0 \quad (n>0)$



ROC: inside a circle

bi-causal $x[n]$

Overlapped ROC



$$f(z) = \sum_{n=n_1}^{\infty} a_n^{\{m\}} (z - z_m)^n$$

$$f(z) = \sum_{n=n_1}^{\infty} a_n z^n \qquad z_m = 0 \qquad a_n^{\{0\}} \rightarrow a_n$$

Laurent Series at $z=0$

$$f(z) = \cdots + a_{-2} z^{-2} + a_{-1} z^{-1} + a_0 z^0 + a_1 z^1 + a_2 z^2 + a_3 z^3 + \cdots$$

Z-transform

Bi-causal $\quad X(z) = \cdots + x[-2] z^2 + x[-1] z^1 + x[0] z^0 + x[1] z^{-1} + x[2] z^{-2} + x[3] z^{-3} + \cdots$

Causal $\quad X(z) = x[0] z^0 + x[1] z^{-1} + x[2] z^{-2} + x[3] z^{-3} + \cdots$

Anti-causal $\quad X(z) = \cdots + x[-2] z^2 + x[-1] z^1 + x[0] z^0$

$$a_n \longleftrightarrow x[-n]$$

$$a_{-n} \longleftrightarrow x[n]$$

$$f(z) = \sum_{n=n_1}^{\infty} a_n^{\{m\}} (z - z_m)^n$$

$$a_n^{\{m\}} = \frac{1}{2\pi i} \oint_c \frac{f(z')}{(z'-z_m)^{n+1}} dz'$$

$$= \sum_k \mathrm{Res}\left(\frac{f(z)}{(z-z_m)^{n+1}}, z_k \right)$$

analytic at z_m

$n_1 \geq 0$ Taylor Series

general n_1 $z_m = 0$ MacLaurin Series

singular at z_m

general n_1 Laurent Series

general n_1 $z_m = 0$ z - Transform

$$f(z) = \sum_{n=n_1}^{\infty} a_n^{\{m\}} (z - z_m)^n$$

$$a_n^{\{m\}} = \frac{1}{2\pi i} \oint_c \frac{f(z')}{(z' - z_m)^{n+1}} \, dz'$$

$$= \sum_k \mathrm{Res}\left(\frac{f(z)}{(z - z_m)^{n+1}} , z_k \right)$$

$z_m = 0$ | $a_{-n}^{\{0\}} = h(n)$ | $n \to -n$

$$H(z) = \sum_{n=n_1}^{\infty} h(-n) \, z^n$$

$$h(n) = \frac{1}{2\pi i} \oint_c \frac{H(z')}{z'^{\,n+1}} \, dz'$$

$$= \sum_k \mathrm{Res}\left(\frac{H(z)}{z^{n+1}} , z_k \right)$$

$$H(z) = \sum_{n=-\infty}^{\infty} h(n) \, z^{-n}$$

$$h(n) = \frac{1}{2\pi i} \oint_c H(z') \, z'^{\,n-1} \, dz'$$

$$= \sum_k \mathrm{Res}\left(H(z) \, z^{n-1} , z_k \right)$$

C is in the same region of analyticity of $f(z)$

typically a circle centered on z_m

z_k within C : singularities of $\frac{f(z)}{(z-z_m)^{n+1}}$

C is in the same region of analyticity of $H(z)$

typically a circle centered on z_m

generally a circle centered on the origin

may enclose any or all singularities of $H(z)$

often the unit circle

z_k within C : singularities of $H(z)\, z^{n-1}$

$$H(z) = \sum_{n=-\infty}^{\infty} h(n)\, z^{-n} \qquad z \in \text{R.O.C}$$

$$h(n) = \frac{1}{2\pi i} \oint_C H(z')\, z'^{\,n-1}\, dz' \qquad C \text{ in R.O.C.}$$

