

# Laurent Series and z-Transform

## - Geometric Series

### Permutations B

20240629 Sat

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$a^n$  $\times$  $R(n)$ 

$a^n$	$a^{-n}$

 $\times$ 

$u(n)$	$u(-n-1)$
$u(-n)$	$u(n-1)$

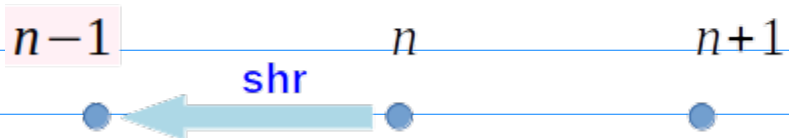
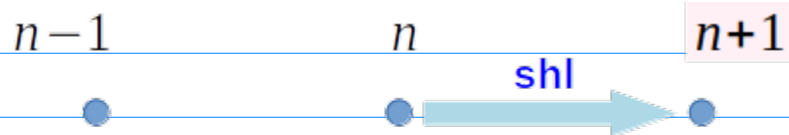
(1)	$a^n u(n)$	$a^{-n} u(n)$	(2)
(3)	$a^n u(-n)$	$a^{-n} u(-n)$	(4)
(5)	$a^n u(-n-1)$	$a^{-n} u(-n-1)$	(6)
(7)	$a^n u(n-1)$	$a^{-n} u(n-1)$	(8)

$b^n$	$shl(b^n)$
$a^n$	$a^{(n+1)}$
$a^{-n}$	$a^{-(n+1)}$

$b^n$	$shr(b^n)$
$a^n$	$a^{(n-1)}$
$a^{-n}$	$a^{-(n-1)}$

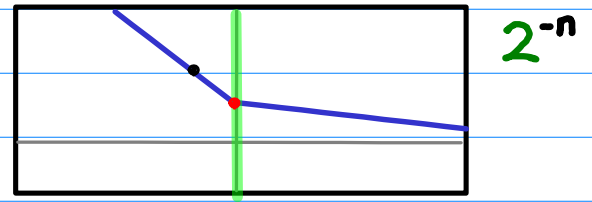
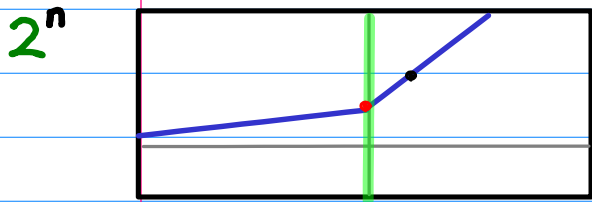
$R(n)$	$shl(R(n))$
$u(n-1)$	$u(n)$
$u(-n)$	$u(-(n+1))$

$R(n)$	$shr(R(n))$
$u(n)$	$u(n-1)$
$u(-(n+1))$	$u(-n)$



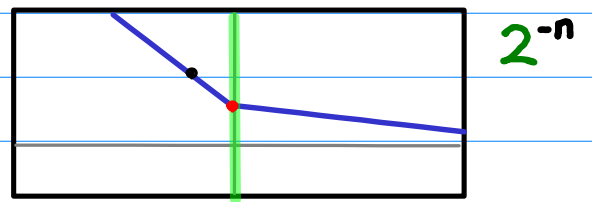
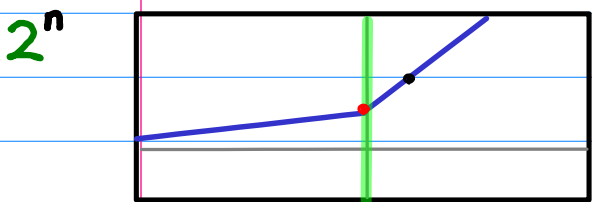
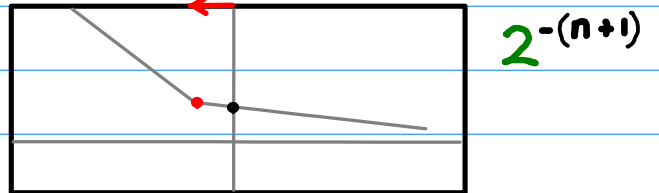
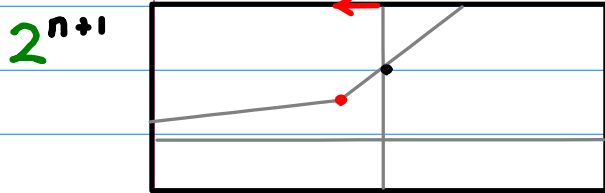
shl( $b^n$ )  
shr( $b^n$ )

shl( $n$ ) =  $n + 1$   
shr( $n$ ) =  $n - 1$



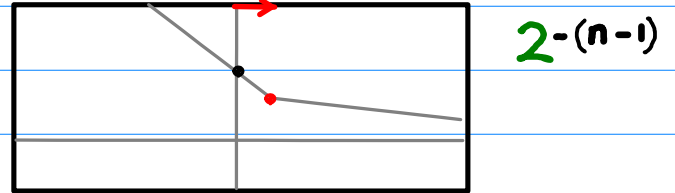
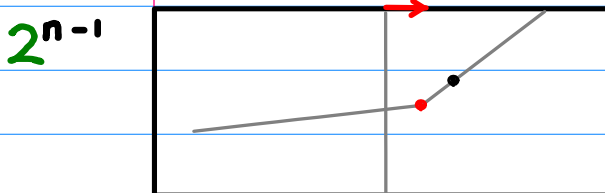
shift left  
 $n \leftarrow n+1$

shift left  
 $n \leftarrow n+1$



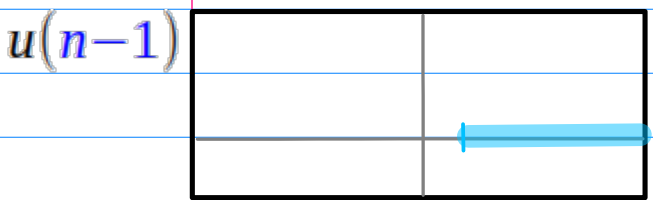
shift right  
 $n \leftarrow n-1$

shift right  
 $n \leftarrow n-1$

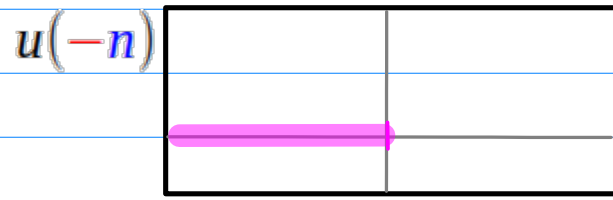
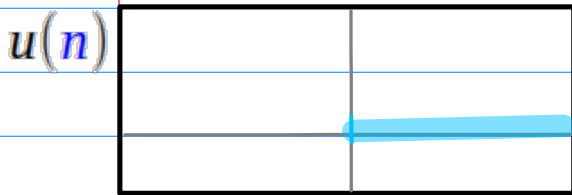


$\text{shl}(R(n))$   
 $\text{shr}(R(n))$

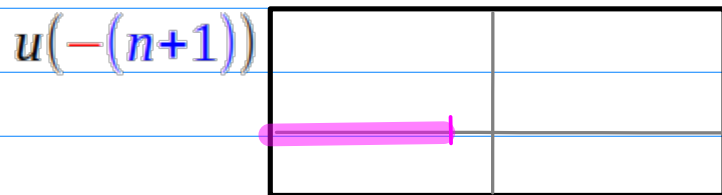
$\text{shl}(n) = n + 1$   
 $\text{shr}(n) = n - 1$



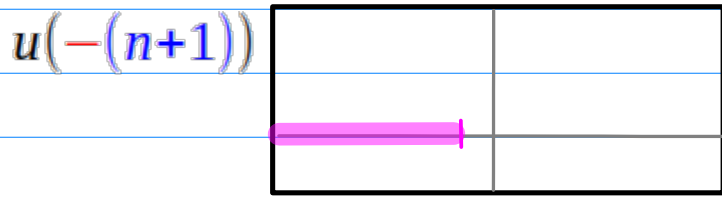
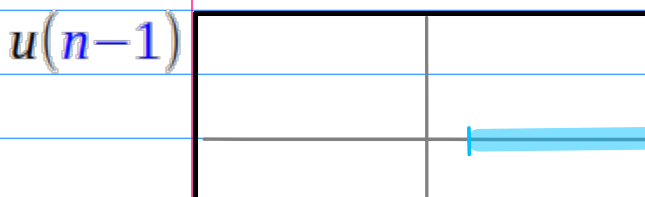
shift left  
 $n \leftarrow n+1$



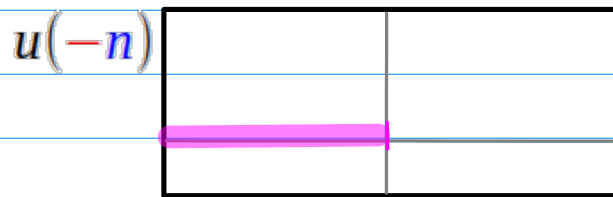
shift left  
 $n \leftarrow n+1$



shift right  
 $n \leftarrow n-1$



shift right  
 $n \leftarrow n-1$



$a^n$  $\times$  $R(n)$ 

$a^{n+1}$	$a^{-n-1}$
$a^{n-1}$	$a^{-n+1}$

 $\times$ 

$u(n)$	$u(-n-1)$
$u(-n)$	$u(n-1)$

(1')	$a^{n+1}u(n)$	$a^{-n-1}u(n)$	(2')
(3')	$a^{n-1}u(-n)$	$a^{-n+1}u(-n)$	(4')
(5')	$a^{n+1}u(-n-1)$	$a^{-n-1}u(-n-1)$	(6')
(7')	$a^{n-1}u(n-1)$	$a^{-n+1}u(n-1)$	(8')

(1'')	$a^{n-1}u(n)$	$a^{-n+1}u(n)$	(2'')
(3'')	$a^{n+1}u(-n)$	$a^{-n-1}u(-n)$	(4'')
(5'')	$a^{n-1}u(-n-1)$	$a^{-n+1}u(-n-1)$	(6'')
(7'')	$a^{n+1}u(n-1)$	$a^{-n-1}u(n-1)$	(8'')

$a^n$  $\times$  $R(n)$ 

$a^{n+1}$	$a^{-n-1}$

 $\times$ 

$u(n)$	$u(-n-1)$
$u(-n)$	$u(n-1)$

$$(1') \quad a^{n+1} u(n) \quad a^{-n-1} u(n) \quad (2')$$

$$(3') \quad a^{n-1} u(-n) \quad a^{-n+1} u(-n) \quad (4')$$

$$(5') \quad a^{n+1} u(-n-1) \quad a^{-n-1} u(-n-1) \quad (6')$$

$$(7') \quad a^{n-1} u(n-1) \quad a^{-n+1} u(n-1) \quad (8')$$



$a^n$  $\times$  $R(n)$ 

$a^{n+1}$	$a^{-n-1}$
$a^{n-1}$	$a^{-n+1}$

 $\times$ 

$u(n)$	$u(-n-1)$
$u(-n)$	$u(n-1)$

(1'')	$a^{n-1} u(n)$	$a^{-n+1} u(n)$	(2'')
(3'')	$a^{n+1} u(-n)$	$a^{-n-1} u(-n)$	(4'')
(5'')	$a^{n-1} u(-n-1)$	$a^{-n+1} u(-n-1)$	(6'')
(7'')	$a^{n+1} u(n-1)$	$a^{-n-1} u(n-1)$	(8'')

