Abstract Algebra Overview II (H.1)

20160621

Linear Functional

Linear form

$V \to k$

From Wikipedia, the free encyclopedia

(Redirected from Linear functional)

In linear algebra, a **linear functional** or **linear form** (also called a **one-form** or **covector**) is a linear map from a vector space to its field of scalars. In $\mathbb{R}^n$, if vectors are represented as column vectors, then linear functionals are represented as row vectors, and their action on vectors is given by the dot product, or the matrix product with the row vector on the left and the column vector on the right. In general, if V is a vector space over a field k, then a linear functional f is a function from V to k that is linear:

$$f(\vec{v}+\vec{w}) = f(\vec{v}) + f(\vec{w}) \text{ for all } \vec{v}, \vec{w} \in V$$
$$f(a\vec{v}) = af(\vec{v}) \text{ for all } \vec{v} \in V, a \in k.$$

The set of all linear functionals from V to k, $\mathrm{Hom}_k(V,k)$, forms a vector space over k with the addition of the operations of addition and scalar multiplication (defined pointwise). This space is called the dual space of V, or sometimes the **algebraic dual space**, to distinguish it from the continuous dual space. It is often written V^* or V' when the field k is understood.

linear functional
linear form
one form
covector

$V \to k$

$f \in \mathrm{Hom}_k(V, k)$ $\qquad$ $f+g \in \mathrm{Hom}_k(V, k)$

$g \in \mathrm{Hom}_k(V, k)$ $\qquad$ $cf \in \mathrm{Hom}_k(V, k)$

$\vec{v} \in V$ $\quad \vec{v}$ col vector $\quad$ Vectors

$\varphi \in V^*$ $\quad \varphi$ row vector $\quad$ linear functionals

a linear functional f

a linear function from V to k

$f(\vec{v}+\vec{w}) = f(\vec{v}) + f(\vec{w}) \qquad \vec{v} \in V, \ \vec{w} \in V$

$f(a\vec{v}) = a f(\vec{v}) \qquad a \in k$

the set of all linear functionals from V to k

$\triangleq \mathrm{Hom}_k(V, k)$: Vector space

$f \in \mathrm{Hom}_k(V, k)$	$f+g \in \mathrm{Hom}_k(V, k)$
$g \in \mathrm{Hom}_k(V, k)$	$cf \in \mathrm{Hom}_k(V, k)$

Dual space : Linear Vector space

Hom-Set

Linear functionals in $\mathbf{R}^n$ [edit]

Suppose that vectors in the real coordinate space $\mathbf{R}^n$ are represented as column vectors

$$\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

Then any linear functional can be written in these coordinates as a sum of the form:

$$f(\vec{x}) = a_1 x_1 + \cdots + a_n x_n.$$

This is just the matrix product of the row vector $[a_1 \ldots a_n]$ and the column vector $\vec{x}$:

$$f(\vec{x}) = [a_1 \ldots a_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

$f(\vec{v} + \vec{w}) = f(\vec{v}) + f(\vec{w})$

$f(a\vec{v}) = a f(\vec{v})$

Linear Function

$V \to W$

Linear map

$V \to V$

From Wikipedia, the free encyclopedia

$V \to k$

In mathematics, a **linear map** (also called a **linear mapping**, **linear transformation** or, in some contexts, **linear function**) is a mapping $V \to W$ between two modules (including vector spaces) that preserves (in the sense defined below) the operations of addition and scalar multiplication. Linear maps can often be represented as matrices, and simple examples include rotation and reflection linear transformations.

An important special case is when $V = W$, in which case the map is called a **linear operator**, or an endomorphism of V. Sometimes the term *linear function* has the same meaning as *linear map*, while in analytic geometry it does not.

A linear map always maps linear subspaces onto linear subspaces (possibly of a lower dimension); for instance it maps a plane through the origin to a plane, straight line or point.

In the language of abstract algebra, a linear map is a module homomorphism. In the language of category theory it is a morphism in the category of modules over a given ring.

Linear Map
Linear Transformation
Linear Function

$V \to W$

Linear Operator
Endomorphism

$V \to V$

linear functional
linear form
one form
covector

$V \to k$

module

V

linear map

linear function

linear transform

module

W

matrix rep

addition

scalar multiplication

module

vector space

V

linear operator

module

vector space

$W = V$

matrix rep

addition

scalar multiplication

vector space

over a field F

V

linear form

linear functional

field

F

matrix rep

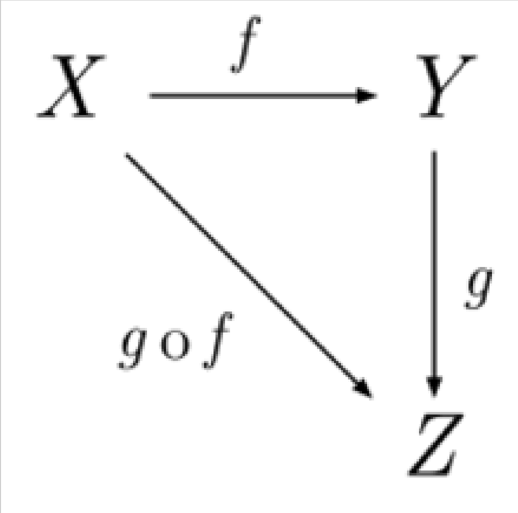
addition

scalar multiplication

Category theory

From Wikipedia, the free encyclopedia

Category theory[1] formalizes mathematical structure and its concepts in terms of a collection of *objects* and of *arrows* (also called morphisms). A category has two basic properties: the ability to compose the arrows associatively and the existence of an identity arrow for each object. The language of category theory has been used to formalize concepts of other high-level abstractions such as sets, rings, and groups.



Schematic representation of a category with objects *X*, *Y*, *Z* and morphisms *f*, *g*, $g \circ f$. (The category's three identity morphisms 1_X, 1_Y and 1_Z, if explicitly represented, would appear as three arrows, next to the letters X, Y, and Z, respectively, each having as its "shaft" a circular arc measuring almost 360 degrees.)