$$= \sum_k \operatorname{Res}\left(H(z)\, z^{n-1}, z_k\right)$$

① a power series representation of a function $f(z)$ of a complex variable z

② a transform $H(z)$ of a sequence of 1

$$X(z) = \frac{z}{z-\frac{1}{2}} \qquad \text{pole } z_0 = \frac{1}{2}$$

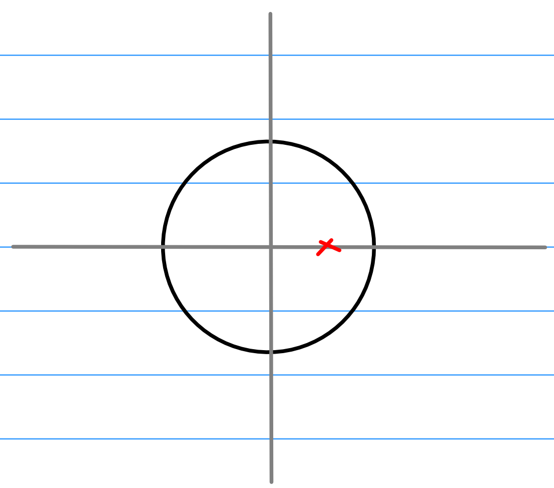
$$x[n] = \text{Res}\left(X(z)z^{n-1}, \; z_0\right) = \text{Res}\left(\frac{z}{z-\frac{1}{2}} z^{n-1}, \frac{1}{2}\right)$$

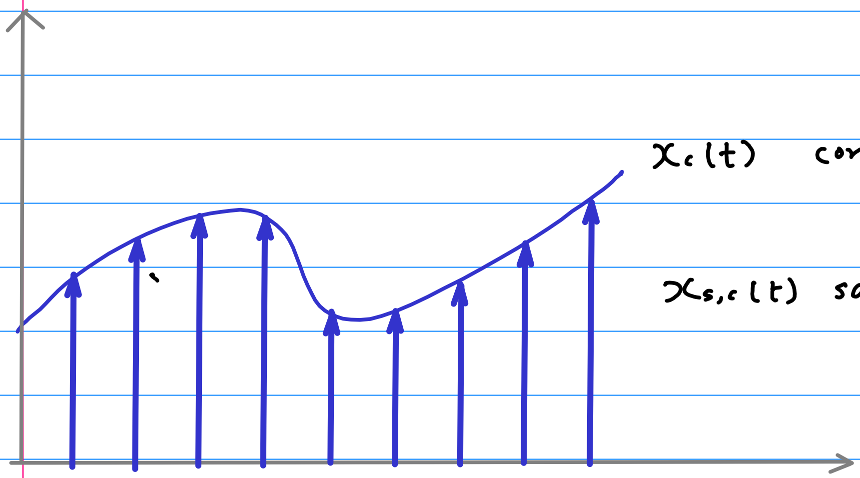
$$= \text{Res}\left(\frac{z^n}{z-\frac{1}{2}}, \frac{1}{2}\right) = \lim_{z \to \frac{1}{2}} \left(z-\frac{1}{2}\right)\frac{z^n}{z-\frac{1}{2}} = \left(\frac{1}{2}\right)^n$$

$$x[n] = \frac{1}{2^n} \qquad n \geq 0$$

$$\left(\frac{1}{2}\right)^0 z^0 + \left(\frac{1}{2}\right)^1 z^{-1} + \left(\frac{1}{2}\right)^2 z^{-2} + \left(\frac{1}{2}\right)^3 z^{-3} + \cdots = \frac{1}{1-\left(\frac{1}{2}z^{-1}\right)}$$

$$= \frac{z}{z-\frac{1}{2}}$$





$$x_{s,c}(t) = \sum_{n=-\infty}^{+\infty} x(n)\, \delta_c(t - n\Delta t)$$

$$X_{s,c}(s) = \mathcal{L}\{x_{s,c}(t)\} = \int_{-\infty}^{\infty} \boxed{\sum_{n=-\infty}^{+\infty} x(n)\, \delta_c(t - n\Delta t)}\, e^{-st}\, dt$$