### Unshifted Sequence

### Shifted Sequence 1

### Shifted Sequence 2

(1)	$a^n u(n)$	$shl(b^n)$	$a^{n+1} u(n)$ (1')	$shl(b^{-n})$	$a^{-n-1} u(n)$ (1'')	$n \leftarrow n+1$
(2)	$a^{-n} u(n)$	$shl(b^n)$	$a^{-n-1} u(n)$ (2')	$shl(b^{-n})$	$a^{n+1} u(n)$ (2'')	$n \leftarrow n+1$
(3)	$a^n u(-n)$	$shr(b^n)$	$a^{n-1} u(-n)$ (3')	$shr(b^{-n})$	$a^{-n+1} u(-n)$ (3'')	$n \leftarrow n-1$
(4)	$a^{-n} u(-n)$	$shr(b^n)$	$a^{-n+1} u(-n)$ (4')	$shr(b^{-n})$	$a^{n-1} u(-n)$ (4'')	$n \leftarrow n-1$
(5)	$a^n u(-n-1)$	$shl(b^n)$	$a^{n+1} u(-n-1)$ (5')	$shl(b^{-n})$	$a^{-n-1} u(-n-1)$ (5'')	$n \leftarrow n+1$
(6)	$a^{-n} u(-n-1)$	$shl(b^n)$	$a^{-n-1} u(-n-1)$ (6')	$shl(b^{-n})$	$a^{n+1} u(-n-1)$ (6'')	$n \leftarrow n+1$
(7)	$a^n u(n-1)$	$shr(b^n)$	$a^{n-1} u(n-1)$ (7')	$shr(b^{-n})$	$a^{-n+1} u(n-1)$ (7'')	$n \leftarrow n-1$
(8)	$a^{-n} u(n-1)$	$shr(b^n)$	$a^{-n+1} u(n-1)$ (8')	$shr(b^{-n})$	$a^{n-1} u(n-1)$ (8'')	$n \leftarrow n-1$

many possible permutations are possible  
but consider these two

## Unshifted Sequence

(1)  $a^n u(n)$

(2)  $a^{-n} u(n)$

(3)  $a^n u(-n)$

(4)  $a^{-n} u(-n)$

(5)  $a^n u(-n-1)$

(6)  $a^{-n} u(-n-1)$

(7)  $a^n u(n-1)$

(8)  $a^{-n} u(n-1)$

## Shifted Sequence 1

$shl(b^n)$   $a^{n+i} u(n)$  (1')

$shl(b^n)$   $a^{-n-i} u(n)$  (2')

$shr(b^n)$   $a^{n-i} u(-n)$  (3')

$shr(b^n)$   $a^{-n+i} u(-n)$  (4')

$shl(b^n)$   $a^{n+i} u(-n-1)$  (5')

$shl(b^n)$   $a^{-n-i} u(-n-1)$  (6')

$shr(b^n)$   $a^{n-i} u(n-1)$  (7')

$shr(b^n)$   $a^{-n+i} u(n-1)$  (8')

**Inter-permutations over unshifted sequence and shifted sequence**

**Intra-permutations over unshifted sequence**

**Intra-permutations over shifted sequence**

## Unshifted Sequence

(1)  $a^n u(n)$

(2)  $a^{-n} u(n)$

(3)  $a^n u(-n)$

(4)  $a^{-n} u(-n)$

(5)  $a^n u(-n-1)$

(6)  $a^{-n} u(-n-1)$

(7)  $a^n u(n-1)$

(8)  $a^{-n} u(n-1)$

## Shifted Sequence 2

$shl(b^{-n})$   $a^{-n-i} u(n)$  (1'')

$shl(b^{-n})$   $a^{n+i} u(n)$  (2'')

$shr(b^{-n})$   $a^{-n+i} u(-n)$  (3'')

$shr(b^{-n})$   $a^{n-i} u(-n)$  (4'')

$shl(b^{-n})$   $a^{-n-i} u(-n-1)$  (5'')

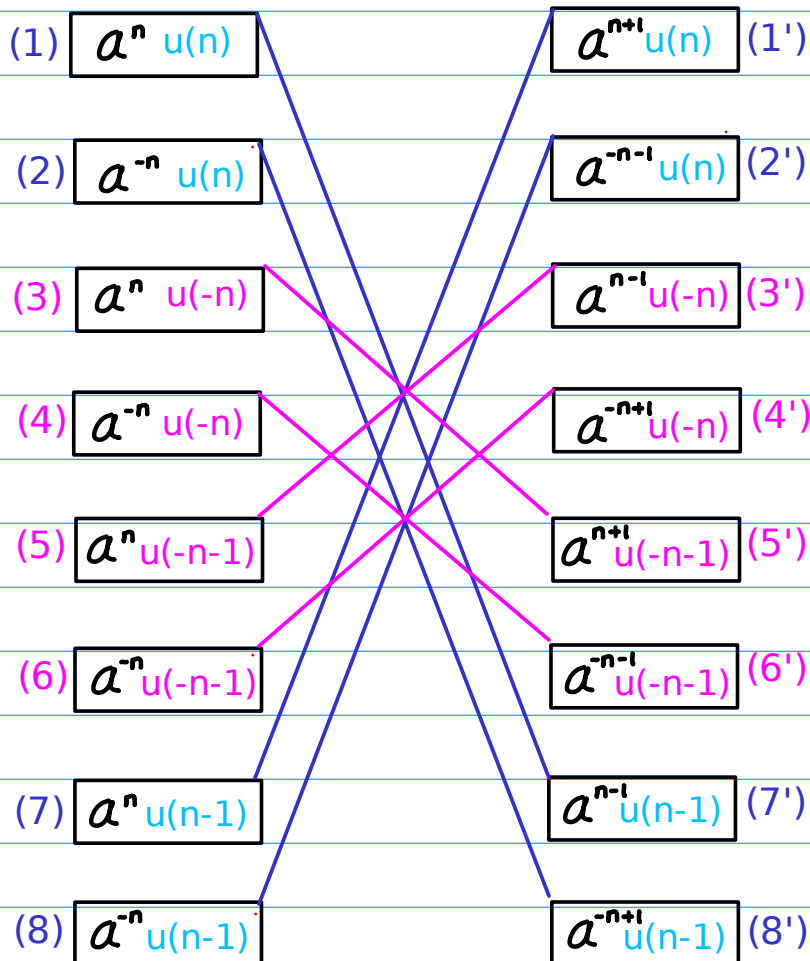
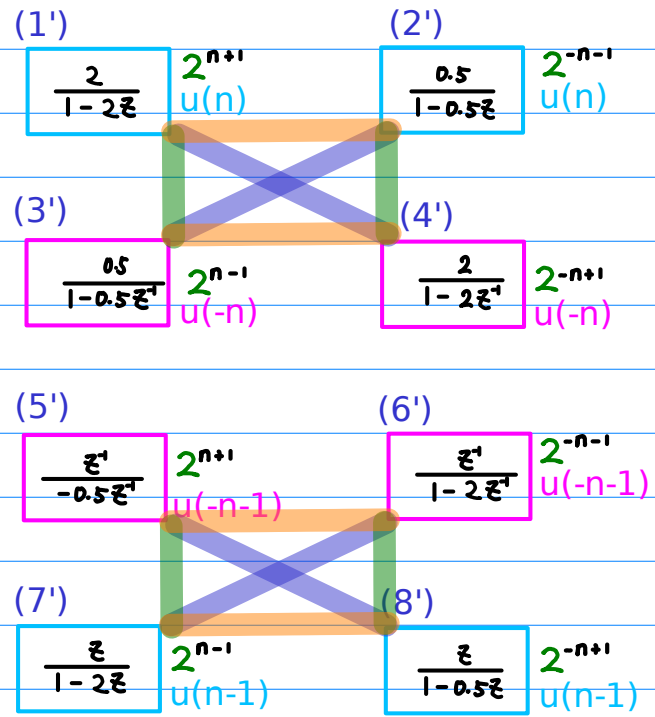
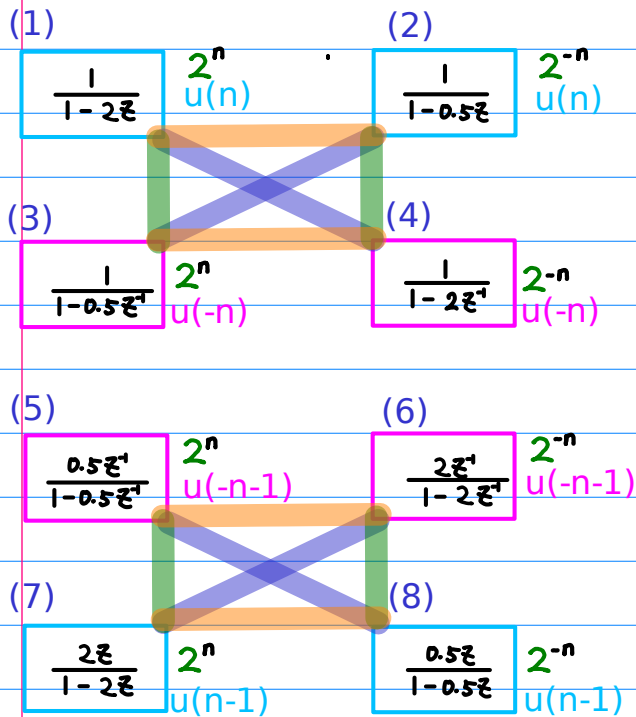
$shl(b^{-n})$   $a^{n+i} u(-n-1)$  (6'')

$shr(b^{-n})$   $a^{-n+i} u(n-1)$  (7'')

$shr(b^{-n})$   $a^{n-i} u(n-1)$  (8'')