Several terms used in category theory, including the term "morphism", are used differently from their uses in the rest of mathematics. In category theory, morphisms obey conditions specific to category theory itself.

Samuel Eilenberg and Saunders Mac Lane introduced the concepts of categories, functors, and natural transformations in 1942–45 in their study of algebraic topology, with the goal of understanding the processes that preserve mathematical structure, and influenced by previous related ideas by Polish and German mathematicians. Category theory has practical applications in programming language theory, in particular for the study of monads in functional programming.

Morphism

From Wikipedia, the free encyclopedia

(Redirected from Hom-set)

In many fields of mathematics, **morphism** refers to a structure-preserving map from one mathematical structure to another. The notion of morphism recurs in much of contemporary mathematics. In set theory, morphisms are functions; in linear algebra, linear transformations; in group theory, group homomorphisms; in topology, continuous functions, and so on.

In category theory, *morphism* is a broadly similar idea, but somewhat more abstract: the mathematical objects involved need not be sets, and the relationship between them may be something more general than a map.

The study of morphisms and of the structures (called "objects") over which they are defined is central to category theory. Much of the terminology of morphisms, as well as the intuition underlying them, comes from concrete categories, where the *objects* are simply *sets with some additional structure*, and *morphisms* are *structure-preserving functions*. In category theory, morphisms are sometimes also called **arrows**.

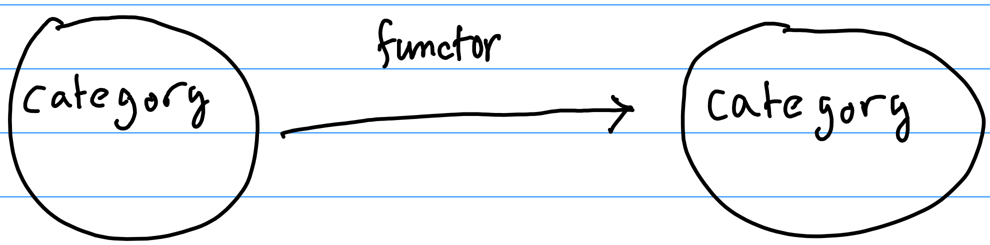
Functor

From Wikipedia, the free encyclopedia

In mathematics, a **functor** is a type of mapping between categories which is applied in category theory. Functors can be thought of as homomorphisms between categories. In the category of small categories, functors can be thought of more generally as morphisms.

Functors were first considered in algebraic topology, where algebraic objects (like the fundamental group) are associated to topological spaces, and algebraic homomorphisms are associated to continuous maps. Nowadays, functors are used throughout modern mathematics to relate various categories. Thus, functors are generally applicable in areas within mathematics that category theory can make an abstraction of.

The word *functor* was borrowed by mathematicians from the philosopher Rudolf Carnap,[1] who used the term in a linguistic context:[2] see function word.



homomorphism

small categories   functors : morphisms algebraic topology

algebraic objects
(fundamental groups)

module

In mathematics, a **module** is one of the fundamental algebraic structures used in abstract algebra. A **module** over a ring is a generalization of the notion of vector space over a field, wherein the corresponding scalars are the elements of an arbitrary given ring (with identity) and a multiplication (on the left and/or on the right) is defined between elements of the ring and elements of the module.

a Vector Space over a field

a Module over a ring

scalars : elements of a ring

multiplication : (elements of a ring) x (elements of a module)

(elements of a module) x (elements of a ring)

> A **ring** is an algebraic system consisting of **a set**,
> **an identity element** for each operation,
> two operations and the inverse operation of the first operation.

Suppose that R is a ring and 1_R is its multiplicative identity. A **left *R*-module** *M* consists of an abelian group $(M, +)$ and an operation $\cdot : R \times M \to M$ such that for all r, s in R and x, y in M, we have:

1. $r \cdot (x + y) = r \cdot x + r \cdot y$
2. $(r + s) \cdot x = r \cdot x + s \cdot x$
3. $(rs) \cdot x = r \cdot (s \cdot x)$
4. $1_R \cdot x = x.$

The operation of the ring on M is called *scalar multiplication*, and is usually written by juxtaposition, i.e. as rx for r in R and x in M, though here it is denoted as $r \cdot x$ to distinguish it from the ring multiplication operation, denoted here by juxtaposition. The notation ${}_RM$ indicates a left R-module M. A **right *R*-module** *M* or M_R is defined similarly, except that the ring acts on the right; i.e., scalar multiplication takes the form $\cdot : M \times R \to M$, and the above axioms are written with scalars r and s on the right of x and y.

R : a ring

M_L : a Left R-module M

scalar multiplication $\cdot : R \times M \to M$

$r, s \in R$ $x, y \in M$

$r \cdot (x + y) = r \cdot x + r \cdot y$

$(r + s) \cdot x = r \cdot x + s \cdot x$

$(rs) \cdot x = r \cdot (s \cdot x)$

$1_R \cdot x = x$

M_R : a Right R-module M

scalar multiplication $\cdot : M \times R \to M$

algebra

In mathematics, an **algebra** is one of the fundamental algebraic structures used in abstract algebra. An **algebra over a field** is a vector space (a module over a field) equipped with a bilinear product. Thus, an algebra over a field is a set, together with operations of multiplication, addition, and scalar multiplication by elements of the underlying field, that satisfy the axioms implied by "vector space" and "bilinear".[1]

The multiplication operation in an algebra may or may not be associative, leading to the notions of associative algebras and nonassociative algebras. Given an integer *n*, the ring of real square matrices of order *n* is an example of an associative algebra over the field of real numbers under matrix addition and matrix multiplication. Three-dimensional Euclidean space with multiplication given by the vector cross product is an example of a nonassociative algebra over the field of real numbers.

an algebra over a field

→ a vector space + bilinear product
a module over a field

→ a set with operations
- multiplication
- addition
- scalar multiplication

Bilinear map

From Wikipedia, the free encyclopedia

In mathematics, a **bilinear map** is a function combining elements of two vector spaces to yield an element of a third vector space, and is linear in each of its arguments. Matrix multiplication is an example.

