

# z Transform (5B)

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# Power Series

A **power series** in powers of  $(z - z_0)$

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n = a_0 + a_1 (z - z_0) + a_2 (z - z_0)^2 + \dots$$

A **power series** in powers of  $z$  ( $z_0 = 0$ )

$$\sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + \dots$$

# Taylor Series

A **power series** in powers of  $(z - z_0)$

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n = a_0 + a_1 (z - z_0) + a_2 (z - z_0)^2 + \dots$$

The **Taylor series** of a function  $f(z)$

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

$$a_n = \frac{1}{n!} f^{(n)}(z_0)$$

$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint_C \frac{f(w)}{(w - z)^{n+1}} dw$$

$$a_n = \frac{1}{n!} f^{(n)}(z_0) \quad \Rightarrow \quad a_n = \frac{1}{2\pi i} \oint_C \frac{f(w)}{(w - z)^{n+1}} dw$$

# Maclaurin Series

A **power series** in powers of  $z$

$$\sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + \dots$$

The **Maclaurin series** of a function  $f(z)$

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

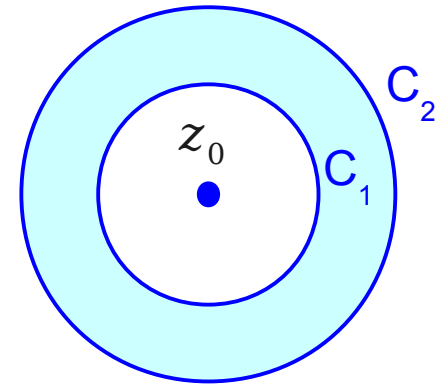
$$a_n = \frac{1}{n!} f^{(n)}(0)$$

$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint_C \frac{f(w)}{(w-z)^{n+1}} dw$$

$$a_n = \frac{1}{n!} f^{(n)}(0) \quad \Rightarrow \quad f^{(n)}(0) = \frac{1}{2\pi i} \oint_C \frac{f(w)}{w^{n+1}} dw$$

# Laurent's Series

$f(z)$  : **analytic** in the region  $R$   
between circles  $C_1, C_2$   
centered at  $z_0$



$$\begin{aligned} \Rightarrow f(z) &= \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} \\ &= a_0 + a_1 (z - z_0) + a_2 (z - z_0)^2 + \dots \\ &\quad + \frac{b_1}{(z - z_0)} + \frac{b_2}{(z - z_0)^2} + \dots \end{aligned}$$

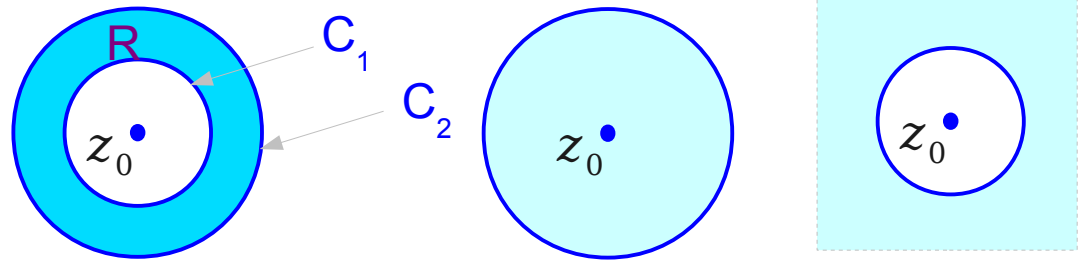
Principal part

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(w)}{(w - z)^{n+1}} dw$$

$$b_n = \frac{1}{2\pi i} \oint_C (w - z)^{n-1} f(w) dw$$

# Laurent's Theorem – Coefficients $a_k$

$f(z)$  : **analytic** in the region  $R$   
 between circles  $C_1, C_2$   
 centered at  $z_0$