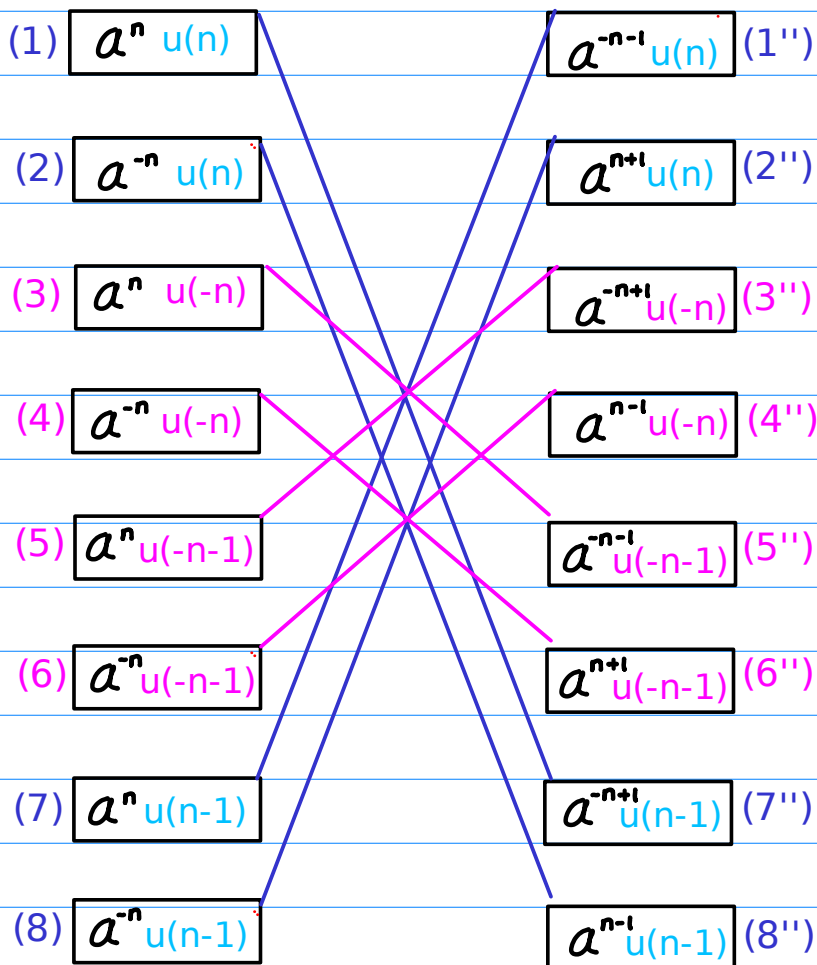
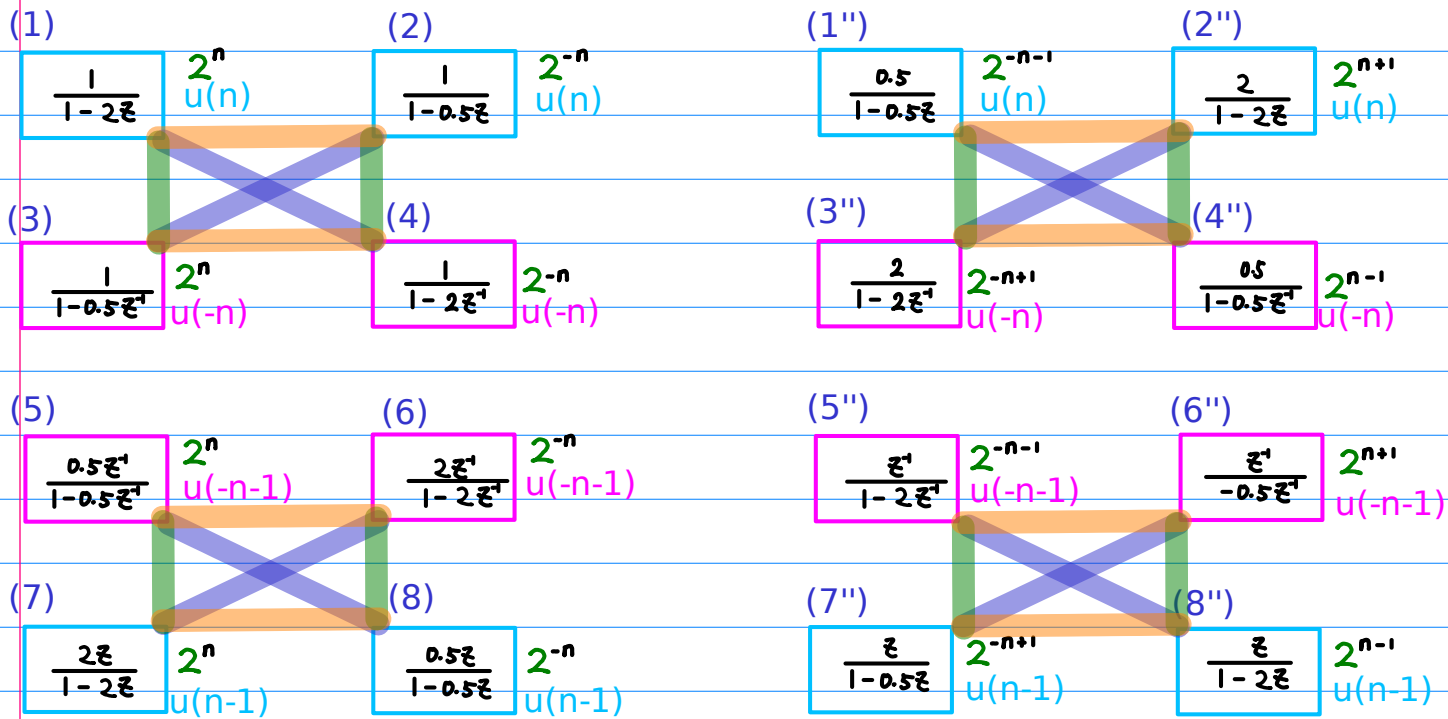
## Unshifted Sequence

## Shifted Sequence 1



## Unshifted Sequence

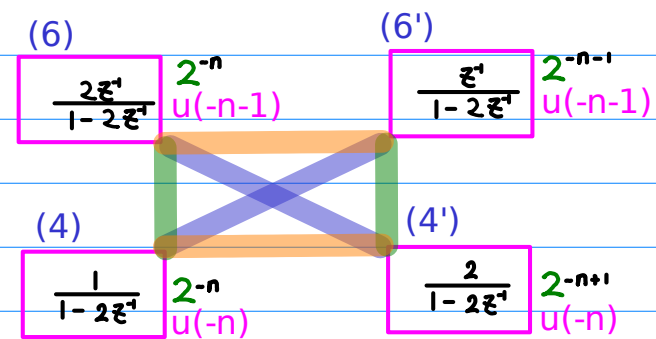
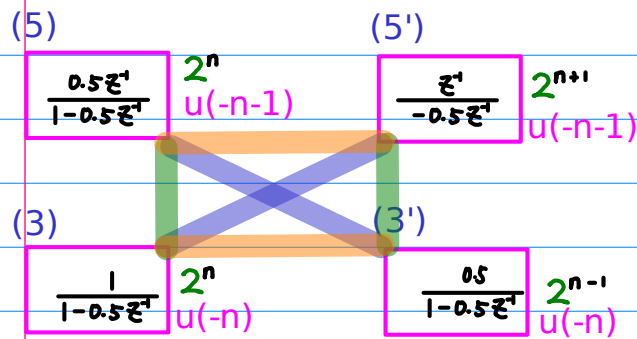
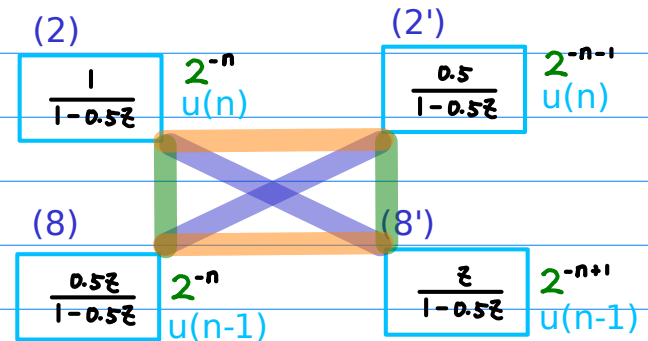
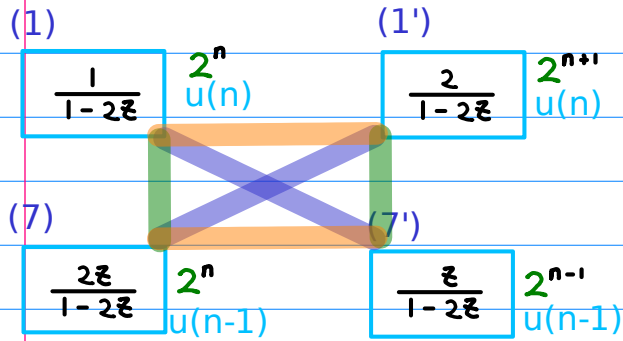
## Shifted Sequence 2



# Inter-permutation

$(x) \quad \dashrightarrow \quad (x')$

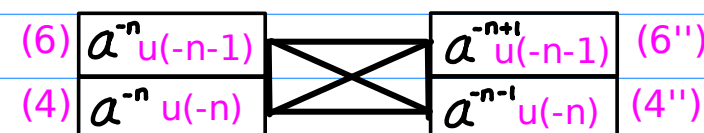
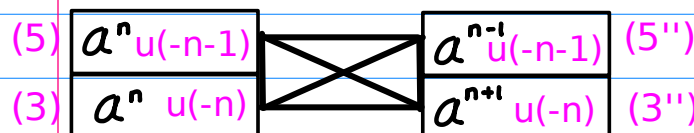
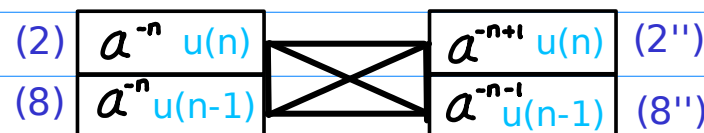
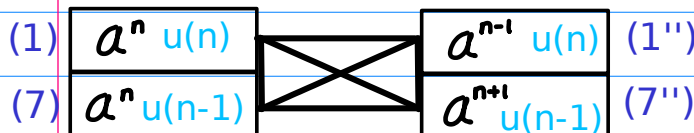
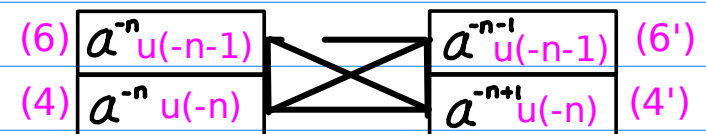
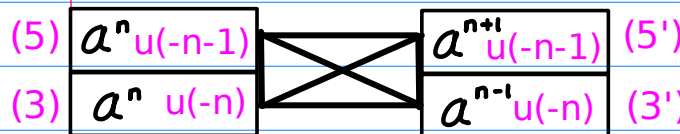
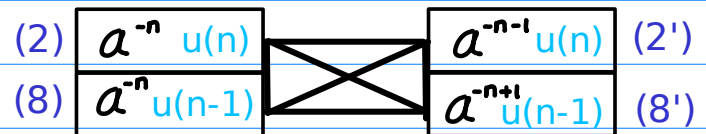
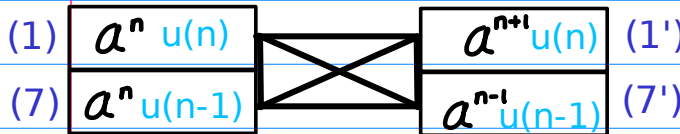
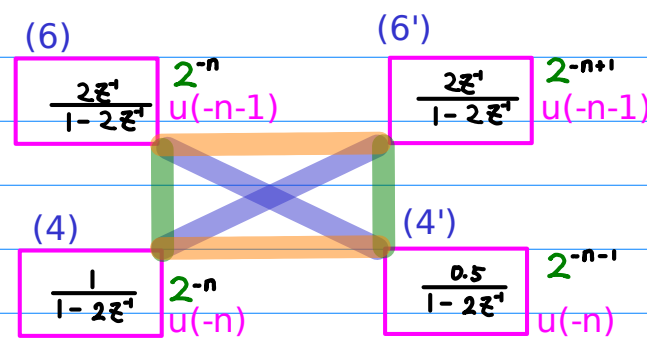
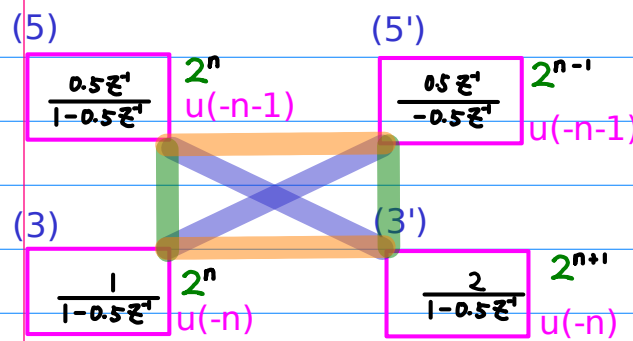
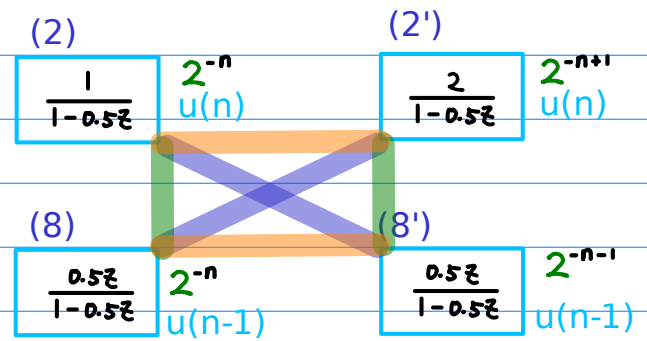
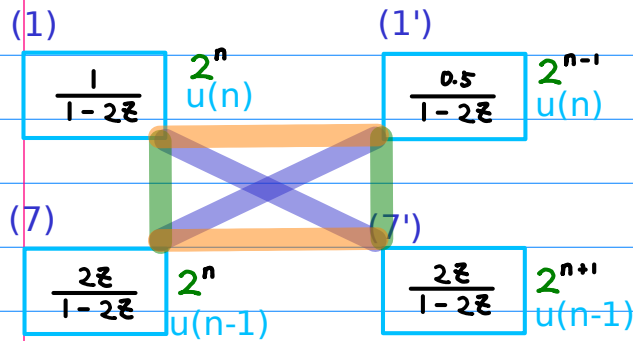
$(1) \sim (8) \quad \dashrightarrow \quad (1') \sim (8')$