Vector spaces [edit]

Let V, W and X be three vector spaces over the same base field F. A bilinear map is a function

$$\boxed{B : V \times W \to X}$$

such that for any w in W the map

$$v \mapsto B(v, w)$$

is a linear map from V to X, and for any v in V the map

$$w \mapsto B(v, w)$$

is a linear map from W to X.

In other words, when we hold the first entry of the bilinear map fixed while letting the second entry vary, the result is a linear operator, and similarly for when we hold the second entry fixed.

If $V = W$ and we have $B(v, w) = B(w, v)$ for all v, w in V, then we say that B is *symmetric*.

The case where X is the base field F, and we have a **bilinear form**, is particularly useful (see for example scalar product, inner product and quadratic form).

Modules [edit]

The definition works without any changes if instead of vector spaces over a field F, we use modules over a commutative ring R. It generalizes to n-ary functions, where the proper term is *multilinear*.

For non-commutative rings R and S, a left R-module M and a right S-module N, a bilinear map is a map $B : M \times N \to T$ with T an (R, S)-bimodule, and for which any n in N, $m \mapsto B(m, n)$ is an R-module homomorphism, and for any m in M, $n \mapsto B(m, n)$ is an S-module homomorphism. This satisfies

$$B(r \cdot m, n) = r \cdot B(m, n)$$
$$B(m, n \cdot s) = B(m, n) \cdot s$$

for all m in M, n in N, r in R and s in S, as well as B being additive in each argument.

Bilinear form

From Wikipedia, the free encyclopedia

In mathematics, more specifically in abstract algebra and linear algebra, a **bilinear form** on a vector space V is a bilinear map $V \times V \to K$, where K is the field of scalars. In other words, a bilinear form is a function $B : V \times V \to K$ which is linear in each argument separately:

- $B(\mathbf{u} + \mathbf{v}, \mathbf{w}) = B(\mathbf{u}, \mathbf{w}) + B(\mathbf{v}, \mathbf{w})$
- $B(\mathbf{u}, \mathbf{v} + \mathbf{w}) = B(\mathbf{u}, \mathbf{v}) + B(\mathbf{u}, \mathbf{w})$
- $B(\lambda\mathbf{u}, \mathbf{v}) = B(\mathbf{u}, \lambda\mathbf{v}) = \lambda B(\mathbf{u}, \mathbf{v})$

The definition of a bilinear form can be extended to include modules over a commutative ring, with linear maps replaced by module homomorphisms.

When K is the field of complex numbers $\mathbf{C}$, one is often more interested in sesquilinear forms, which are similar to bilinear forms but are conjugate linear in one argument.

Topological Space

In topology and related branches of mathematics, a **topological space** may be defined as a set of points, along with a set of neighbourhoods for each point, that satisfy a set of axioms relating points and neighbourhoods. The definition of a topological space relies only upon set theory and is the most general notion of a mathematical space that allows for the definition of concepts such as continuity, connectedness, and convergence.[1] Other spaces, such as manifolds and metric spaces, are specializations of topological spaces with extra structures or constraints. Being so general, topological spaces are a central unifying notion and appear in virtually every branch of modern mathematics. The branch of mathematics that studies topological spaces in their own right is called point-set topology or general topology.

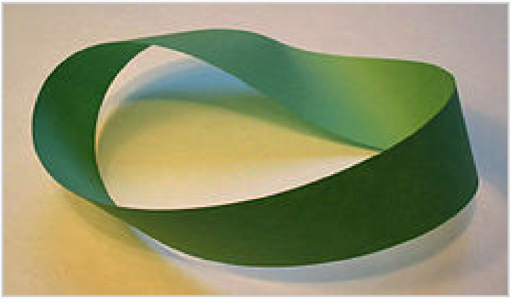
a set of points
a set of neighborhoods for each point

Satisfying a set of axioms

continuity
connectedness
convergence

Topology

In mathematics, **topology** (from the Greek τόπος, *place*, and λόγος, *study*) is concerned with the properties of space that are preserved under continuous deformations, such as stretching and bending, but not tearing or gluing. This can be studied by considering a collection of subsets, called open sets, that satisfy certain properties, turning the given set into what is known as a topological space. Important topological properties include connectedness and compactness.[1]



Möbius strips, which have only one surface and one edge, are a kind of object studied in topology.

Continuous deformation

eg) { Stretching, bending   { tearing (X), gluing (X)

properties of space preserved

eg) connectedness
compactness

Collection of subsets (open sets)

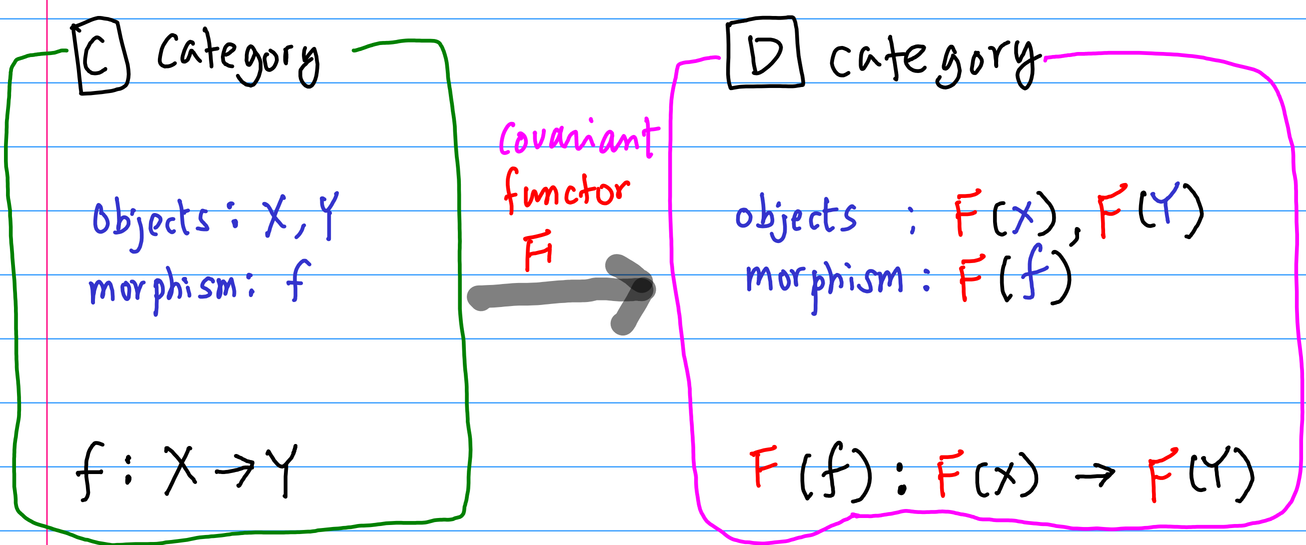
Functor

Definition [edit]