$$= \sum_{n=-\infty}^{+\infty} x(n) \int_{-\infty}^{\infty} \delta_c(t - n\Delta t)\, e^{-st}\, dt$$

$$= \sum_{n=-\infty}^{+\infty} x(n)\, e^{-sn\Delta t} \qquad e^{s\Delta t} \triangleq z$$

$$X_{s,c}(s) = \sum_{n=-\infty}^{+\infty} x(n)\, z^{-n} \Big|_{z = e^{s\Delta t}}$$

$$X_{s,c}(s) = X(z) \Big|_{z = e^{s\Delta t}}$$

$$X_{s.c}(s) = \mathcal{L}\{x_{s.c}(t)\} = X(z)\Big|_{z=e^{s\Delta t}}$$

$x_{s.c}(t)$ an impulse train

whose coefficients are given by $x[n] = x_c(n\Delta t)$

Z - transform : a special Laurent series

$\boxed{z_m = 0}$ $\boxed{a_{-n}^{\{0\}} = h(n)}$ $\boxed{n \to -n}$

$$\boxed{\begin{aligned} f(z) &= \sum_{n=n_1}^{\infty} a_n^{\{m\}} (z - z_m)^n \\ a_n^{\{m\}} &= \frac{1}{2\pi i} \oint_c \frac{f(z')}{(z'-z_m)^{n+1}} dz' \\ &= \sum_k \mathrm{Res}\left(\frac{f(z)}{(z-z_m)^{n+1}}, z_k \right) \end{aligned}}$$

Time Reversal ← Laplace Transform

the transform functions

$X(s) = \int$ over negative powers e^{-st} for $t > 0$

$X(z) = \int$ over negative powers z^{-n} for $n > 0$

the time expansion functions

$x(t) = \int$ over negative powers e^{-st} for $t > 0$

$x[n] = \int$ over negative powers z^{-n} for $n > 0$

Time Reversal ← z^{-1} : unit delay , char eq (modes in z^k)

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

Stable System : $h[n]$ must be asbsolutely summable

$$|e^{j\omega n}| = 1$$

$$|z^n| \qquad z = 1$$

$$\infty > M_h > \sum_{n=-\infty}^{\infty} |h[n]| \qquad \text{asolutely summable}$$

$$= \sum_{n=-\infty}^{+\infty} |h[n]\, e^{-j\omega n}|$$

$$\geq \left| \sum_{n=-\infty}^{+\infty} h[n]\, e^{-j\omega n} \right|$$

$$= \left| \, H(z)\big|_{z=e^{j\omega}} \right|$$

$$\infty > \left| \, H(z)\big|_{z=e^{j\omega}} \right|$$

a stable system,

$H(z)$ must converge on the unit circle $|z| = 1$

ROC (Region of Convergence) must include the unit circle

regardless of causality of $h[n]$

$$H(z)\Big|_{|z|=1} = H(e^{j\hat{\omega}})$$ DTFT of $h[n]$

discrete — all stable sequence must have convergent DTFTs

continuous — all stable signal must have convergent CTFTs

$C \leftarrow$ unit circle $\quad z = e^{j\hat{\omega}}$

ZT^{-1} $\quad DTFT^{-1}$ identical formulas

$h[n]$ causal

$$H(z) = \sum_{n=-\infty}^{+\infty} h[n]\, z^{-n} = \sum_{n=0}^{+\infty} h[n]\, z^{-n} \qquad n \in [0, \infty)$$

for finite values of n,

each term must be finite as long as $z \neq 0$

For the sum to converge,

$h[n]\, z^{-n}$ must vanish as $n \to \infty$

$|z| > r_h \qquad z_h = r_h e^{j\theta}$

z_h^n is the largest magnitude

geometrically increasing component

$n^m z_e^n$: the most general term

for impulse responses

$n \to \infty \qquad z_e^n$ dominant over n^m for finite m

geometric components — as poles

$$\mathcal{Z}\{z_\ell^n u[n]\} = \frac{1}{1-\left(\frac{z_\ell}{z}\right)} = \frac{z}{z - z_\ell}$$