# Inter-permutation

(x)  $\dashrightarrow$  (x')

(1)~(8)  $\dashrightarrow$  (1')~(8')



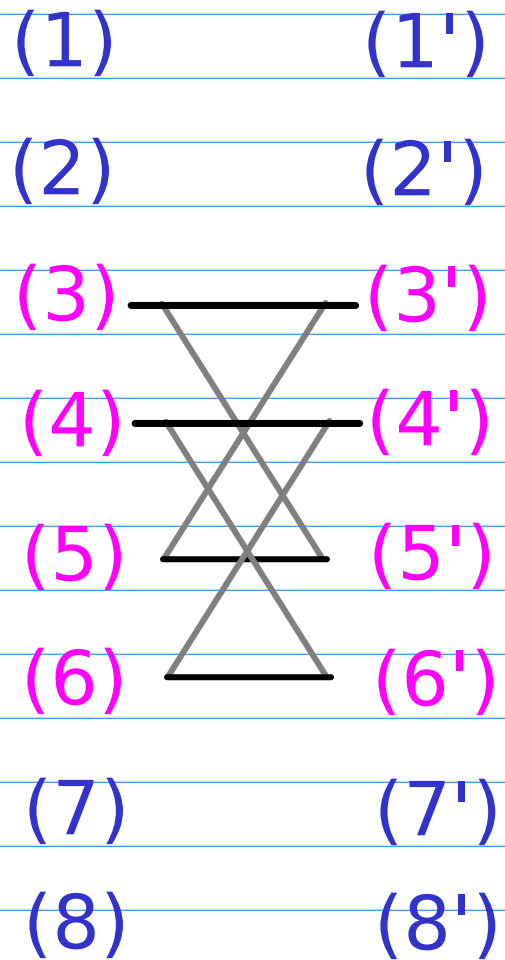
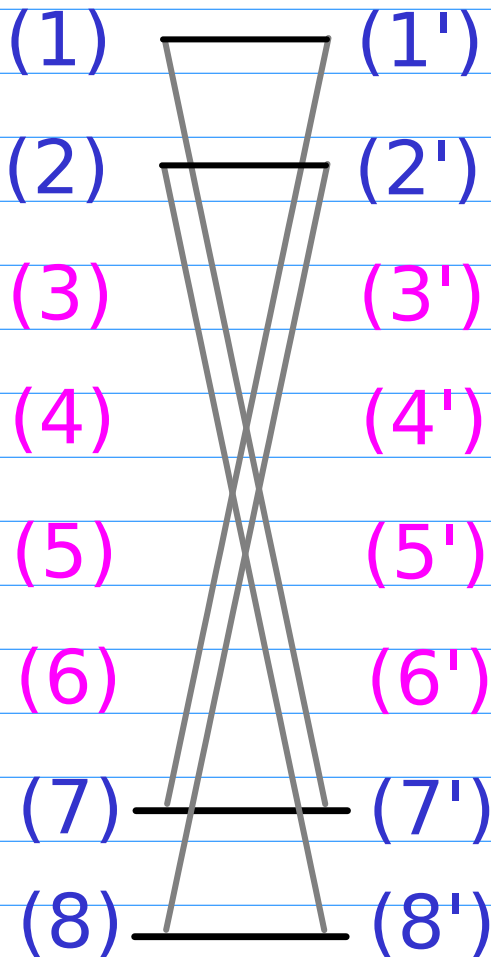
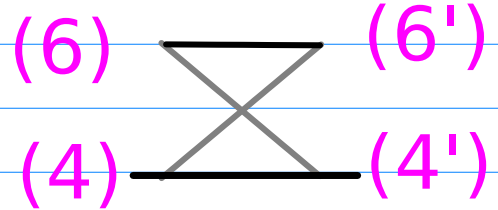
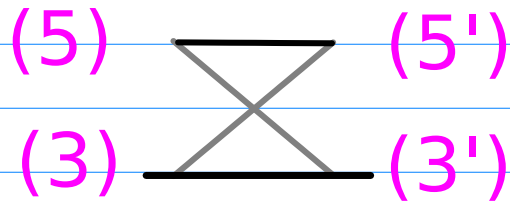
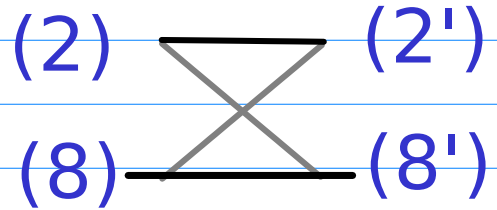
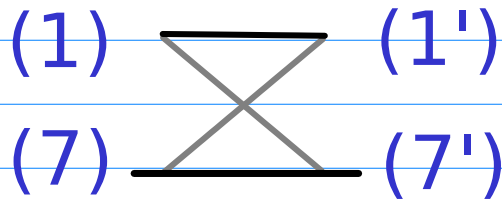


(1) - (1')  
(7) - (7')  
(5) - (5')  
(3) - (3')

(2) - (2')  
(8) - (8')  
(6) - (6')  
(4) - (4')

(1) - (1')  
(2) - (2')  
(3) - (3')  
(4) - (4')

(5) - (5')  
(6) - (6')  
(7) - (7')  
(8) - (8')



# Butterfly Relations

$(x) \quad \dashrightarrow \quad (x')$   
 $(1) \sim (8) \quad \dashrightarrow \quad (1') \sim (8')$

(\*) unit starting

(1)  $\xrightarrow{*a \leftarrow (SL, id)}$  (1') SL

$/z \leftarrow (SL, SL)$

$*z \Rightarrow (SR, SR)$

(7)  $\xrightarrow{/a \rightarrow (SR, id)}$  (7') SR

$/a \rightarrow (SR, id)$

C.R. starting

(7)

(\*) unit starting

(2)  $\xrightarrow{/a \leftarrow (SL, id)}$  (2') SL

$/z \leftarrow (SL, SL)$

$*z \Rightarrow (SR, SR)$

(8)  $\xrightarrow{*a \rightarrow (SR, id)}$  (8') SR

$*a \rightarrow (SR, id)$

C.R. starting

(8)

(\*) C.R. starting

(5)  $\xrightarrow{*a \leftarrow (SL, id)}$  (5') SL

$/z \leftarrow (SL, SL)$

$*z \Rightarrow (SR, SR)$

(3)  $\xrightarrow{/a \rightarrow (SR, id)}$  (3') SR

$/a \rightarrow (SR, id)$

unit starting

(3)