Let *C* and *D* be categories. A **functor** *F* from *C* to *D* is a mapping that[3]

- associates to each object X in *C* an object $F(X)$ in *D*,
- associates to each morphism $f : X \to Y$ in *C* a morphism $F(f) : F(X) \to F(Y)$ in *D* such that the following two conditions hold:
  - $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ for every object X in *C*,
  - $F(g \circ f) = F(g) \circ F(f)$ for all morphisms $f : X \to Y$ and $g : Y \to Z$ in *C*.

That is, functors must preserve identity morphisms and composition of morphisms.



preserve identity morphism
composite morphism

Covariance and contravariance [edit]

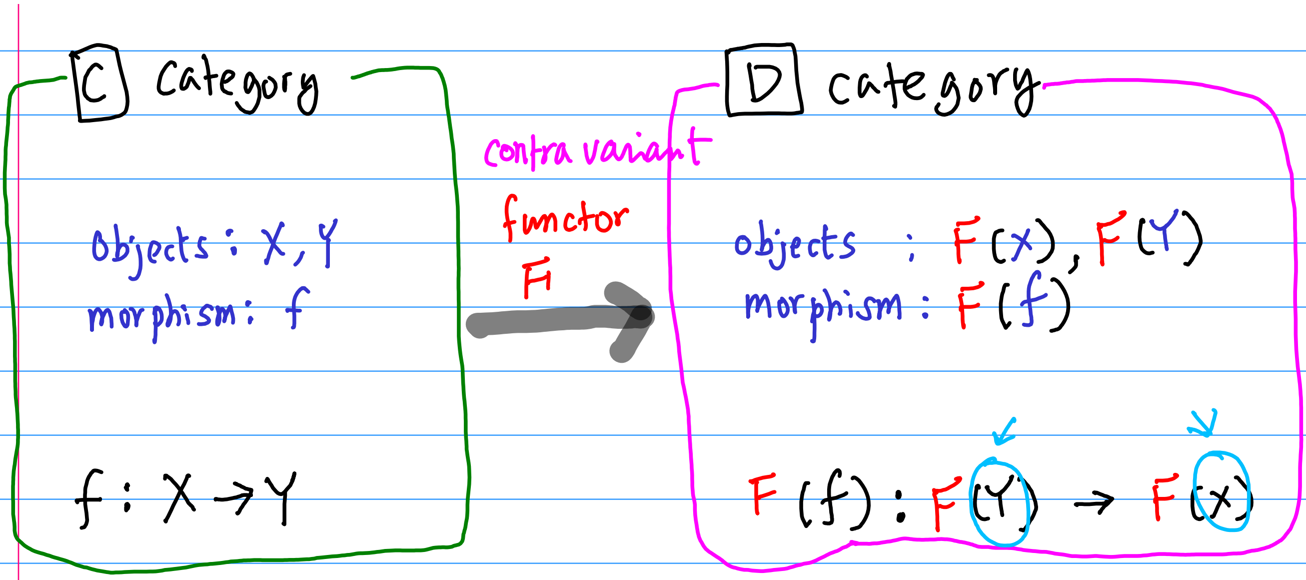
There are many constructions in mathematics that would be functors but for the fact that they "turn morphisms around" and "reverse composition". We then define a **contravariant functor** *F* from *C* to *D* as a mapping that

- associates to each object X in *C* an object $F(X)$ in *D*,
- associates to each morphism $f : X \to Y$ in *C* a morphism $F(f) : F(Y) \to F(X)$ in *D* such that
  - $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ for every object X in *C*,
  - $F(g \circ f) = F(f) \circ F(g)$ for all morphisms $f : X \to Y$ and $g : Y \to Z$ in *C*.

Note that contravariant functors reverse the direction of composition.

Ordinary functors are also called **covariant functors** in order to distinguish them from contravariant ones. Note that one can also define a contravariant functor as a *covariant* functor on the opposite category C^{op}.[4] Some authors prefer to write all expressions covariantly. That is, instead of saying $F : C \to D$ is a contravariant functor, they simply write $F : C^{\mathrm{op}} \to D$ (or sometimes $F : C \to D^{\mathrm{op}}$) and call it a functor.

Contravariant functors are also occasionally called *cofunctors*.[5]



Covariant Functor

$$g \circ f \longrightarrow F(g \circ f) = \underline{F(g)} \circ \underline{F(f)}$$

Contravariant Functor

$$g \circ f \longrightarrow F(g \circ f) = \underline{F(f)} \circ \underline{F(g)}$$

Homomorphism

From Wikipedia, the free encyclopedia

In abstract algebra, a **homomorphism** is a structure-preserving map between two algebraic structures (such as groups, rings, or vector spaces). The word *homomorphism* comes from the ancient Greek language: *ὁμός (homos)* meaning "same" and *μορφή (morphe)* meaning "form" or "shape". Isomorphisms, automorphisms, and endomorphisms are special types of homomorphisms.

A homomorphism is a map that preserves selected structure between two algebraic structures, with the structure to be preserved being given by the naming of the homomorphism.

algebraic structure $\xrightarrow{\text{homomorphism}}$ algebraic structure

structure preserving map

groups
rings
vector spaces

Isomorphism
automorphism
endomorphism

Definition of Morphism

A category C consists of two classes, one of *objects* and the other of *morphisms*.

