z Transform (5B)

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Power Series

A **power series** in powers of $(z - z_0)$

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n = a_0 + a_1 (z - z_0) + a_2 (z - z_0)^2 + \cdots$$

A **power series** in powers of z $\quad (z_0 = 0)$

$$\sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + \cdots$$

Taylor Series

A **power series** in powers of $(z-z_0)$

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$

The **Taylor series** of a function $f(z)$

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$$

$$a_n = \frac{1}{n!} f^{(n)}(z_0)$$

$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint_C \frac{f(w)}{(w-z)^{n+1}} dw$$

$$a_n = \frac{1}{n!} f^{(n)}(z_0) \Rightarrow a_n = \frac{1}{2\pi i} \oint_C \frac{f(w)}{(w-z)^{n+1}} dw$$

Maclaurin Series

A **power series** in powers of z

$$\sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + \cdots$$

The **Maclaurin series** of a function $f(z)$

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \qquad a_n = \frac{1}{n!} f^{(n)}(0)$$

$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint_C \frac{f(w)}{(w-z)^{n+1}} dw$$

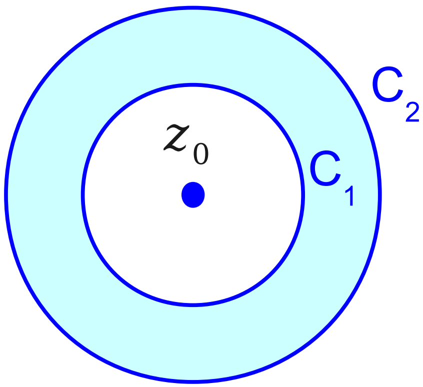
$$a_n = \frac{1}{n!} f^{(n)}(0) \quad \Rightarrow \quad f^{(n)}(0) = \frac{1}{2\pi i} \oint_C \frac{f(w)}{w^{n+1}} dw$$

Laurent's Series

$f(z)$: **analytic** in the region R

between circles C_1, C_2

centered at z_0



$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0) + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$

$$= a_0 + a_1 (z - z_0) + a_2 (z - z_0)^2 + \cdots$$

$$+ \frac{b_1}{(z - z_0)} + \frac{b_2}{(z - z_0)^2} + \cdots$$

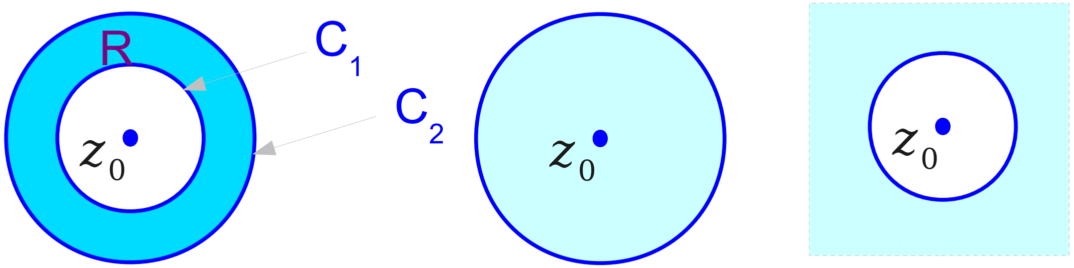
Principal part

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(w)}{(w - z)^{n+1}} dw$$

$$b_n = \frac{1}{2\pi i} \oint_C (w - z)^{n-1} f(w)\, dw$$

Laurent's Theorem – Coefficients a_k

$f(z)$: **analytic** in the region R
between circles C_1, C_2
centered at z_0



$$f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots + \frac{b_1}{(z-z_0)} + \frac{b_2}{(z-z_0)^2} + \cdots$$

: **convergent** in the region R

$$a_n = \frac{1}{2\pi i}\oint_C \frac{f(z)\,dz}{(z-z_0)^{n+1}}$$

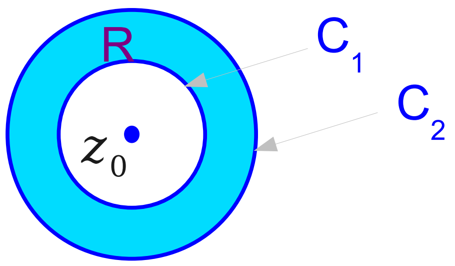
$$b_n = \frac{1}{2\pi i}\oint_C \frac{f(z)\,dz}{(z-z_0)^{-n+1}}$$

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z-z_0)^k$$

$$a_k = \frac{1}{2\pi i}\oint_C \frac{f(z)\,dz}{(z-z_0)^{k+1}}$$

Residue

$f(z)$: **analytic** in the region R between circles C_1, C_2 centered at z_0



$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k$$

$$a_k = \frac{1}{2\pi i} \oint_C \frac{f(z)\,dz}{(z-z_0)^{k+1}}$$

coefficient a_{-1} of $\dfrac{1}{(z-z_0)}$

$$a_{-1} = Res(f(z), z_0)$$

: **residue** of the function $f(z)$ at the isolated singularity z_0

$$a_{-1} = \frac{1}{2\pi i} \oint_C f(z)\,dz$$

$$\oint_C f(z)\,dz = 2\pi i\, Res(f(z), z_0)$$

z Transform (1)

Laurent Series The power series representation of f(z)

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k$$

$$a_k = \frac{1}{2\pi i} \oint_C \frac{f(z)\,dz}{(z-z_0)^{k+1}}$$

$$z_0 = 0$$

$$f(z) = \sum_{k=-\infty}^{\infty} a_k z^k$$

$$a_k = \frac{1}{2\pi i} \oint_C \frac{f(z)\,dz}{z^{k+1}}$$

$$n = -k$$

$$f(z) = \sum_{n=-\infty}^{\infty} a_{-n} z^{-n}$$

$$a_{-n} = \frac{1}{2\pi i} \oint_C f(z) z^{n-1}\,dz$$

z Transform of a_{-k} A transform of a sequence of numbers a_{-k}

z Transform (2)

Laurent Series The power series representation of f(z)

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k$$

$$a_k = \frac{1}{2\pi i} \oint_C \frac{f(z)dz}{(z-z_0)^{k+1}}$$

$z_0 = 0$ $n = -k$

$$f(z) = \sum_{n=-\infty}^{\infty} a_{-n} z^{-n}$$

$$a_{-n} = \frac{1}{2\pi i} \oint_C f(z) z^{n-1} dz$$

z Transform of a_{-k} A transform of a sequence of numbers a_{-k}

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$x[n] = \frac{1}{2\pi i} \oint_C X(z) z^{n-1} dz$$

z Transform of $x[n]$

z Transform (3)

Laurent Series

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k$$

$$a_k = \frac{1}{2\pi i} \oint_C \frac{f(z)\,dz}{(z - z_0)^{k+1}}$$

z Transform of a_{-k} $\qquad z_0 = 0 \qquad x[n] = \frac{1}{2\pi i} \oint_C \frac{X(z)\,dz}{z^{-n+1}}$

z Transform of $x[n]$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$x[n] = \frac{1}{2\pi i} \oint_C X(z) z^{n-1}\, dz$$

Inverse z Transform

$$x[n] = \frac{1}{2\pi i} \oint_C X(z) z^{n-1} dz$$

References

[1] http://en.wikipedia.org/
[2] J.H. McClellan, et al., Signal Processing First, Pearson Prentice Hall, 2003
[3] A “graphical interpretation” of the DFT and FFT, by Steve Mann