(\*) C.R. starting

(6)  $\xrightarrow{/a \leftarrow (SL, id)}$  (6') SL

$/z \leftarrow (SL, SL)$

$*z \Rightarrow (SR, SR)$

(4)  $\xrightarrow{*a \rightarrow (SR, id)}$  (4') SR

$*a \rightarrow (SR, id)$

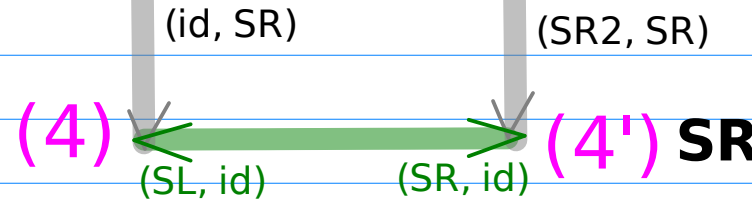
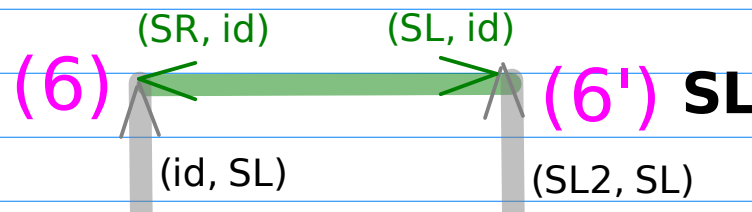
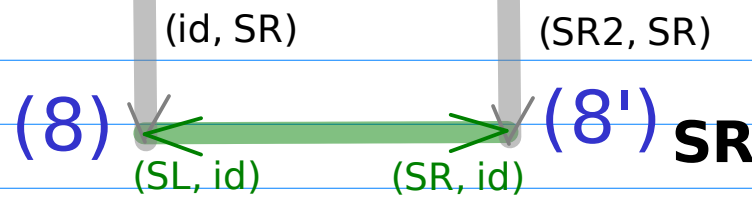
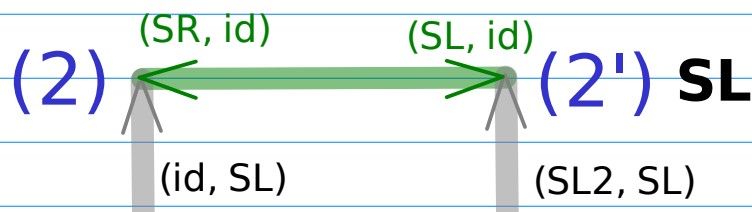
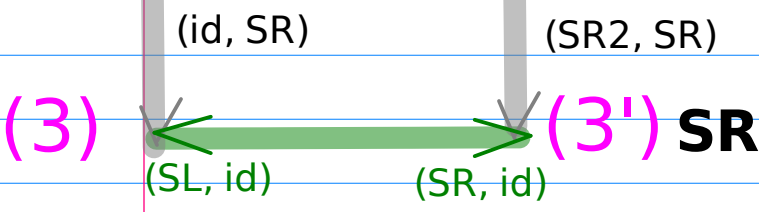
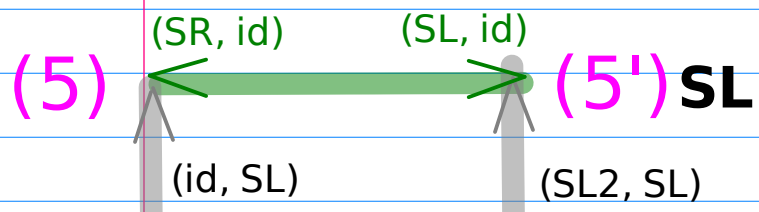
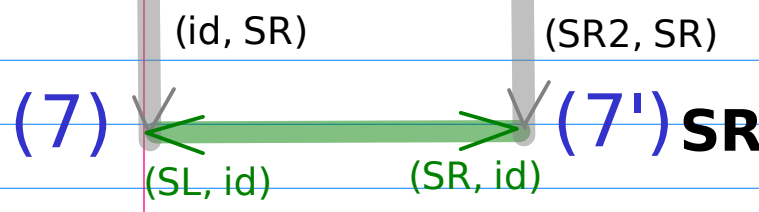
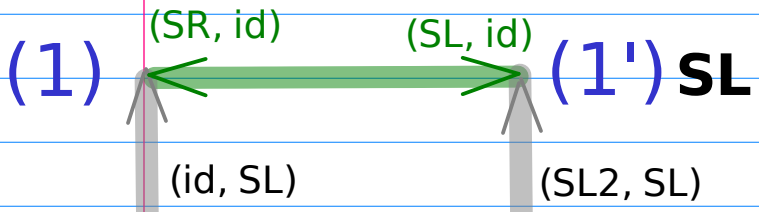
unit starting

(4)

**(Exp Shift, Range Shift)**

# Butterfly Relations

$(x) \quad \dashrightarrow \quad (x')$   
 $(1) \sim (8) \quad \dashrightarrow \quad (1') \sim (8')$



**(Exp Shift, Range Shift)**

# Decomposition of Exp and Rng Shifts (1)

$$(id, SR) + (SR, id) = (SR, SR)$$

(\*) unit starting

(1)

(1') SL

(id, SR)

\*z ⇒ (SR, SR)

C.R. starting

(7)

(7') SR

/a ⇒ (SR, id)

(\*) unit starting

(2)

(2') SL

(id, SR)

\*z ⇒ (SR, SR)

C.R. starting

(8)

(8') SR

\*a ⇒ (SR, id)

(\*) C.R. starting

(5)

(5') SL

(id, SR)

\*z ⇒ (SR, SR)

unit starting

(3)

(3') SR

/a ⇒ (SR, id)

(\*) C.R. starting

(6)

(6') SL

(id, SR)

\*z ⇒ (SR, SR)

unit starting

(4)

(4') SR

\*a ⇒ (SR, id)

# Decomposition of Exp and Rng Shifts (2)

$$(id, SL) + (SL, id) = (SL, SL)$$

(\*) unit starting

(1)  $\xrightarrow{*a \leftarrow (SL, id)}$  (1') SL

(id, SL)

$\xrightarrow{/z \leftarrow (SL, SL)}$

C.R. starting

(7)  $\xrightarrow{\uparrow}$  (7') SR

(\*) unit starting

(2)  $\xrightarrow{/a \leftarrow (SL, id)}$  (2') SL

(id, SL)

$\xrightarrow{/z \leftarrow (SL, SL)}$

C.R. starting

(8)  $\xrightarrow{\uparrow}$  (8') SR

(\*) C.R. starting

(5)  $\xrightarrow{*a \leftarrow (SL, id)}$  (5') SL

(id, SL)

$\xrightarrow{/z \leftarrow (SL, SL)}$

unit starting

(3)  $\xrightarrow{\uparrow}$  (3') SR

(\*) C.R. starting

(6)  $\xrightarrow{/a \leftarrow (SL, id)}$  (6') SL

(id, SL)

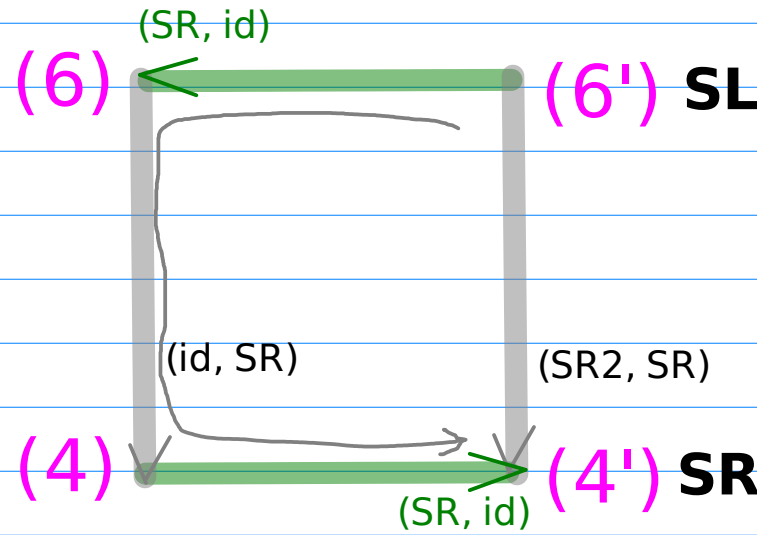
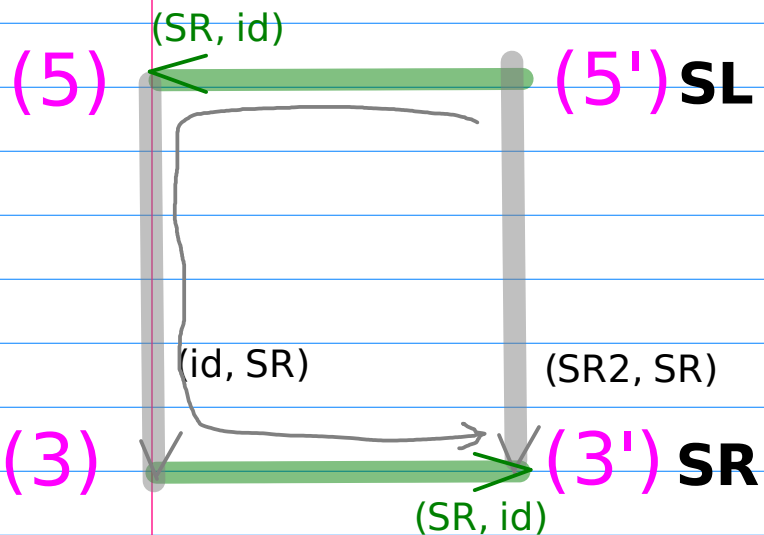
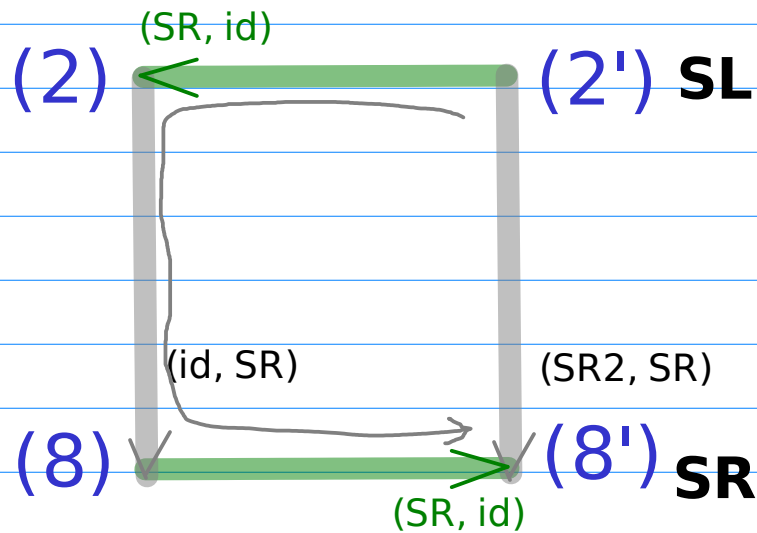
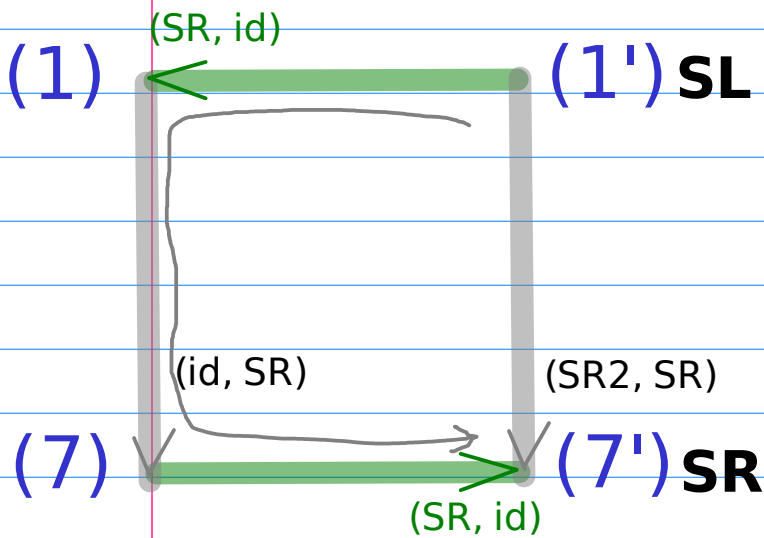
$\xrightarrow{/z \leftarrow (SL, SL)}$

unit starting

(4)  $\xrightarrow{\uparrow}$  (4') SR

# Decomposition of Exp and Rng Shifts (3)

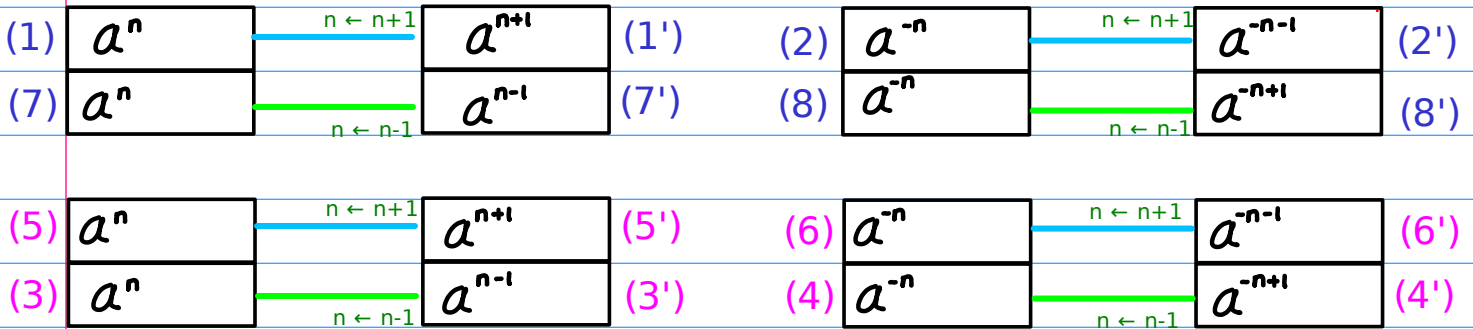
$$(SR, id) + (id, SR) + (SR, id) = (SR2, SR)$$



