Higher Order ODE's (3B)

Initial Value Problems, and Boundary Value Problems

Please send corrections (or suggestions) to youngwlim@hotmail.com.

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Complex Exponentials

$$c_1 e^{-\sigma t} e^{+i\omega t} + c_2 e^{-\sigma t} e^{-i\omega t}$$

$$(c_1 + c_2) = c_3$$

$$i(c_1 - c_2) = c_4$$

$$= e^{-\sigma t}(c_1 e^{+i\omega t} + c_2 e^{-i\omega t})$$

$$= e^{-\sigma t}[c_1(\cos(\omega t) + i\sin(\omega t)) + c_2(\cos(\omega t) - i\sin(\omega t))]$$

$$= e^{-\sigma t}[(c_1 + c_2)\cos(\omega t) + i(c_1 - c_2)\sin(\omega t)]$$

$$= c_3 e^{-\sigma t}\cos(\omega t) + c_4 e^{-\sigma t}\sin(\omega t)$$

$$c_3 e^{-\sigma t}\cos(\omega t) + c_4 e^{-\sigma t}\sin(\omega t)$$

$$\frac{(c_3 - c_4 i)}{2} = c_1$$

$$\frac{(c_3 + c_4 i)}{2} = c_2$$

$$= c_3 e^{-\sigma t}(e^{+i\omega t} + e^{-i\omega t})/2 + c_4 e^{-\sigma t}(e^{+i\omega t} - e^{-i\omega t})/2i$$

$$= c_3 e^{-\sigma t}(e^{+i\omega t} + e^{-i\omega t})/2 + c_4 e^{-\sigma t}(-i e^{+i\omega t} + i e^{-i\omega t})/2$$

$$= \frac{(c_3 - c_4 i)}{2} e^{-\sigma t} e^{+i\omega t} + \frac{(c_3 + c_4 i)}{2} e^{-\sigma t} e^{-i\omega t}$$

$$= c_1 e^{-\sigma t} e^{+i\omega t} + c_2 e^{-\sigma t} e^{-i\omega t}$$

Complex Exponentials

$$c_1 e^{-\sigma t} e^{+i\omega t} + c_2 e^{-\sigma t} e^{-i\omega t} \quad \rightarrow \quad c_3 e^{-\sigma t} \cos(\omega t) + c_4 e^{-\sigma t} \sin(\omega t)$$

$$(c_1 + c_2) = c_3$$

$$i(c_1 - c_2) = c_4$$

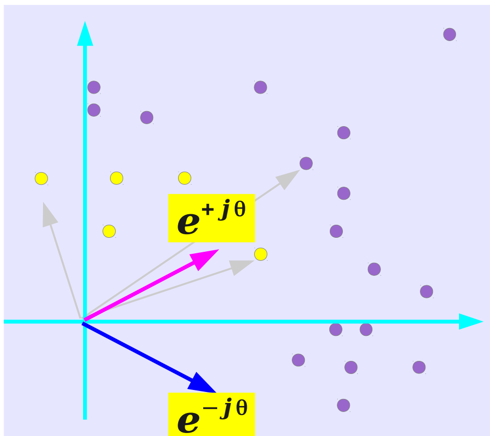
$$c_1 e^{-\sigma t} e^{+i\omega t} + c_2 e^{-\sigma t} e^{-i\omega t} \quad \leftarrow \quad c_3 e^{-\sigma t} \cos(\omega t) + c_4 e^{-\sigma t} \sin(\omega t)$$

$$c_1 = \frac{(c_3 - c_4 i)}{2}$$

$$c_2 = \frac{(c_3 + c_4 i)}{2}$$

Basis of the Complex Plane

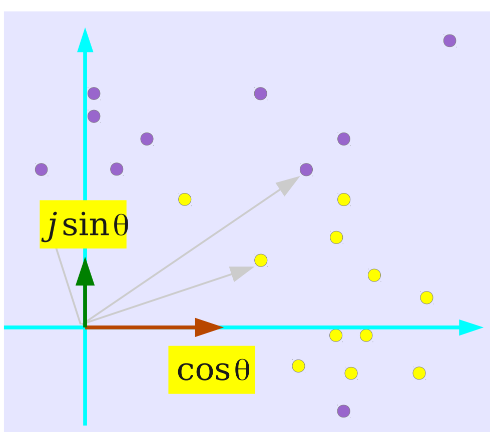
Basis : a set of linear independent spanning vectors



every complex number can be represented by

$$k_1 e^{+j\theta} + k_2 e^{+j\theta}$$

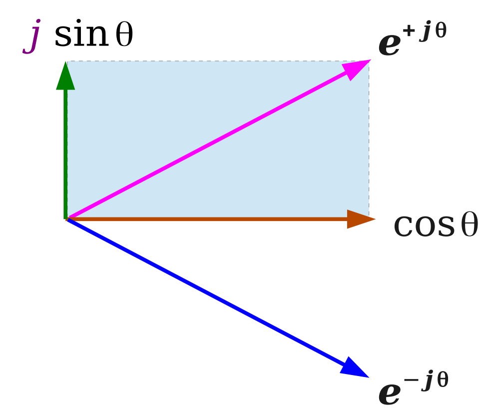
linear combination of $e^{+j\theta}$ and $e^{+j\theta}$
which are one set of linear independent
two vectors