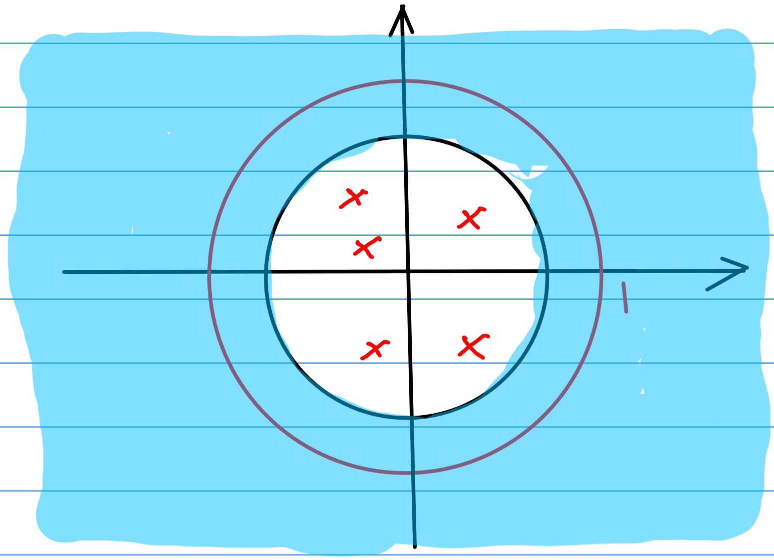
ROC of a causal sequence $h[n]$
outside the radius of the largest magnitude pole of $H(z)$

ROC of a causal signal $h(t)$
to the right of the rightmost pole of $H_c(s)$

if $h[n]$ is a stable, causal sequence,
the unit circle must be included in the ROC

- Causal $h[n]$

ROC : outside of a circle



- Stable $h[n]$

all poles inside the unit circle

ROC circle must be smaller than the unit circle

$\Rightarrow$ all the geometric components of $h[n]$: modes must decay with increasing n

all the poles of $H(z)$ must be within the unit circle

all the poles of $H_c(s)$ must be in the left half plane

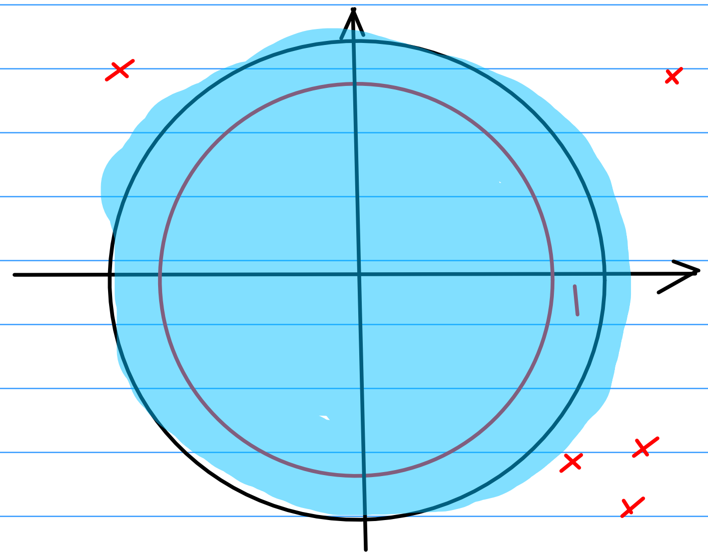
- anti-Causal $h[n]$

ROC: inside of a circle

- Stable $h[n]$

all poles outside the unit circle

ROC circle must be larger than the unit circle



$\Rightarrow$ all the geometric components of $h[n]$: modes must decay with decreasing n