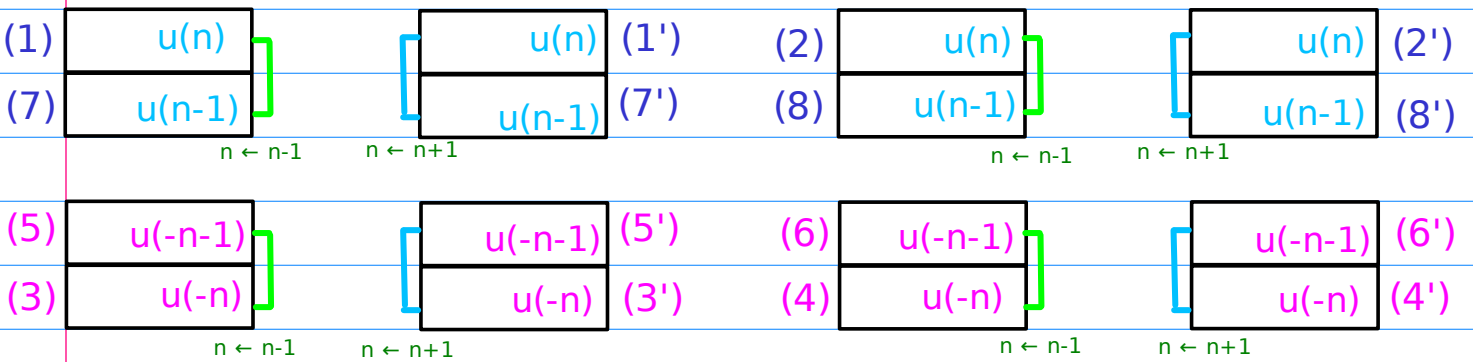
$(id, SR) + (SR, id) = (SR, SR)$   
 $(id, SL) + (SL, id) = (SL, SL)$

$(\mathbf{x}) \quad \dashrightarrow \quad (\mathbf{x}')$   
 $(1) \sim (8) \quad \dashrightarrow \quad (1') \sim (8')$

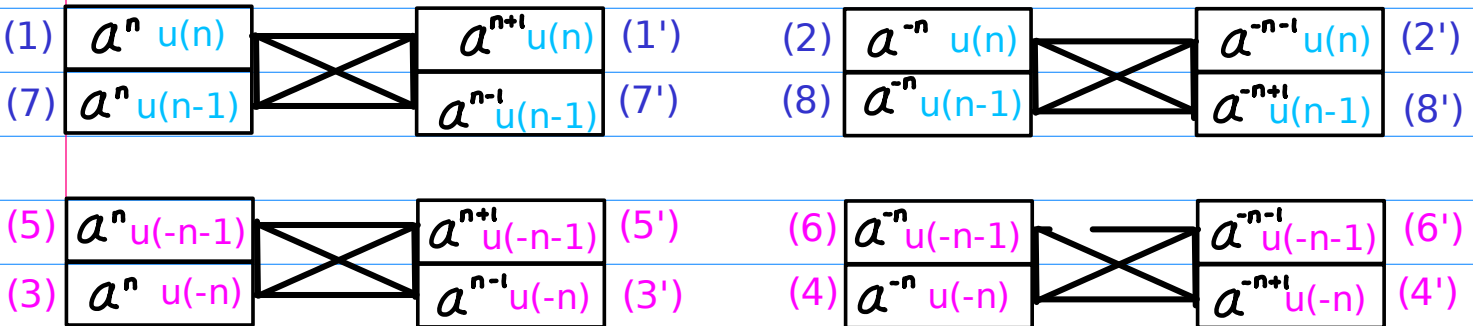
### Exponent Shifts



### Range Shifts



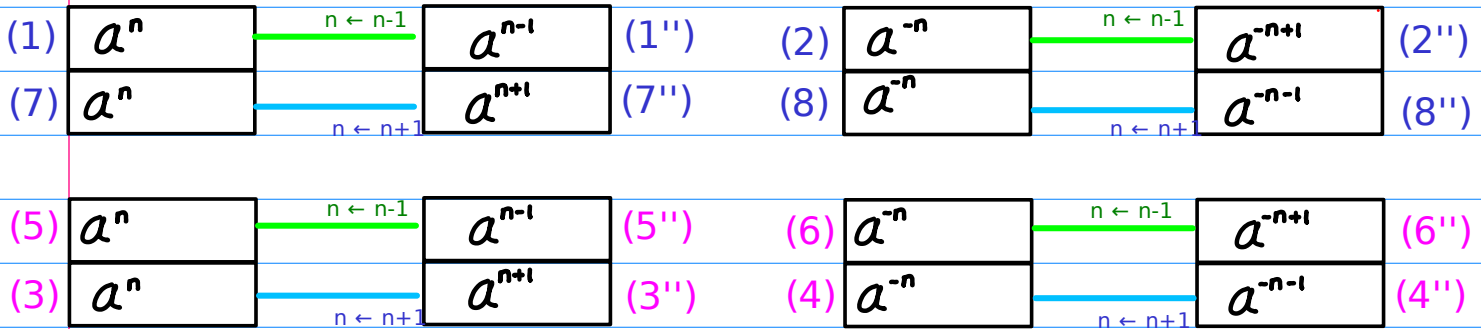
### Exponent & Range Permutations



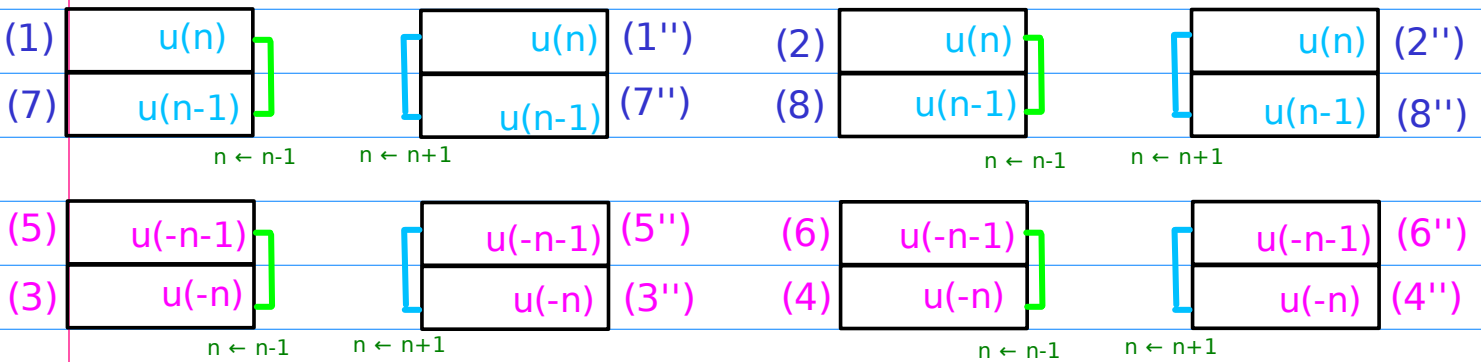
$$\begin{aligned}
 (\text{id}, \text{SR}) + (\text{SL}, \text{id}) &= (\text{SL}, \text{SR}) \\
 (\text{id}, \text{SL}) + (\text{SR}, \text{id}) &= (\text{SR}, \text{SL})
 \end{aligned}$$

$$\begin{aligned}
 (\mathbf{x}) &\dashrightarrow (\mathbf{x}'') \\
 (1) \sim (8) &\dashrightarrow (1'') \sim (8'')
 \end{aligned}$$

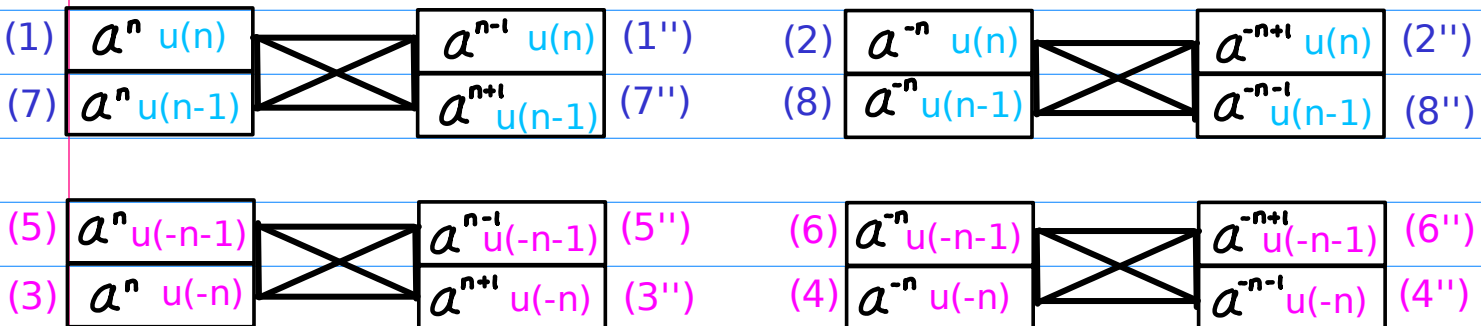
## Exponent Shifts



## Range Shifts



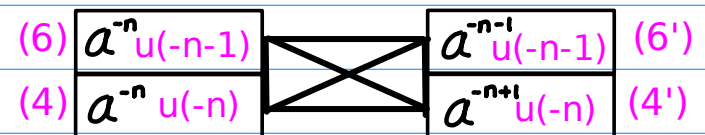
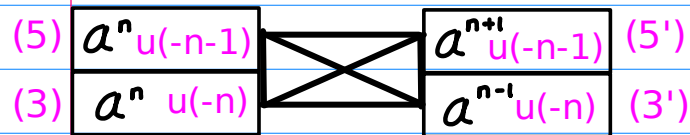
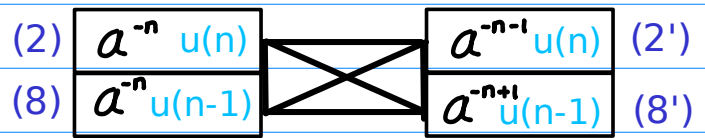
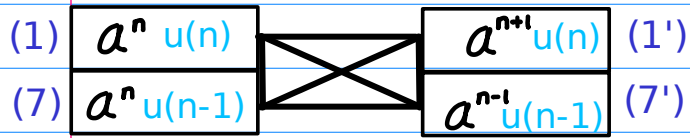
## Exponent & Range Permutations