every complex number can also be represented by

$$l_1 \cos\theta + l_2 j \sin\theta$$

$$l_1 \cos\theta + l_2 \; j\sin\theta$$



Real Coefficients C_1 & C_2

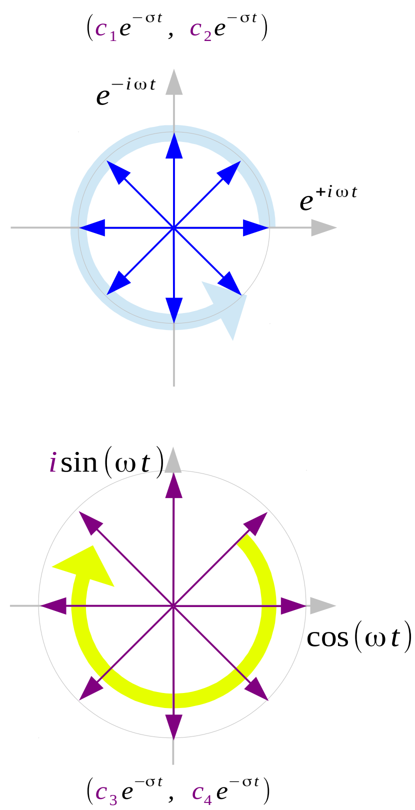
$c_1 e^{+i\omega t} + c_2 e^{-i\omega t}$ ⟷ $c_3 \cos(\omega t) + c_4 \sin(\omega t)$

real number $c_1 = (c_3 - c_4 i)/2$

real number $c_2 = (c_3 + c_4 i)/2$

$c_3 = (c_1 + c_2)$ *real number*

$c_4 = i(c_1 - c_2)$ *imaginary number*



$1 \cdot e^{+i\omega t} + 0 \cdot e^{-i\omega t}$	$1 \cdot \cos(\omega t) + 1 i \cdot \sin(\omega t)$
$\frac{1}{\sqrt{2}} \cdot e^{+i\omega t} + \frac{1}{\sqrt{2}} \cdot e^{-i\omega t}$	$\sqrt{2} \cdot \cos(\omega t) + 0 i \cdot \sin(\omega t)$
$0 \cdot e^{+i\omega t} + 1 \cdot e^{-i\omega t}$	$1 \cdot \cos(\omega t) - 1 i \cdot e^{-\sigma t} \sin(\omega t)$
$\frac{-1}{\sqrt{2}} \cdot e^{+i\omega t} + \frac{1}{\sqrt{2}} \cdot e^{-i\omega t}$	$0 \cdot \cos(\omega t) - \sqrt{2} i \cdot \sin(\omega t)$
$-1 \cdot e^{+i\omega t} + 0 \cdot e^{-i\omega t}$	$-1 \cdot \cos(\omega t) - 1 i \cdot \sin(\omega t)$
$\frac{-1}{\sqrt{2}} \cdot e^{+i\omega t} + \frac{-1}{\sqrt{2}} \cdot e^{-i\omega t}$	$-\sqrt{2} \cdot \cos(\omega t) - 0 i \cdot \sin(\omega t)$
$0 \cdot e^{+i\omega t} - 1 \cdot e^{-i\omega t}$	$-1 \cdot \cos(\omega t) + 1 i \cdot \sin(\omega t)$
$\frac{1}{\sqrt{2}} \cdot e^{+i\omega t} + \frac{-1}{\sqrt{2}} \cdot e^{-i\omega t}$	$0 \cdot \cos(\omega t) + \sqrt{2} i \cdot \sin(\omega t)$

Complex Plane Basis

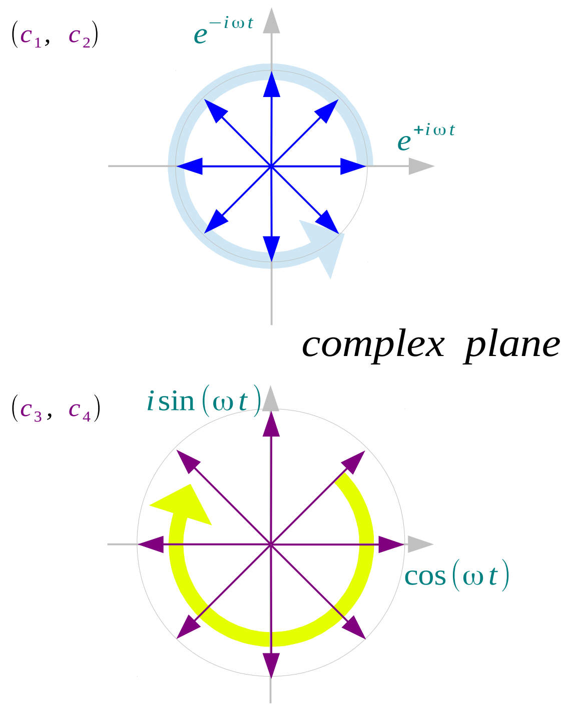
$c_1 e^{+i\omega t} + c_2 e^{-i\omega t}$ ⟷ $c_3' \cos(\omega t) + c_4' i \sin(\omega t)$