There are two objects that are associated to every morphism, the *source* and the *target*.

For many common categories, objects are sets (usually with more structure) and morphisms are functions from an object to another object. Therefore the source and the target of a morphism are often called respectively *domain* and *codomain*.

A morphism f with source X and target Y is written $f : X \to Y$. Thus a morphism is represented by an *arrow* from its source to its target.

Morphisms are equipped with a partial binary operation, called *composition*. The composition of two morphism f and g is defined if and only if the target of g is the source of f, and is denoted $f \circ g$. The source of $f \circ g$ is the source of g, and the target of $f \circ g$ is the target of f. The composition satisfies two axioms:

Identity: for every object X, there exists a morphism $\mathrm{id}_X : X \to X$ called the **identity morphism** on X, such that for every morphism $f : A \to B$ we have $\mathrm{id}_B \circ f = f = f \circ \mathrm{id}_A$.

Associativity: $h \circ (g \circ f) = (h \circ g) \circ f$ whenever the operations are defined, that is when the target of f is the source of g, and the target of g is the source of h.

Types [edit]

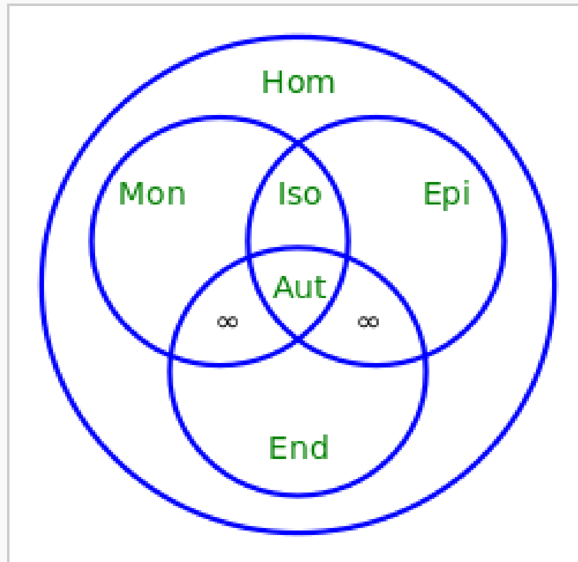
[Proof 1] [show]

[Proof 2] [show]

[Proof 3] [show]

In abstract algebra, several specific kinds of homomorphisms are defined as follows:

- An **isomorphism** is a bijective homomorphism.
- An **epimorphism** (sometimes called a cover) is a surjective homomorphism. Equivalently, [note 1] f: $A \to B$ is an epimorphism if it has a right inverse g: $B \to A$, i.e. if $f(g(b)) = b$ for all $b \in B$.
- A **monomorphism** (sometimes called an embedding or extension) is an injective homomorphism. Equivalently, [note 1] f: $A \to B$ is a monomorphism if it has a left inverse g: $B \to A$, i.e. if $g(f(a)) = a$ for all $a \in A$.
- An **endomorphism** is a homomorphism from an algebraic structure to itself.
- An **automorphism** is an endomorphism which is also an isomorphism, i.e., an isomorphism from an algebraic structure to itself.[1]
- The **trivial homomorphism** between unital magmas is the constant map onto the identity element of the codomain.[2]



Relationships between different kinds of homomorphisms.
Hom = set of Homomorphisms,
Mon = set of Monomorphisms,
Epi = set of Epimorphisms,
Iso = set of Isomorphisms,
End = set of Endomorphism,
Aut = set of Automorphisms.
Notice that: $Mon \cap Epi = Iso$,
$Iso \cap End = Aut$.
The sets $(Mon \cap End) \setminus Aut$ and $(Epi \cap End) \setminus Aut$ contain only homomorphisms from some infinite structures to themselves.

Isomorphism

In mathematics, an **isomorphism** (from the Ancient Greek: ἴσος *isos* "equal", and μορφή *morphe* "form" or "shape") is a homomorphism or morphism (i.e. a mathematical mapping) that admits an inverse.[note 1] Two mathematical objects are **isomorphic** if an isomorphism exists between them. An *automorphism* is an isomorphism whose source and target coincide. The interest of isomorphisms lies in the fact that two isomorphic objects cannot be distinguished by using only the properties used to define morphisms; thus isomorphic objects may be considered the same as long as one considers only these properties and their consequences.

For most algebraic structures, including groups and rings, a homomorphism is an isomorphism if and only if it is bijective.

In topology, where the morphisms are continuous functions, isomorphisms are also called *homeomorphisms* or *bicontinuous functions*. In mathematical analysis, where the morphisms are differentiable functions, isomorphisms are also called *diffeomorphisms*.

In mathematics, an **endomorphism** is a morphism (or homomorphism) from a mathematical object to itself. For example, an endomorphism of a vector space, V is a linear map, $f: V \to V$, and an endomorphism of a group, G, is a group homomorphism $f: G \to G$. In general, we can talk about endomorphisms in any category. In the category of sets, endomorphisms are functions from a set S to itself.

In any category, the composition of any two endomorphisms of X is again an endomorphism of X. It follows that the set of all endomorphisms of X forms a monoid, denoted $\mathrm{End}(X)$ (or $\mathrm{End}_C(X)$ to emphasize the category C).

An invertible endomorphism of X is called an automorphism. The set of all automorphisms is a subset of $\mathrm{End}(X)$ with a group structure, called the automorphism group of X and denoted $\mathrm{Aut}(X)$. In the following diagram, the arrows denote implication:

Automorphism	⇒	Isomorphism
⇓		⇓
Endomorphism	⇒	(Homo)morphism

In mathematics, an **automorphism** is an isomorphism from a mathematical object to itself. It is, in some sense, a symmetry of the object, and a way of mapping the object to itself while preserving all of its structure. The set of all automorphisms of an object forms a group, called the **automorphism group**. It is, loosely speaking, the symmetry group of the object.