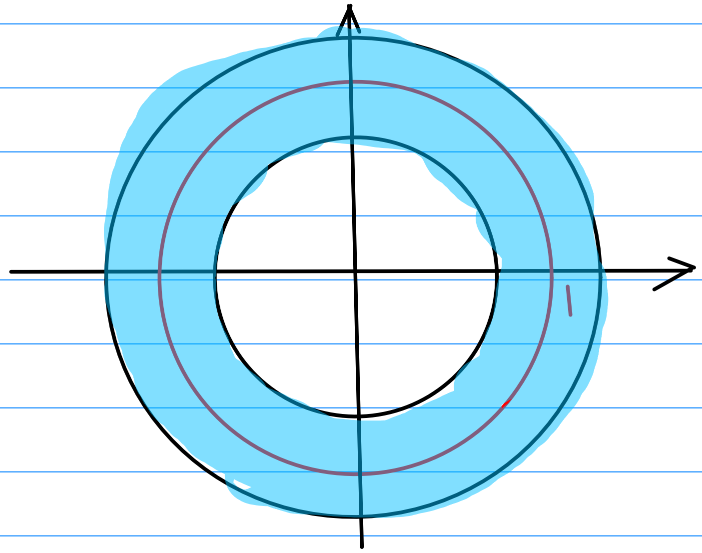
- bi-Causal $h[n]$

  $h_c[n] + h_{ac}[n]$

  outside inside

  max mag < min mag

  Overlapped ROC



- Stable $h[n]$

  all poles <u>outside</u>

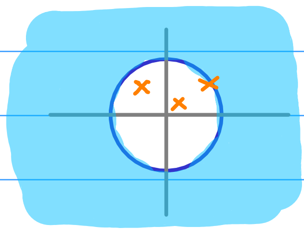
  the unit circle

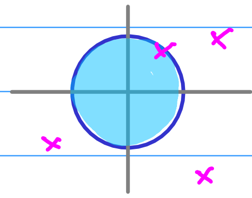
  ROC circle must include the unit circle

- bi-causal $h[n]$

$h[n] = h_c[n] + h_{ac}[n]$

causal comp.   anti-causal comp

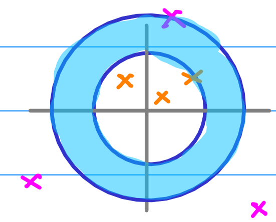




outside a circle   inside a circle

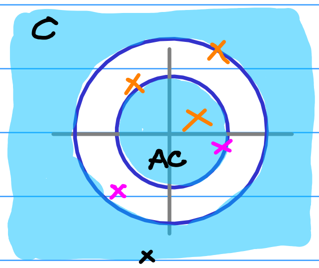
max mag < min mag

Overlapped ROC



max mag > min mag

non-overlapping ROC



- Stable $h[n]$

all poles outside the large circle
inside the small circle

ROC circle must include the unit circle
only one annulus include the unit circle
only one stable sequence

Existence of the z-Transform

$$X(z) = \sum_{n=0}^{\infty} x[n] z^{-n} = \sum_{n=0}^{\infty} \frac{x[n]}{z^n}$$

the existence of the z-transform is guaranteed if

$$|X(z)| \leq \sum_{n=0}^{\infty} \frac{|x[n]|}{|z^n|} < \infty \quad \text{for some } |z|$$

any signal $x[n]$ that grows no faster than an exponential signal r_0^n, for some r_0 satisfies the above condition

if $|x[n]| \leq r_0^n$ for some r_0

then $$|X(z)| \leq \sum_{n=0}^{\infty} \left(\frac{r_0}{|z|}\right)^n = \frac{1}{1 - \frac{r_0}{|z|}} \quad |z| > r_0$$

therefore $X(z)$ exists for $|z| > r_0$

Almost all practical signal satisfy this condition $|x[n]| \leq r_0^n$ for some r_0 and z-transformable

Some signal models (e.g. r^{n^2}) grows faster than the exponential signal r_0^n (for any r_0) and do not satisfy this condition and are not z-transformable