$$c_1 = (c_3' + c_4')/2 \quad \textit{real number}$$
$$c_2 = (c_3' - c_4')/2 \quad \textit{real number}$$

$$C^1$$

$$c_3' = (c_1 + c_2) \quad \textit{real number}$$
$$c_4' = (c_1 - c_2) \quad \textit{real number}$$

$$1 \cdot e^{+i\omega t} + 0 \cdot e^{-i\omega t}$$
$$\tfrac{1}{\sqrt{2}} \cdot e^{+i\omega t} + \tfrac{1}{\sqrt{2}} \cdot e^{-i\omega t}$$
$$0 \cdot e^{+i\omega t} + 1 \cdot e^{-i\omega t}$$
$$\tfrac{-1}{\sqrt{2}} \cdot e^{+i\omega t} + \tfrac{1}{\sqrt{2}} \cdot e^{-i\omega t}$$
$$-1 \cdot e^{+i\omega t} + 0 \cdot e^{-i\omega t}$$
$$\tfrac{-1}{\sqrt{2}} \cdot e^{+i\omega t} + \tfrac{-1}{\sqrt{2}} \cdot e^{-i\omega t}$$
$$0 \cdot e^{+i\omega t} - 1 \cdot e^{-i\omega t}$$
$$\tfrac{1}{\sqrt{2}} \cdot e^{+i\omega t} + \tfrac{-1}{\sqrt{2}} \cdot e^{-i\omega t}$$



$$1 \cdot \cos(\omega t) + 1 i \cdot \sin(\omega t)$$
$$\sqrt{2} \cdot \cos(\omega t) + 0 i \cdot \sin(\omega t)$$
$$1 \cdot \cos(\omega t) - 1 i \cdot \sin(\omega t)$$
$$0 \cdot \cos(\omega t) - \sqrt{2} i \cdot \sin(\omega t)$$
$$-1 \cdot \cos(\omega t) - 1 i \cdot \sin(\omega t)$$
$$-\sqrt{2} \cdot \cos(\omega t) - 0 i \cdot \sin(\omega t)$$
$$-1 \cdot \cos(\omega t) + 1 i \cdot \sin(\omega t)$$
$$0 \cdot \cos(\omega t) + \sqrt{2} i \cdot \sin(\omega t)$$

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9/3/14

Real Coefficients C_3 & C_4

$c_1 e^{+i\omega t} + c_2 e^{-i\omega t}$ ⟷ $c_3 \cos(\omega t) + c_4 \sin(\omega t)$

$c_1 = (c_3 - c_4 i)/2$
$c_2 = (c_3 + c_4 i)/2$

conjugate complex number

R^2 : $+2 * real\ part$, $-2 * imag\ part$

$c_3 = (c_1 + c_2)$ *real number*
$c_4 = i(c_1 - c_2)$ *real number*

$\frac{(+1-0i)}{2} \cdot e^{+i\omega t} + \frac{(+1+0i)}{2} \cdot e^{-i\omega t}$ — $1 \cdot \cos(\omega t) + 0 \cdot \sin(\omega t)$

$\frac{(+1-i)}{2\sqrt{2}} \cdot e^{+i\omega t} + \frac{(+1+i)}{2\sqrt{2}} \cdot e^{-i\omega t}$ — $\frac{1}{\sqrt{2}} \cdot \cos(\omega t) + \frac{1}{\sqrt{2}} \cdot \sin(\omega t)$

$\frac{(0-i)}{2} \cdot e^{+i\omega t} + \frac{(0+i)}{2} \cdot e^{-i\omega t}$ — $0 \cdot \cos(\omega t) + 1 \cdot \sin(\omega t)$

$\frac{(-1-i)}{2\sqrt{2}} \cdot e^{+i\omega t} + \frac{(-1+i)}{2\sqrt{2}} \cdot e^{-i\omega t}$ — $\frac{-1}{\sqrt{2}} \cdot \cos(\omega t) + \frac{1}{\sqrt{2}} \cdot \sin(\omega t)$

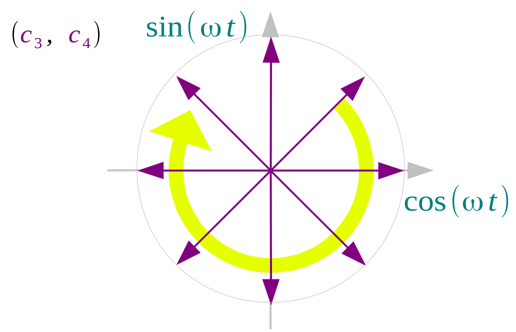
$\frac{(-1-0i)}{2} \cdot e^{+i\omega t} + \frac{(-1+0i)}{2} \cdot e^{-i\omega t}$ — $-1 \cdot \cos(\omega t) + 0 \cdot \sin(\omega t)$