Category C

- object
  - source
  - target
- morphism

- sets
  - domain
  - codomain
- functions

- object
  - source X
  - target Y
- morphism f

$$f : X \to Y$$

For a concrete category (that is the objects are sets with additional structure, and of the morphisms as structure-preserving functions), the identity morphism is just the identity function, and composition is just the ordinary composition of functions. *Associativity* then follows, because the composition of functions is associative.

The composition of morphisms is often represented by a commutative diagram. For example,

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "g \circ f"'] & Y \arrow[d, "g"] \\ & Z \end{tikzcd}$$

The collection of all morphisms from *X* to *Y* is denoted $\hom_C(X,Y)$ or simply hom(*X*, *Y*) and called the **hom-set** between *X* and *Y*. Some authors write $\mathrm{Mor}_C(X,Y)$, Mor(*X*, *Y*) or C(*X*, *Y*). Note that the term hom-set is something of a misnomer as the collection of morphisms is not required to be a set. A category where hom(*X*, *Y*) is a set for all objects *X* and *Y* is called locally small.

Hom - Set

$\mathrm{Hom}(X, Y)$

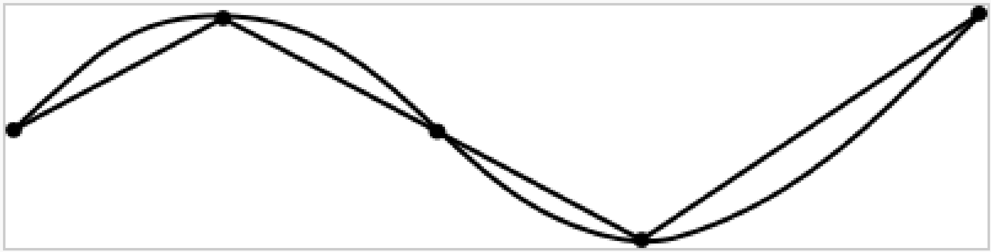
the collection of all morphisms from X to Y

Note that the domain and codomain are in fact part of the information determining a morphism. For example, in the category of sets, where morphisms are functions, two functions may be identical as sets of ordered pairs (may have the same range), while having different codomains. The two functions are distinct from the viewpoint of category theory. Thus many authors require that the hom-classes hom(X, Y) be disjoint. In practice, this is not a problem because if this disjointness does not hold, it can be assured by appending the domain and codomain to the morphisms, (say, as the second and third components of an ordered triple).

Functional (mathematics)

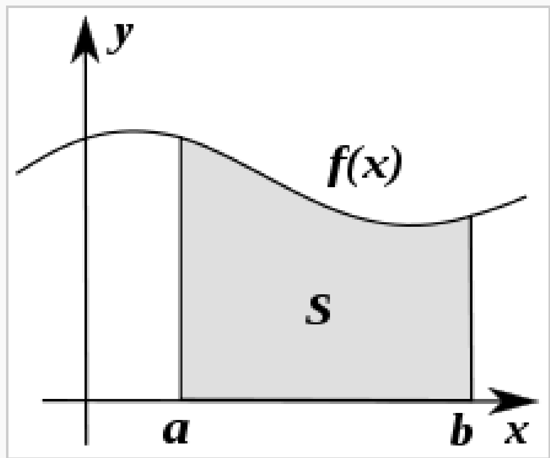
From Wikipedia, the free encyclopedia

Not to be confused with functional notation.



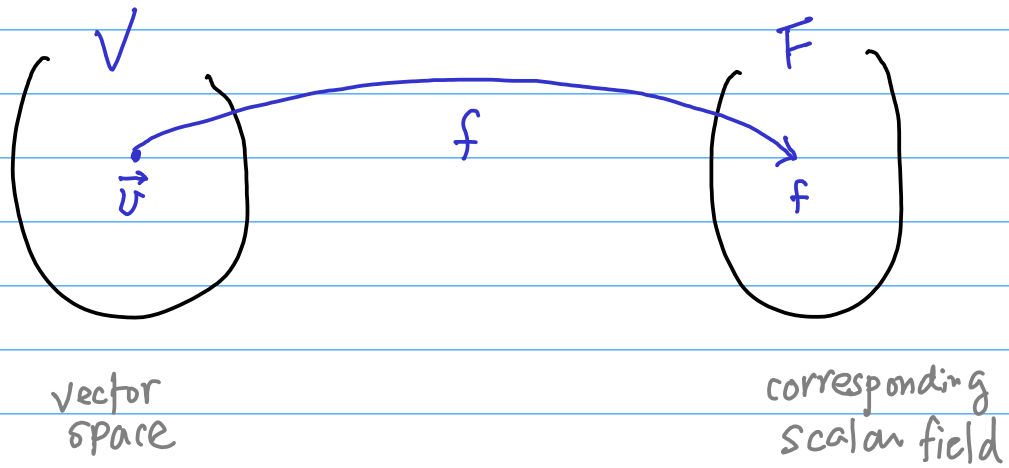
The arc length functional has as its domain the vector space of rectifiable curves (a subspace of $C([0,1],\mathbb{R}^3)$), and outputs a real scalar. This is an example of a non-linear functional.

In mathematics, and particularly in functional analysis and the calculus of variations, a **functional** is a function from a vector space into its underlying scalar field, or a set of functions of the real numbers. In other words, it is a function that takes a vector as its input argument, and returns a scalar. Commonly the vector space is a space of functions, thus the functional takes a function for its input argument, then it is sometimes considered a *function of a function* (a higher-order function). Its use originates in the calculus of variations where one searches for a function that minimizes a certain functional. A particularly important application in physics is searching for a state of a system that minimizes the energy functional.