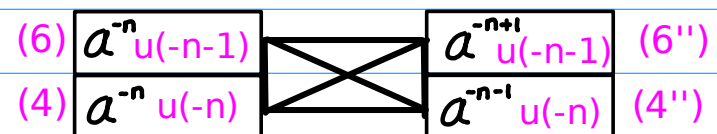
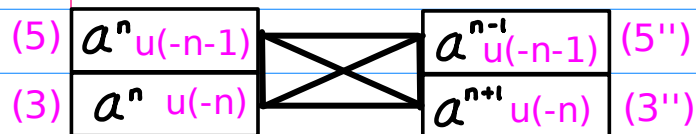
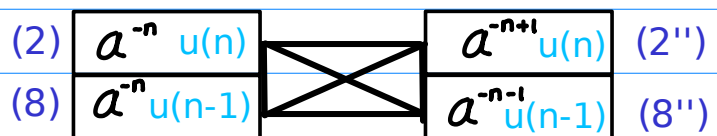
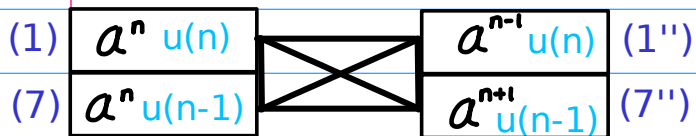
$$\begin{aligned}
 (\text{id}, \text{SR}) + (\text{SR}, \text{id}) &= (\text{SR}, \text{SR}) \\
 (\text{id}, \text{SL}) + (\text{SL}, \text{id}) &= (\text{SL}, \text{SL})
 \end{aligned}$$

$$\begin{aligned}
 (\mathbf{x}) &\quad \dashrightarrow \quad (\mathbf{x}') \\
 (1) \sim (8) &\quad \dashrightarrow \quad (1') \sim (8')
 \end{aligned}$$



$$\begin{aligned}
 (\text{id}, \text{SR}) + (\text{SL}, \text{id}) &= (\text{SL}, \text{SR}) \\
 (\text{id}, \text{SL}) + (\text{SR}, \text{id}) &= (\text{SR}, \text{SL})
 \end{aligned}$$

$$\begin{aligned}
 (\mathbf{x}) &\quad \dashrightarrow \quad (\mathbf{x}'') \\
 (1) \sim (8) &\quad \dashrightarrow \quad (1'') \sim (8'')
 \end{aligned}$$





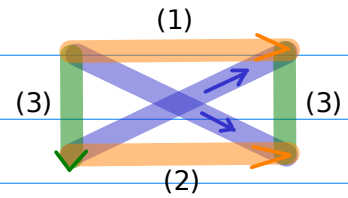
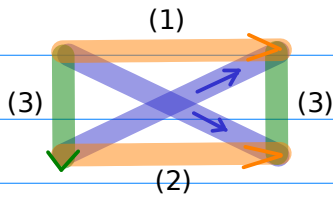
# Exponent Permutations

$$\begin{array}{l} (1) \\ (7) \end{array} \begin{array}{|c|} \hline a^n u(n) \\ \hline a^n u(n-1) \\ \hline \end{array} \begin{array}{|c|} \hline \diagdown \quad \diagup \\ \hline \diagup \quad \diagdown \\ \hline \end{array} \begin{array}{|c|} \hline a^{n+1} u(n) \\ \hline a^{n-1} u(n-1) \\ \hline \end{array} \begin{array}{l} (1') \\ (7') \end{array}$$

$$\begin{array}{l} (2) \\ (8) \end{array} \begin{array}{|c|} \hline a^{-n} u(n) \\ \hline a^{-n} u(n-1) \\ \hline \end{array} \begin{array}{|c|} \hline \diagdown \quad \diagup \\ \hline \diagup \quad \diagdown \\ \hline \end{array} \begin{array}{|c|} \hline a^{-n-1} u(n) \\ \hline a^{-n+1} u(n-1) \\ \hline \end{array} \begin{array}{l} (2') \\ (8') \end{array}$$

$$\begin{array}{l} (5) \\ (3) \end{array} \begin{array}{|c|} \hline a^n u(-n-1) \\ \hline a^n u(-n) \\ \hline \end{array} \begin{array}{|c|} \hline \diagdown \quad \diagup \\ \hline \diagup \quad \diagdown \\ \hline \end{array} \begin{array}{|c|} \hline a^{n+1} u(-n-1) \\ \hline a^{n-1} u(-n) \\ \hline \end{array} \begin{array}{l} (5') \\ (3') \end{array}$$

$$\begin{array}{l} (6) \\ (4) \end{array} \begin{array}{|c|} \hline a^{-n} u(-n-1) \\ \hline a^{-n} u(-n) \\ \hline \end{array} \begin{array}{|c|} \hline \diagdown \quad \diagup \\ \hline \diagup \quad \diagdown \\ \hline \end{array} \begin{array}{|c|} \hline a^{-n-1} u(-n-1) \\ \hline a^{-n+1} u(-n) \\ \hline \end{array} \begin{array}{l} (6') \\ (4') \end{array}$$



- (1) shift left exponent
- (2) shift right exponent
- (3) shift right range

- (1) shift left exponent
- (2) shift right exponent
- (3) shift right range

(SR, id) shift right exponent  
 (id, SR) shift right range  
 (SR, SR)

(SR, id) shift right exponent  
 (id, SR) shift right range  
 (SR, SR)

(SL, id) shift left exponent  
 (id, SL) shift left range  
 (SL, SL)

(SL, id) shift left exponent  
 (id, SL) shift left range  
 (SL, SL)

$$\begin{aligned} (\mathbf{SR, id}) + (\mathbf{id, SR}) &= (\mathbf{SR, SR}) \\ (\mathbf{SL, id}) + (\mathbf{id, SL}) &= (\mathbf{SL, SL}) \end{aligned}$$

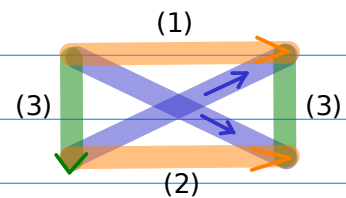
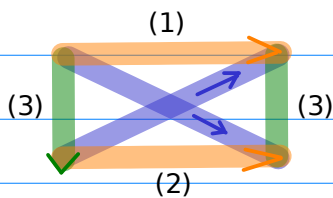
$$\begin{aligned} (\mathbf{SR, id}) + (\mathbf{id, SR}) &= (\mathbf{SR, SR}) \\ (\mathbf{SL, id}) + (\mathbf{id, SL}) &= (\mathbf{SL, SL}) \end{aligned}$$

$$\begin{array}{l} (1) \ a^n u(n) \\ (7) \ a^n u(n-1) \end{array} \begin{array}{c} \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array} \begin{array}{l} a^{n-1} u(n) \quad (1'') \\ a^{n+1} u(n-1) \quad (7'') \end{array}$$

$$\begin{array}{l} (2) \ a^{-n} u(n) \\ (8) \ a^{-n} u(n-1) \end{array} \begin{array}{c} \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array} \begin{array}{l} a^{-n+1} u(n) \quad (2'') \\ a^{-n-1} u(n-1) \quad (8'') \end{array}$$

$$\begin{array}{l} (5) \ a^n u(-n-1) \\ (3) \ a^n u(-n) \end{array} \begin{array}{c} \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array} \begin{array}{l} a^{n-1} u(-n-1) \quad (5'') \\ a^{n+1} u(-n) \quad (3'') \end{array}$$

$$\begin{array}{l} (6) \ a^{-n} u(-n-1) \\ (4) \ a^{-n} u(-n) \end{array} \begin{array}{c} \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array} \begin{array}{l} a^{-n+1} u(-n-1) \quad (6'') \\ a^{-n-1} u(-n) \quad (4'') \end{array}$$



**(1) shift right exponent**

**(2) shift left exponent**

**(3) shift right range**

**(1) shift right exponent**

**(2) shift left exponent**

**(3) shift right range**

(SL, id) shift right exponent

(id, SR) shift right range

(SL SR)

(SL, id) shift right exponent

(id, SR) shift right range

(SL, SR)

(SR, id) shift left exponent

(id, SL) shift left range

(SR, SL)

(SR, id) shift left exponent

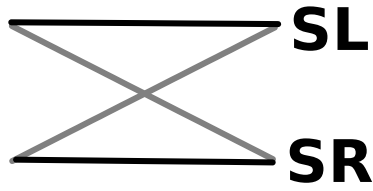
(id, SL) shift left range

(SR, SL)

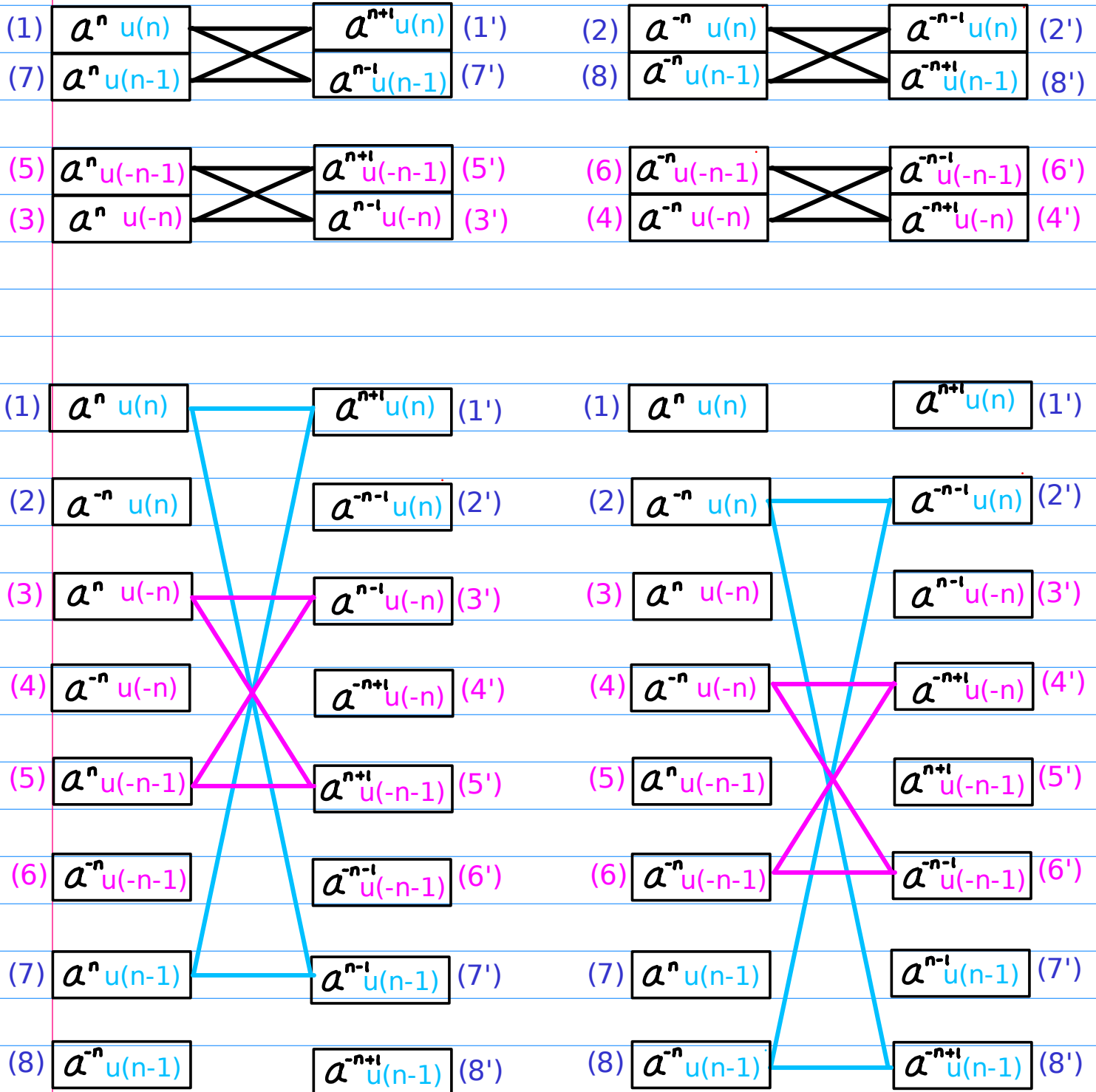
$$\begin{aligned} (\mathbf{SL, id}) + (\mathbf{id, SR}) &= (\mathbf{SL, SR}) \\ (\mathbf{SR, id}) + (\mathbf{id, SL}) &= (\mathbf{SR, SL}) \end{aligned}$$

$$\begin{aligned} (\mathbf{SL, id}) + (\mathbf{id, SR}) &= (\mathbf{SL, SR}) \\ (\mathbf{SR, id}) + (\mathbf{id, SL}) &= (\mathbf{SR, SL}) \end{aligned}$$