$\frac{(-1+i)}{2\sqrt{2}} \cdot e^{+i\omega t} + \frac{(-1-i)}{2\sqrt{2}} \cdot e^{-i\omega t}$ — $\frac{-1}{\sqrt{2}} \cdot \cos(\omega t) + \frac{-1}{\sqrt{2}} \cdot \sin(\omega t)$

$\frac{(0+i)}{2} \cdot e^{+i\omega t} + \frac{(0-i)}{2} \cdot e^{-i\omega t}$ — $0 \cdot \cos(\omega t) - 1 \cdot \sin(\omega t)$

$\frac{(+1+i)}{2\sqrt{2}} \cdot e^{+i\omega t} + \frac{(+1-i)}{2\sqrt{2}} \cdot e^{-i\omega t}$ — $\frac{1}{\sqrt{2}} \cdot \cos(\omega t) - \frac{1}{\sqrt{2}} \cdot \sin(\omega t)$

real plane



Real Coefficients C_3 & C_4

$$c_1 e^{+i\omega t} + c_2 e^{-i\omega t} \quad \Longleftrightarrow \quad c_3 \cos(\omega t) + c_4 \sin(\omega t)$$

$$c_1 = (c_3 - c_4 i)/2$$
$$c_2 = (c_3 + c_4 i)/2$$

conjugate complex number

R^2 $+2*real\ part$ $-2*imag\ part$

$$c_3 = (c_1 + c_2)$$ *real number*

$$c_4 = i(c_1 - c_2)$$ *real number*

$$A\cos(\omega t - \varphi) \quad \Longleftrightarrow \quad c_3 \cos(\omega t) + c_4 \sin(\omega t)$$

$$\sqrt{c_3^2 + c_4^2} = A$$

$$\frac{c_3}{\sqrt{c_3^2 + c_4^2}} = \cos(\varphi)$$

$$\frac{c_4}{\sqrt{c_3^2 + c_4^2}} = \sin(\varphi)$$

C^1 and R^2 Spaces

$c_1 e^{+i\omega t} + c_2 e^{-i\omega t}$ ⟷ $c_3 \cos(\omega t) + c_4 i \sin(\omega t)$

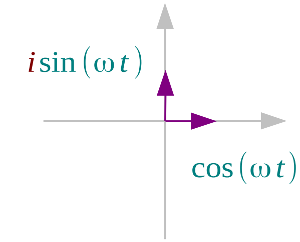
$c_1 = (c_3 + c_4)/2$ *real number*

$c_2 = (c_3 - c_4)/2$ *real number*

$c_3 = (c_1 + c_2)$ *real number*

$c_4 = (c_1 - c_2)$ *real number*

C^1



$c_1 e^{+i\omega t} + c_2 e^{-i\omega t}$ ⟷ $c_3 \cos(\omega t) + c_4 \sin(\omega t)$

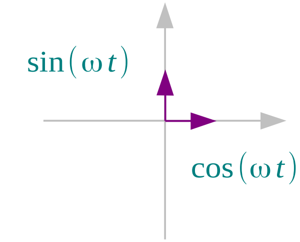
$c_1 = (c_3 - c_4 i)/2$ *conjugate*

$c_2 = (c_3 + c_4 i)/2$ *complex number*

$+2 * real\ part$ $c_3 = (c_1 + c_2)$ *real number*

$-2 * imag\ part$ $c_4 = i(c_1 - c_2)$ *real number*

R^2



Signal Spaces and Phasors

$c_1 e^{+i\omega t} + c_2 e^{-i\omega t}$ ⟺ $c_3 \cos(\omega t) + c_4 \sin(\omega t)$

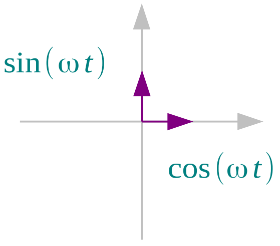
$$c_1 = (c_3 - c_4 i)/2$$
$$c_2 = (c_3 + c_4 i)/2$$

conjugate complex number

$+2*real\ part$: $c_3 = (c_1 + c_2)$ — *real number*

$-2*imag\ part$: $c_4 = i(c_1 - c_2)$ — *real number*

R^2



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Roots of the Auxiliary Equation

Homogeneous Second Order DEs with Constant Coefficients

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

$$a\,y'' + b\,y' + c\,y = 0$$

auxiliary equation

try a solution $y = e^{mx}$ → $(a m^2 + b m + c) = 0$

$$m_1 = \left(-b + \sqrt{b^2 - 4ac}\right)/2a$$

$$m_2 = \left(-b - \sqrt{b^2 - 4ac}\right)/2a$$

$y_1 = e^{m_1 x}, \quad y_2 = e^{m_2 x}$	**(A)**	$b^2 - 4ac > 0$	Real, distinct m_1, m_2
$y_1 = e^{m_1 x} = y_2 = e^{m_2 x}$	**(B)**	$b^2 - 4ac = 0$	Real, equal m_1, m_2
$y_1 = e^{m_1 x}, \quad y_2 = e^{m_2 x}$	**(C)**	$b^2 - 4ac < 0$	Conjugate complex m_1, m_2