The Riemann integral is a linear functional on the vector space of Riemann-integrable functions from $\mathbb{R}$ to $\mathbb{R}$.

functional



Vetor space
= a space of functions

function space → Vector space
commonly

functional = function of functions
higher order function

a set of real function → real number set

Functional details [edit]

Duality [edit]

The mapping

$$x_0 \mapsto f(x_0)$$

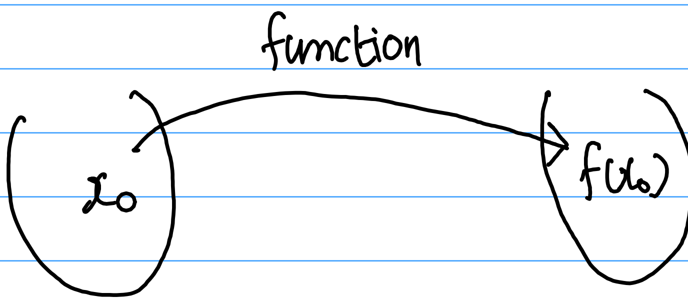
is a function, where x_0 is an argument of a function f. At the same time, the mapping of a function to the value of the function at a point

$$f \mapsto f(x_0)$$

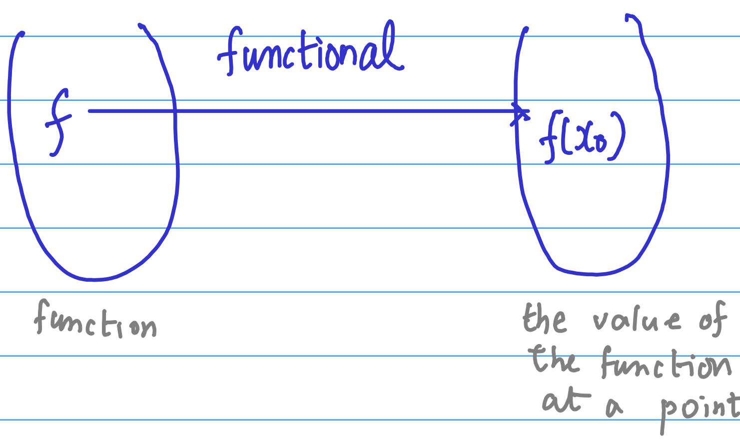
is a *functional*, here x_0 is a parameter.

Provided that f is a linear function from a linear vector space to the underlying scalar field, the above linear maps are dual to each other, and in functional analysis both are called linear functionals.

linear functional
linear form
one form
covector



x_0 : argument



x_0 : parameter

Definite integral [edit]

Integrals such as

$$f \mapsto I[f] = \int_\Omega H(f(x), f'(x), \ldots)\ \mu(\mathrm{d}x)$$

form a special class of functionals. They map a function f into a real number, provided that H is real-valued. Examples include

- the area underneath the graph of a positive function f

$$f \mapsto \int_{x_0}^{x_1} f(x)\ \mathrm{d}x$$

- L^p norm of functions

$$f \mapsto \left(\int |f|^p\ \mathrm{d}x \right)^{1/p}$$

- the arclength of a curve in 2-dimensional Euclidean space

$$f \mapsto \int_{x_0}^{x_1} \sqrt{1 + |f'(x)|^2}\ \mathrm{d}x$$

Vector scalar product [edit]

Given any vector $\vec{x}$ in a vector space X, the scalar product with another vector $\vec{y}$, denoted $\vec{x} \cdot \vec{y}$ or $\langle \vec{x}, \vec{y} \rangle$, is a scalar. The set of vectors $\vec{x}$ such that $\vec{x} \cdot \vec{y}$ is zero is a vector subspace of X, called the *null space* or kernel of X.

functional : function of functions

$$f \xrightarrow{\text{area}} \int_{x_0}^{x_1} f(x)\, dx$$

$$g \longrightarrow \int_{x_0}^{x_1} g(x)\, dx$$

$$f \xrightarrow{L^p \text{ norm}} \int_{x_0}^{x_1} |f(x)|^p\, dx$$

$$f \xrightarrow{\text{arc length}} \int_{x_0}^{x_1} \sqrt{1+|f'(x)|^2}\, dx$$

$$x \xrightarrow{\text{function}} \int_{x_0}^{x_1} f(x)\,dx$$

x : argument

$$f \xrightarrow{\text{functional}} \int_{x_0}^{x_1} f(x)\,dx$$

x : parameter

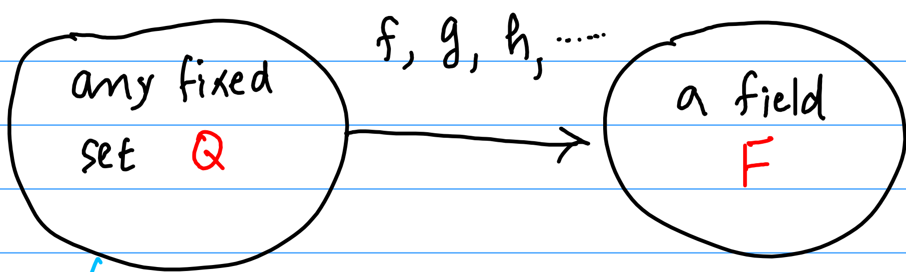
Function spaces [edit]

Functions from any fixed set Ω to a field F also form vector spaces, by performing addition and scalar multiplication pointwise. That is, the sum of two functions f and g is the function $(f + g)$ given by

$$(f + g)(w) = f(w) + g(w),$$

and similarly for multiplication. Such function spaces occur in many geometric situations, when Ω is the real line or an interval, or other subsets of $\mathbf{R}$. Many notions in topology and analysis, such as continuity, integrability or differentiability are well-behaved with respect to linearity: sums and scalar multiples of functions possessing such a property still have that property.[14] Therefore, the set of such functions are vector spaces. They are studied in greater detail using the methods of functional analysis, see below. Algebraic constraints also yield vector spaces: the vector space $F[x]$ is given by polynomial functions:

$$f(x) = r_0 + r_1x + \ldots + r_{n-1}x^{n-1} + r_nx^n,$$ where the coefficients $r_0, \ldots, r_n$ are in F.[15]