Such signals are of little practical or theoretical interest

Even such signals over a finite interval are z-transformable

Region of Convergence

Laplace Transform $Ae^{\alpha t}u(t)$ $\alpha > 0$

z - Transform $A\alpha^n u[n]$ $|\alpha| > 0$

DTFT (X)

$$X(z) = A \sum_{n=-\infty}^{\infty} \alpha^n u[n] z^{-n} = A \sum_{n=0}^{\infty} \alpha^n z^{-n} = A \sum_{n=0}^{\infty} \left(\frac{\alpha}{z}\right)^n$$

converge $\left|\frac{\alpha}{z}\right| < 1$ $|z| > |\alpha|$

open exterior of
a circle of radius $|\alpha|$

the sum of a geometric series

$$X(z) = A \frac{1}{1 - \frac{\alpha}{z}} = \frac{A}{1 - \alpha z^{-1}} = A \frac{z}{z - \alpha} \qquad |z| > |\alpha|$$

DTFT

$$X(j\hat{\omega}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\hat{\omega}n}$$

DTFT

DTFT of the unit sequence $u[n]$

$$X(e^{-j\hat{\omega}n}) = \sum_{n=-\infty}^{+\infty} u[n] e^{-j\hat{\omega}n} = \sum_{n=0}^{\infty} e^{-j\hat{\omega}n}$$

not converge

$\hat{\omega} = 0$ $\quad \sum_{n=0}^{\infty} 1^n$ diverge

$\hat{\omega} = \pi$ $\quad \sum_{n=0}^{\infty} (-1)^n$ oscillates

$\hat{\omega} = \frac{\pi}{2}$ $\quad \sum_{n=0}^{\infty} (j)^n$

The DTFTs of some commonly used functions do not exist in the strict sense.

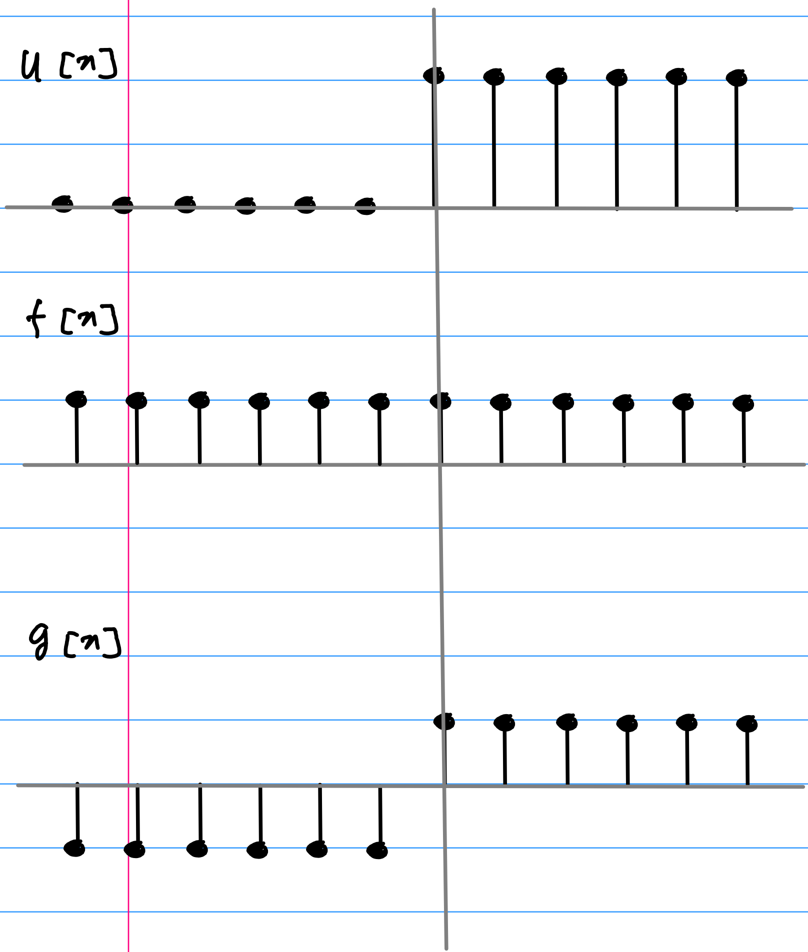
But even though the DTFT does not exist, the z-transform does exist.