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9/3/14

Homogeneous Second Order DEs with Constant Coefficients

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

$$ay'' + by' + cy = 0$$

$$\frac{d^2y}{dx^2} + \frac{b}{a}\frac{dy}{dx} + \frac{c}{a}y = 0$$

$$\frac{d^2y}{dx^2} + 2\lambda\frac{dy}{dx} + \omega^2 y = 0$$

$$\lambda = \frac{b}{2a} \qquad \omega^2 = \frac{c}{a}$$

$$D = \left(\frac{b}{a}\right)^2 - 4\left(\frac{c}{a}\right)$$

$$D = (2\lambda)^2 - 4\omega^2$$

$$D = (b^2 - 4ac)$$

$$D = \lambda^2 - \omega^2$$

$$m_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

$$m_1 = -\lambda + \sqrt{\lambda^2 - \omega^2}$$

$$m_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

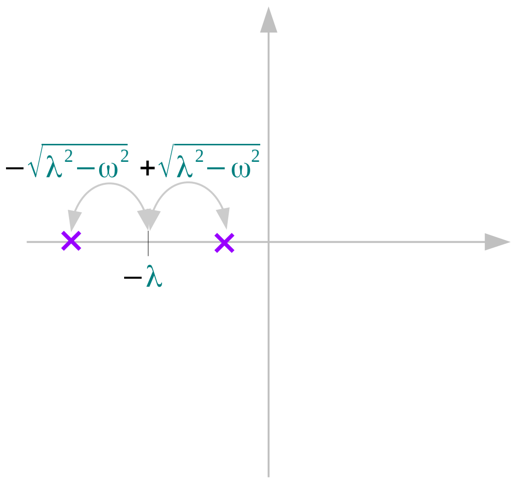
$$m_2 = -\lambda - \sqrt{\lambda^2 - \omega^2}$$

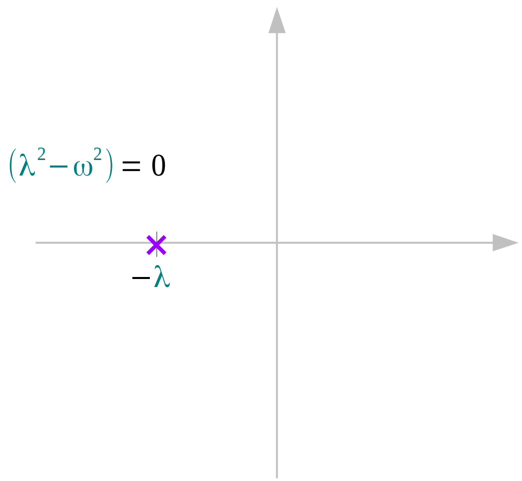
Homogeneous Second Order DEs with Constant Coefficients

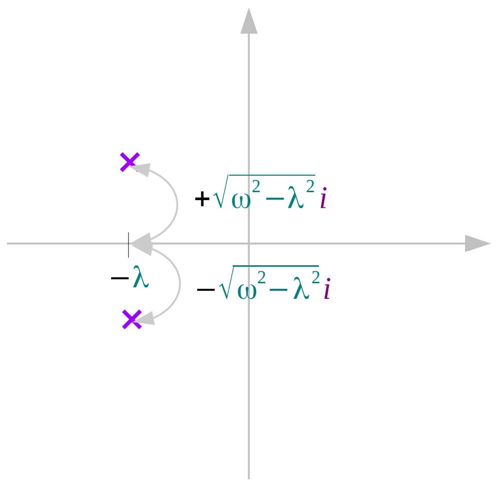
$$\frac{d^2 y}{dx^2} + \frac{b}{a}\frac{dy}{dx} + \frac{c}{a} y = 0$$

$$\frac{d^2 y}{dx^2} + 2\lambda\frac{dy}{dx} + \omega^2 y = 0$$

$$\lambda = \frac{b}{2a} \qquad \omega^2 = \frac{c}{a}$$







overdamped $y(t)$

$$e^{-\lambda t}\left(c_1 e^{+\sqrt{\lambda^2-\omega^2}\,t} + c_2 e^{-\sqrt{\lambda^2-\omega^2}\,t}\right)$$