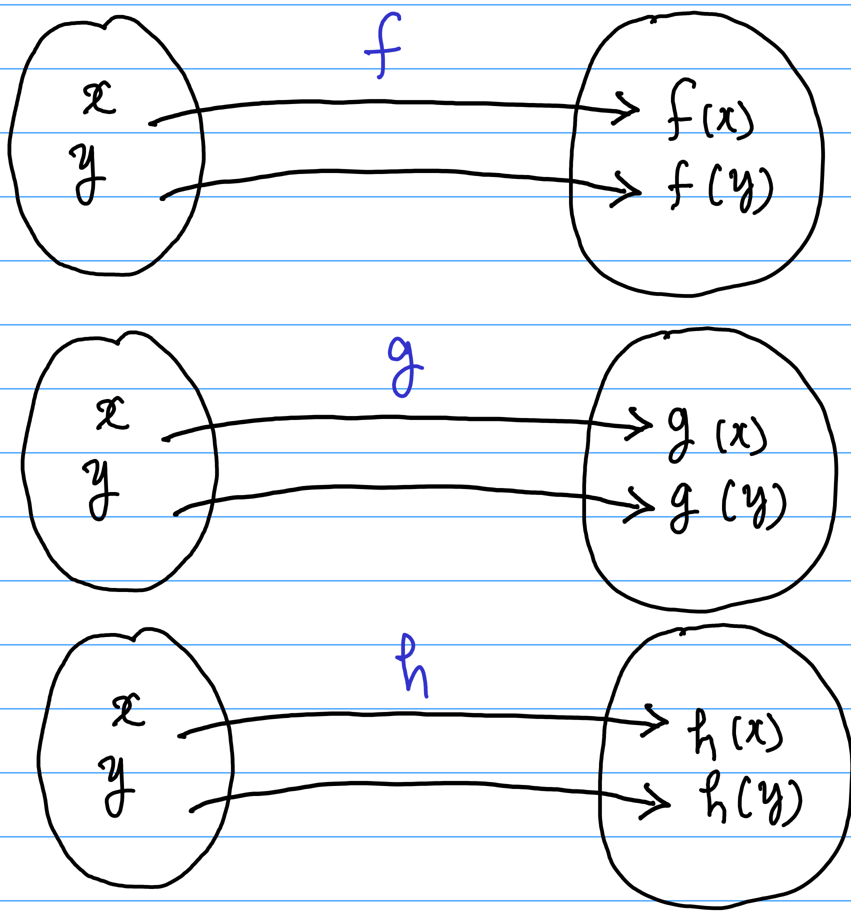


the real line
an interval
other subsets of R

$$\{f, g, h, \ldots\} = V \quad \text{vectorspace}$$

- sums of such functions $\in V$
- scalar multiplication $\in V$

any fixed set Q a field F



Let $\boxed{\begin{aligned} m &= f+g \\ n &= cf \end{aligned}}$ $m(x) = f(x) + g(x)$

$n(x) = c f(x)$

Vector Space

$$\begin{aligned} f \in V \\ g \in V \end{aligned} \Rightarrow \begin{aligned} m \in V \\ n \in V \end{aligned}$$

$x \in \mathbb{R}$ $\qquad f(x) \in F$

$f \in F[x]$ $\qquad f+g \in F[x]$

$g \in F[x]$ $\qquad cf \in F[x]$

$$f(x) = r_0 + r_1 x^1 + \cdots + r_{n-1} x^{n-1} + r_n x^n$$

$$g(x) = s_0 + s_1 x^1 + \cdots + s_{n-1} x^{n-1} + s_n x^n$$

$$(f+g)(x) = t_0 + t_1 x^1 + \cdots + t_{n-1} x^{n-1} + t_n x^n$$

$$(cf)(x) = u_0 + u_1 x^1 + \cdots + u_{n-1} x^{n-1} + u_n x^n$$

Function space

From Wikipedia, the free encyclopedia

In mathematics, a **function space** is a set of functions of a given kind from a set X to a set Y. It is called a space because in many applications it is a topological space (including metric spaces), a vector space, or both. Namely, if Y is a field, functions have inherent vector structure with two operations of pointwise addition and multiplication to a scalar. Topological and metrical structures of function spaces are more diverse.

Function

$$x \mapsto f(x)$$

By domain and codomain

$\mathbf{X} \longrightarrow \mathbf{B}$ $\quad$ $\mathbf{B}^n \longrightarrow \mathbf{B}$

$\mathbf{X} \longrightarrow \mathbf{Z} \longrightarrow \mathbf{X}$

$\mathbf{X} \longrightarrow \mathbf{R} \longrightarrow \mathbf{X}$ $\quad$ $\mathbf{R}^n \longrightarrow \mathbf{X}$

$\mathbf{X} \longrightarrow \mathbf{C} \longrightarrow \mathbf{X}$ $\quad$ $\mathbf{C}^n \longrightarrow \mathbf{X}$

Classes/properties

Constant · Identity · Linear · Polynomial · Rational · Algebraic · Analytic · Smooth · Continuous · Measurable · Injective · Surjective · Bijective

Constructions

Restriction · Composition · λ · Inverse

Generalizations

Partial · Multivalued · Implicit

V · T · E

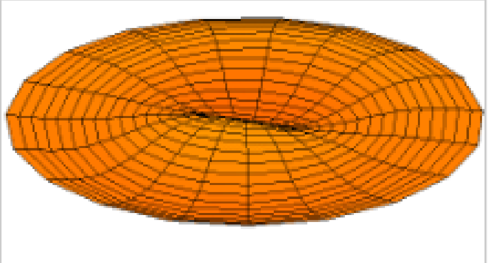
Contents [hide]

Functional analysis

From Wikipedia, the free encyclopedia

For the assessment and treatment of human behavior, see Functional analysis (psychology).

Functional analysis is a branch of mathematical analysis, the core of which is formed by the study of vector spaces endowed with some kind of limit-related structure (e.g. inner product, norm, topology, etc.) and the linear operators acting upon these spaces and respecting these structures in a suitable sense. The historical roots of functional analysis lie in the study of spaces of functions and the formulation of properties of transformations of functions such as the Fourier transform as transformations defining continuous, unitary etc. operators between function spaces. This point of view turned out to be particularly useful for the study of differential and integral equations.



One of the possible modes of vibration of an idealized circular drum head. These modes are eigenfunctions of a linear operator on a function space, a common construction in functional analysis.