$$f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \dots$$

$$+ \frac{b_1}{(z-z_0)} + \frac{b_2}{(z-z_0)^2} + \dots$$

: **convergent** in the region  $R$

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z) dz}{(z-z_0)^{n+1}}$$

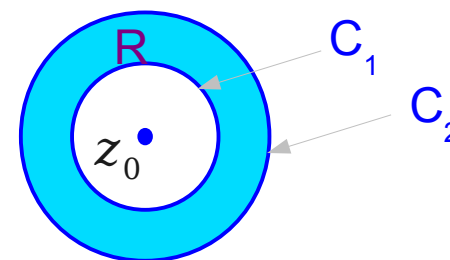
$$b_n = \frac{1}{2\pi i} \oint_C \frac{f(z) dz}{(z-z_0)^{-n+1}}$$

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z-z_0)^k$$

$$a_k = \frac{1}{2\pi i} \oint_C \frac{f(z) dz}{(z-z_0)^{k+1}}$$

# Residue

$f(z)$  : **analytic** in the region  $R$   
 between circles  $C_1, C_2$   
 centered at  $z_0$



$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k$$

$$a_k = \frac{1}{2\pi i} \oint_C \frac{f(z) dz}{(z - z_0)^{k+1}}$$

coefficient  $a_{-1}$  of  $\frac{1}{(z - z_0)}$

$$a_{-1} = \text{Res}(f(z), z_0)$$

: **residue** of the function  $f(z)$  at the isolated singularity  $z_0$

$$\oint_C f(z) dz = 2\pi i \text{Res}(f(z), z_0)$$

$$a_{-1} = \frac{1}{2\pi i} \oint_C f(z) dz$$



# z Transform (1)

## Laurent Series

The power series representation of  $f(z)$

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k$$

$$a_k = \frac{1}{2\pi i} \oint_C \frac{f(z) dz}{(z - z_0)^{k+1}}$$



$$z_0 = 0$$



$$f(z) = \sum_{k=-\infty}^{\infty} a_k z^k$$

$$a_k = \frac{1}{2\pi i} \oint_C \frac{f(z) dz}{z^{k+1}}$$



$$n = -k$$



$$f(z) = \sum_{n=-\infty}^{\infty} a_{-n} z^{-n}$$

$$a_{-n} = \frac{1}{2\pi i} \oint_C f(z) z^{n-1} dz$$

**z Transform** of  $a_{-k}$  A transform of a sequence of numbers  $a_{-k}$

# z Transform (2)

## Laurent Series

The power series representation of  $f(z)$

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k$$

$$a_k = \frac{1}{2\pi i} \oint_C \frac{f(z) dz}{(z - z_0)^{k+1}}$$



$$z_0 = 0$$

$$n = -k$$



$$f(z) = \sum_{n=-\infty}^{\infty} a_{-n} z^{-n}$$

$$a_{-n} = \frac{1}{2\pi i} \oint_C f(z) z^{n-1} dz$$

**z Transform** of  $a_{-k}$  A transform of a sequence of numbers  $a_{-k}$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$x[n] = \frac{1}{2\pi i} \oint_C X(z) z^{n-1} dz$$

**z Transform** of  $x[n]$

# z Transform (3)

## Laurent Series

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k$$

$$a_k = \frac{1}{2\pi i} \oint_C \frac{f(z) dz}{(z - z_0)^{k+1}}$$

**z Transform** of  $a_{-k}$        $z_0 = 0$

$$x[n] = \frac{1}{2\pi i} \oint_C \frac{X(z) dz}{z^{-n+1}}$$

**z Transform** of  $x[n]$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$x[n] = \frac{1}{2\pi i} \oint_C X(z) z^{n-1} dz$$

# Inverse z Transform

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Inverse z Transform

$$x[n] = \frac{1}{2\pi i} \oint_C X(z) z^{n-1} dz$$





## References

- [1] <http://en.wikipedia.org/>
- [2] J.H. McClellan, et al., Signal Processing First, Pearson Prentice Hall, 2003
- [3] A “graphical interpretation” of the DFT and FFT, by Steve Mann