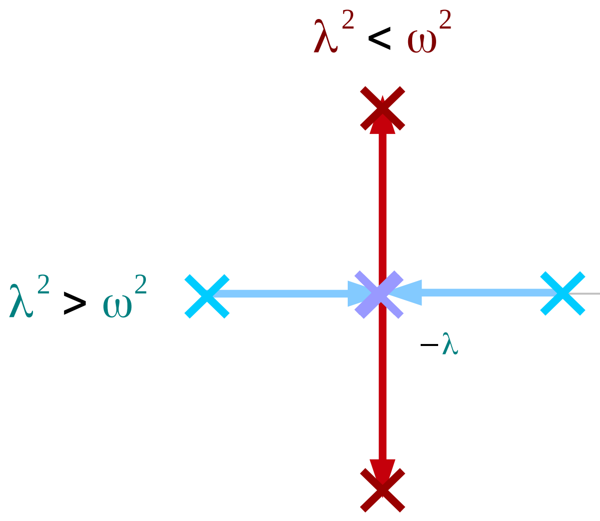
critically damped $y(t)$

$$e^{-\lambda t}\left(c_1 + c_2 t\right)$$

underdamped $y(t)$

$$e^{-\lambda t}\left(c_1 \cos\sqrt{\lambda^2-\omega^2}\,t + c_2 \sin\sqrt{\lambda^2-\omega^2}\,t\right)$$

for a given λ
increasing ω



underdamped $y(t)$ $\lambda^2 < \omega^2$

$$e^{-\lambda t}\left(c_1 \cos\sqrt{\lambda^2-\omega^2}\,t + c_2 \sin\sqrt{\lambda^2-\omega^2}\,t\right)$$

complex m_1, m_2

overdamped $y(t)$ $\lambda^2 > \omega^2$

$$e^{-\lambda t}\left(c_1 e^{+\sqrt{\lambda^2-\omega^2}\,t} + c_2 e^{-\sqrt{\lambda^2-\omega^2}\,t}\right)$$

real m_1, m_2

critically damped $y(t)$ $\lambda^2 = \omega^2$

$$e^{-\lambda t}\left(c_1 + c_2 t\right)$$

real $m_1 = m_2$

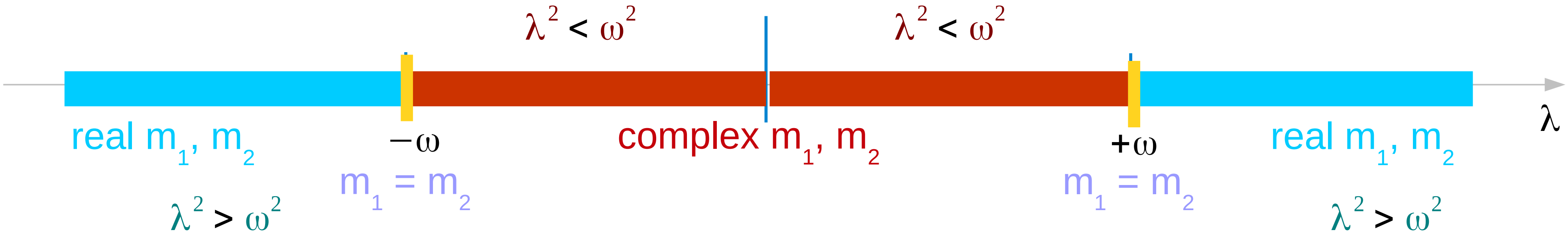
critically damped $y(t)$

$$e^{-\lambda t}(c_1 + c_2 t)$$

$$\lambda^2 = \omega^2$$

underdamped $y(t)$

$$e^{-\lambda t}\left(c_1 \cos\sqrt{\lambda^2-\omega^2}\,t + c_2 \sin\sqrt{\lambda^2-\omega^2}\,t\right)$$



overdamped $y(t)$

$$e^{-\lambda t}\left(c_1 e^{+\sqrt{\lambda^2-\omega^2}\,t} + c_2 e^{-\sqrt{\lambda^2-\omega^2}\,t}\right)$$