$$X(z) = \sum_{n=-\infty}^{+\infty} u[n] z^{-n} = \sum_{n=0}^{\infty} z^{-n}$$

$|z| > 1$ $\quad X(z) = \frac{z}{z-1} = \frac{1}{1-z^{-1}}$

$X(z) = \frac{z}{z-1}$ $\quad$ pole $z=1$, zero $z=0$

$X(z) = \frac{1}{1-z^{-1}}$ $\quad$ useful when a system is synthesized from a z-domain transfer function http://math.stackexchange.com/questions/1772285/dtft-of-the-unit-step-function



$$f[n] = \frac{1}{2} \qquad -\infty < n < \infty$$

$$g[n] = \begin{cases} \frac{1}{2} & n \geq 0 \\ -\frac{1}{2} & n < 0 \end{cases}$$

$$u[n] = f[n] + g[n]$$

$$\delta[n] = g[n] - g[n-1]$$

$$1 = G(e^{j\hat{\omega}}) - e^{-j\hat{\omega}} G(e^{j\hat{\omega}})$$

$$G(e^{j\hat{\omega}}) = \frac{1}{1 - e^{-j\hat{\omega}}}$$

$$F(e^{j\hat{\omega}}) = \pi \sum_{k=-\infty}^{+\infty} \delta(\omega - 2\pi k) \qquad \text{(impulse train)}$$

$$U(e^{j\hat{\omega}}) = \frac{1}{1 - e^{-j\hat{\omega}}} + \pi \sum_{k=-\infty}^{+\infty} \delta(\omega - 2\pi k)$$

Discrete Time Exponential γ^n

continuous time exponential $e^{\lambda t}$

$$e^{\lambda t} = \gamma^t \qquad (e^{\lambda})^t = \gamma^t$$

$$e^{\lambda} = \gamma$$

$$\lambda = \ln \gamma$$

$$e^{-0.3t} = (0.7408)^t$$

$$4^t = e^{1.386t}$$

continuous time analysis $e^{\lambda t}$

discrete time analysis x^n

$$e^{\lambda n} = \gamma^n \qquad (e^{\lambda})^n = \gamma^n$$

$$e^{\lambda} = \gamma$$

$$\lambda = \ln \gamma$$

$e^{\lambda n}$

exponentially grows if $Re\,\lambda > 0$ (λ in RHP)
exponentially decays if $Re\,\lambda < 0$ (λ in LHP)
oscillates or constant if $Re\,\lambda = 0$ (λ in imag axis)

the location of λ in the complex plain indicates whether

① $e^{\lambda t}$ will grow exponentially
② $e^{\lambda t}$ will decay exponentially
③ $e^{\lambda t}$ will oscillates with constant amplitude

constant signal : oscillation with zero frequency

$e^{j\Omega n}$ $\quad \lambda = j\Omega$ imaginary axis
constant amplitude oscillating signal

$e^{j\Omega n} = (e^{j\Omega})^n = \gamma^n \qquad \gamma = e^{j\Omega} \qquad |\gamma| = 1$

$\lambda = j\Omega$ imaginary axis $\rightarrow$ $|\gamma| = 1$ unit circle

if γ lies on the unit circle,
γ^n oscillates with constant amplitude

the imaginary axis in the λ plane
the unit circle in the γ plane

$e^{\lambda n}$ $\lambda = a + jb$ in the LHP $(a<0)$

exponentially decaying

$\gamma = e^{\lambda} = e^{a+jb} = e^{a} e^{jb}$

$|\gamma| = |e^{\lambda}| = |e^{a}| \cdot |e^{jb}| = |e^{a}| = e^{a}$

$|\gamma| = e^{a} < 1$ inside the Unit circle

γ^{n} : exponentially decaying

$|\gamma| = e^{a} > 1$ outside the Unit circle

γ^{n} : exponentially growing

λ-plane		r-plane
the imaginary axis	$\longrightarrow$	the unit circle
the LHP	$\longrightarrow$	inside of the unit circle
the RHP	$\longrightarrow$	outside of the unit circle