for a given ω
increasing λ

Homogeneous Second Order DEs with Constant Coefficients

$$\frac{d^2y}{dx^2} + \frac{b}{a}\frac{dy}{dx} + \frac{c}{a}y = 0$$

$$\frac{d^2y}{dx^2} + 2\lambda\frac{dy}{dx} + \omega^2 y = 0$$

$$\lambda = \frac{b}{2a} \qquad \omega^2 = \frac{c}{a}$$

$\lambda = 1 \qquad \omega = 1/2$

$$y'' + 2y' + y/4 = 0$$

$$y = c_1 e^{(-1+\sqrt{3}/2)t} + c_2 e^{(-1-\sqrt{3}/2)t}$$

$\lambda = 1 \qquad \omega = 1$

$$y'' + 2y' + y = 0$$

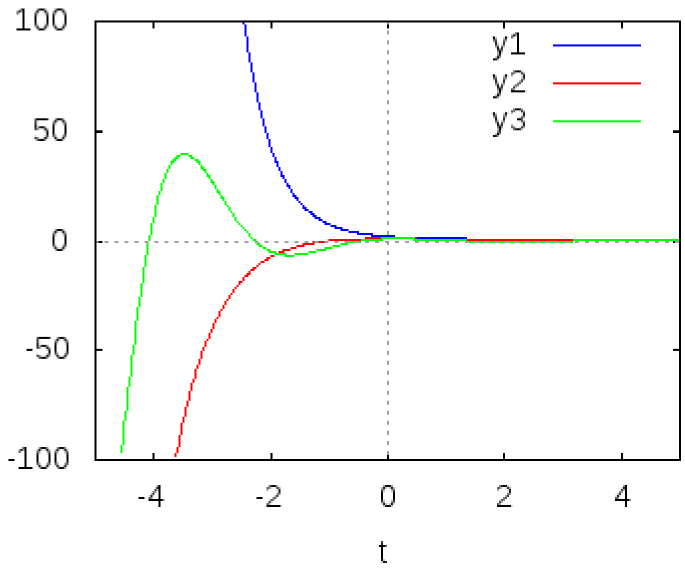
$$y = c_1 e^{-t} + c_2 t e^{-t}$$

$\lambda = 1 \qquad \omega = 2$

$$y'' + 2y' + 4y = 0$$

$$y = c_1 e^{(-1+\sqrt{3}i)t} + c_2 e^{(-1-\sqrt{3}i)t}$$

$$= e^{-t}\left(c_3\cos(\sqrt{3}t) + c_4\sin(\sqrt{3}t)\right)$$



$\lambda = 1 \qquad \omega = 1/2$

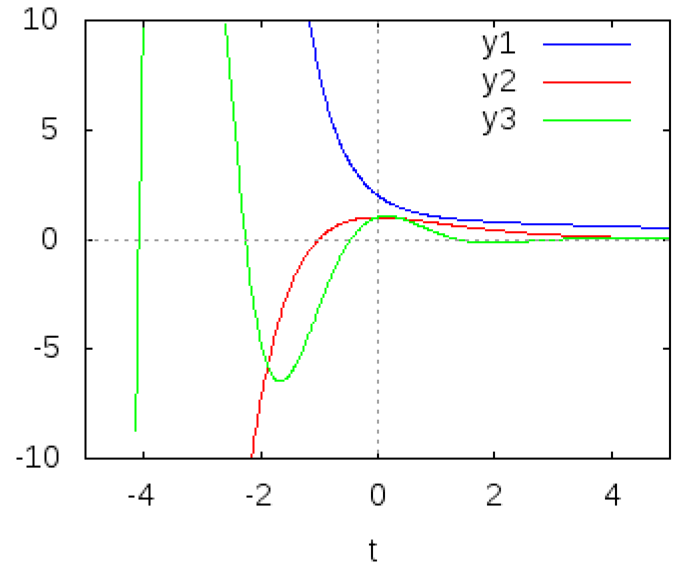
$$y_1(t) = e^{(-1+\sqrt{3}/2)t} + e^{(-1-\sqrt{3}/2)t}$$

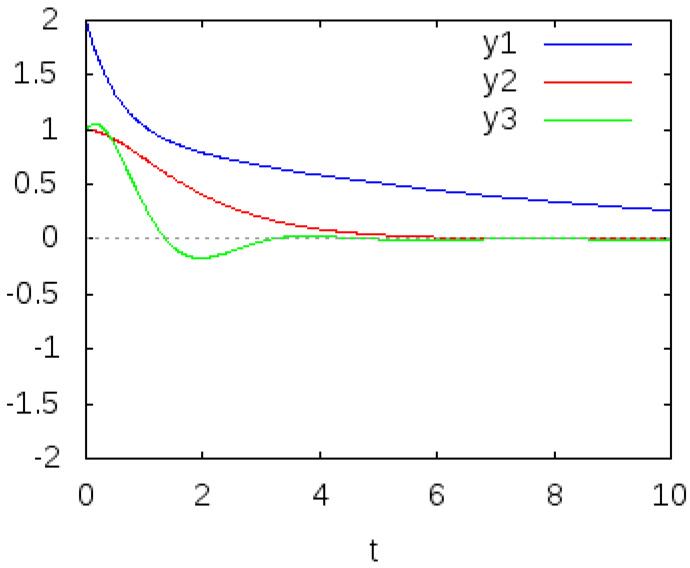
$\lambda = 1 \qquad \omega = 1$

$$y_2(t) = e^{-t} + t\,e^{-t}$$

$\lambda = 1 \qquad \omega = 2$

$$y_3(t) = e^{-t}(\cos(\sqrt{3}t) + \sin(\sqrt{3}t))$$





Homogeneous Second Order DEs with Constant Coefficients

$$\frac{d^2y}{dx^2} + \frac{b}{a}\frac{dy}{dx} + \frac{c}{a}y = 0$$

$$\frac{d^2y}{dx^2} + 2\lambda\frac{dy}{dx} + \omega^2 y = 0 \qquad \lambda = \frac{b}{2a} \qquad \omega^2 = \frac{c}{a}$$

$\lambda = 2 \qquad \omega = 1$

$$y'' + 4y' + y = 0$$

$$y = c_1 e^{(-2+\sqrt{3})t} + c_2 e^{(-2-\sqrt{3})t}$$

$\lambda = 1 \qquad \omega = 1$

$$y'' + 2y' + y = 0$$

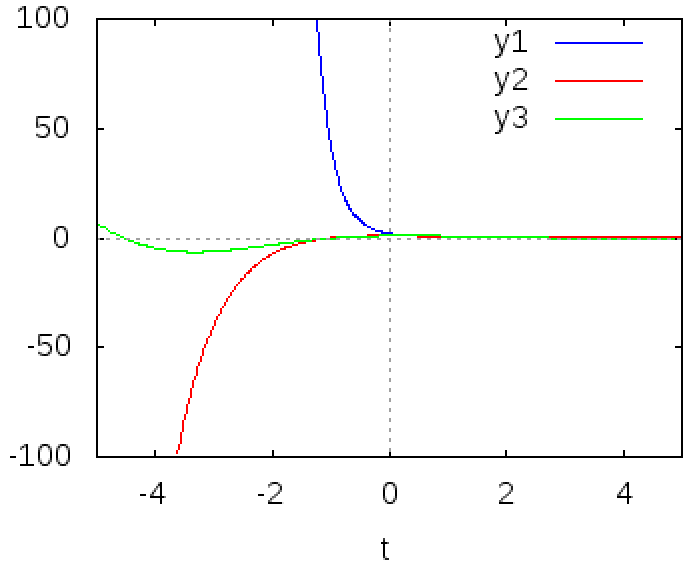
$$y = c_1 e^{-t} + c_2 t e^{-t}$$

$\lambda = 1/2 \qquad \omega = 1$

$$y'' + y' + y = 0$$

$$y = c_1 e^{(-1+\sqrt{3}i)/2t} + c_2 e^{(-1-\sqrt{3}i)/2t}$$

$$= e^{-t/2}\left(c_3 \cos\left(\sqrt{3}/2\ t\right) + c_4 \sin\left(\sqrt{3}/2\ t\right)\right)$$



$\lambda = 2 \qquad \omega = 1$

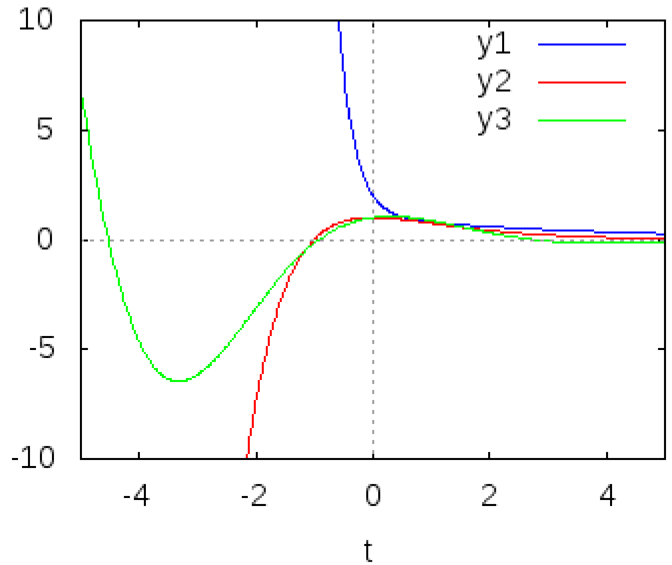
$$y_1(t) = c_1 e^{(-2+\sqrt{3})t} + c_2 e^{(-2-\sqrt{3})t}$$

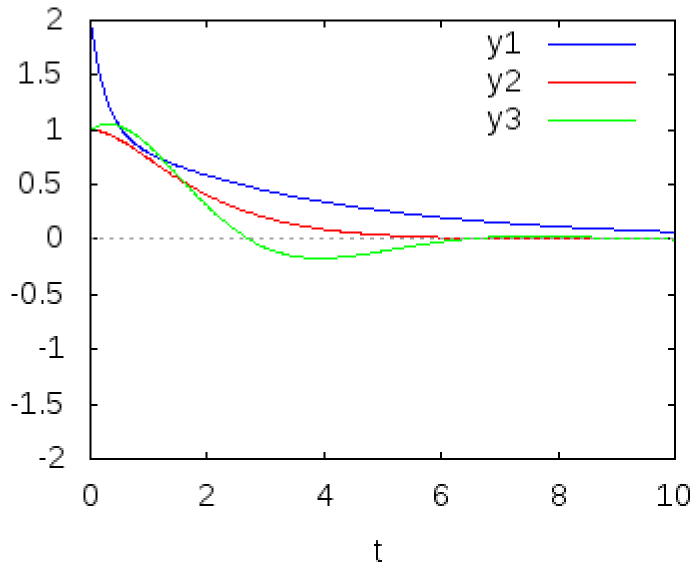
$\lambda = 1 \qquad \omega = 1$

$$y_2(t) = c_1 e^{-t} + c_2 t e^{-t}$$

$\lambda = 1/2 \qquad \omega = 1$

$$y_3(t) = e^{-t/2}\left(c_3 \cos\left(\sqrt{3}/2\ t\right) + c_4 \sin\left(\sqrt{3}/2\ t\right)\right)$$





References

[1] http://en.wikipedia.org/
[2] M.L. Boas, “Mathematical Methods in the Physical Sciences”
[3] E. Kreyszig, “Advanced Engineering Mathematics”
[4] D. G. Zill, W. S. Wright, “Advanced Engineering Mathematics”