Unshifted Sequence



Shifted Sequence





# Intra-Permutations over (1) ~ (8)

(1) $a^n u(n)$	$a^{n+i} u(n)$ (1')
(2) $a^{-n} u(n)$	$a^{-n-i} u(n)$ (2')
(3) $a^n u(-n)$	$a^{-i} u(-n)$ (3')
(4) $a^{-n} u(-n)$	$a^{-n+i} u(-n)$ (4')
(5) $a^n u(-n-1)$	$a^{n+i} u(-n-1)$ (5')
(6) $a^{-n} u(-n-1)$	$a^{-n-i} u(-n-1)$ (6')
(7) $a^n u(n-1)$	$a^{-i} u(n-1)$ (7')
(8) $a^{-n} u(n-1)$	$a^{-n+i} u(n-1)$ (8')

## A. Flipping

Base Inverting  
Range Flipping

## B. Range Shifting = Range Flipping + Range Complementing

Range Flipping  
Range Complementing

## C. Complementary Inverting

Base Inverting  
Range Complementing

# Intra-Permutations over (1') ~ (8')

(1)  $a^n u(n)$

(2)  $a^{-n} u(n)$

(3)  $a^n u(-n)$

(4)  $a^{-n} u(-n)$

(5)  $a^n u(-n-1)$

(6)  $a^{-n} u(-n-1)$

(7)  $a^n u(n-1)$

(8)  $a^{-n} u(n-1)$

$a^{n+i} u(n)$  (1')

$a^{-n-i} u(n)$  (2')

$a^{n-i} u(-n)$  (3')

$a^{-n+i} u(-n)$  (4')

$a^{n+i} u(-n-1)$  (5')

$a^{-n-i} u(-n-1)$  (6')

$a^{n-i} u(n-1)$  (7')

$a^{-n+i} u(n-1)$  (8')

## D. Flipping2

Base Inverting

Shifted Range Flipping = Exponent Shifting2 + Range Flipping

## E. Shifting2 = Exponent Shifting2 + Range Shifting

Shifted Range Flipping = Exponent Shifting2 + Range Flipping

Range Complementing

## F. Complementary Inverting

Base Inverting

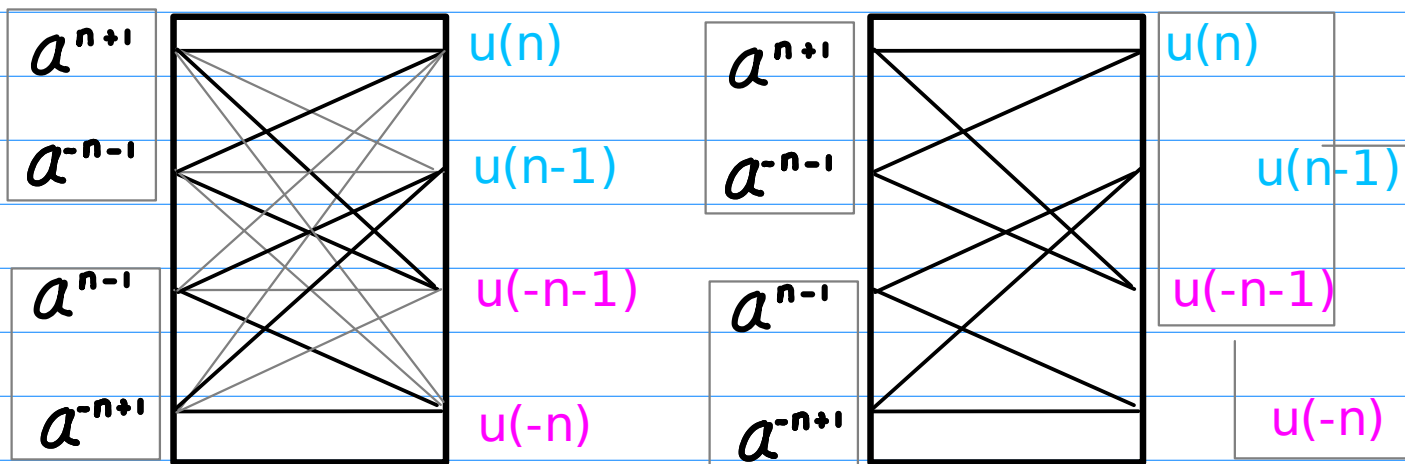
Range Complementing





$$a^n \times R(n)$$

$a^{n+1}$	$a^{-n-1}$	$\times$	$u(n)$	$u(-n-1)$
$a^{n-1}$	$a^{-n+1}$	$\times$	$u(n-1)$	$u(-n)$



$n \leftarrow n+2$ or $n \leftarrow n-2$
---

(1') $a^{n+1}u(n)$	$a^{-n-1}u(n)$	(2')
(3') $a^{n-1}u(-n)$	$a^{-n+1}u(-n)$	(4')
(5') $a^{n+1}u(-n-1)$	$a^{-n-1}u(-n-1)$	(6')
(7') $a^{n-1}u(n-1)$	$a^{-n+1}u(n-1)$	(8')

(1'') $a^{n-1}u(n)$	$a^{-n+1}u(n)$	(2'')
(3'') $a^{n+1}u(-n)$	$a^{-n-1}u(-n)$	(4'')
(5'') $a^{n-1}u(-n-1)$	$a^{-n+1}u(-n-1)$	(6'')
(7'') $a^{n+1}u(n-1)$	$a^{-n-1}u(n-1)$	(8'')



## permutation over (1) ~ (8)

### A. Flipping

Base Inverting  
Range Flipping

### B. Range Shifting = Range Flipping + Range Complementing

Range Flipping  
Range Complementing

### C. Complementary Inverting

Base Inverting  
Range Complementing

## permutation over (1') ~ (8')

### D. Flipping<sub>2</sub>

Base Inverting  
Shifted Range Flipping = Exponent Shifting<sub>2</sub> + Range Flipping

### E. Shifting<sub>2</sub> = Exponent Shifting<sub>2</sub> + Range Shifting

Shifted Range Flipping = Exponent Shifting<sub>2</sub> + Range Flipping  
Range Complementing

### F. Complementary Inverting

Base Inverting  
Range Complementing

**Range Shifting = Range Flipping + Range Complementing**

**Shifted Range Flipping = Exponent Shifting<sub>2</sub> + Range Flipping**

**Shifting<sub>2</sub> = Shifted Range Flipping + Range Complementing**

**= Exponent Shifting<sub>2</sub> + Range (Flipping+Complementing)**

**= Exponent Shifting<sub>2</sub> + Range Shifting**

## permutation over (1) ~ (8)

<b>A. Flipping</b>	(1) - (4)	(5) - (8)
<b>Base Inverting</b>	(2) - (3)	(6) - (7)
<b>Range Flipping</b>	(3) - (2)	(7) - (6)
	(4) - (1)	(8) - (5)
<b>B. Range Shifting</b>	(1) - (7)	(5) - (3)
<b>Range Flipping</b>	(2) - (8)	(6) - (4)
<b>Range Complementing</b>	(3) - (5)	(7) - (1)
	(4) - (6)	(8) - (2)
<b>C. Complementary Inverting</b>	(1) - (6)	(5) - (2)
<b>Base Inverting</b>	(6) - (1)	(6) - (1)
<b>Range Complementing</b>	(2) - (5)	(7) - (4)
	(5) - (2)	(8) - (3)

## permutation over (1') ~ (8')

<b>D. Flipping2</b>	(1') - (4')	(5') - (8')
<b>Base Inverting</b>	(2') - (3')	(6') - (7')
<b>Shifted Range Flipping</b>	(3') - (2')	(7') - (6')
	(4') - (1')	(8') - (5')
<b>E. Shifting2</b>	(1') - (7')	(5') - (3')
<b>Shifted Range Flipping</b>	(2') - (8')	(6') - (4')
<b>Range Complementing</b>	(3') - (5')	(7') - (1')
	(4') - (6')	(8') - (2')
<b>F. Complementary Inverting</b>	(1') - (6')	(5') - (2')
<b>Base Inverting</b>	(6') - (1')	(6') - (1')
<b>Range Complementing</b>	(2') - (5')	(7') - (4')
	(5') - (2')	(8') - (3')



## permutation over (1) ~ (8)

	A	B	C
Base Inverting	X		X
Range Flipping	X	X	
Range Complementing		X	X

## permutation over (1') ~ (8')

	D	E	F
Base Inverting	X		X
Shifted Range Flipping	X	X	
Range Complementing		X	X

**Range Shifting = Range Flipping + Range Complementing**  
**Shifted Range Flipping = Exponent Shifting2 + Range Flipping**

**Shifting2 = Shifted Range Flipping + Range Complementing**  
**= Exponent Shifting2 + Range (Flipping+Complementing)**  
**= Exponent Shifting2 + Range Shifting**





**Over (1) ~ (8)**

**Base Inverting**

$$a^n \longleftrightarrow a^{-n}$$

**Range Flipping**

$$R(n) \longleftrightarrow R(-n)$$

**Range Complementing**

$$R(n) \longleftrightarrow \overline{R(n)}$$

**Over (1') ~ (8')**

**Base Inverting**

$$a^n \longleftrightarrow a^{-n}$$

**Shifted Range Flipping**

$$a^n R(n) \longleftrightarrow a^{sh2(n)} R(-n)$$

**Range Complementing**

$$R(n) \longleftrightarrow \overline{R(n)}$$

**A.I Flipping**  
**Base Inverting**  
**Range Flipping**

$$a^n \longleftrightarrow a^{-n}$$

$$R(n) \longleftrightarrow R(-n)$$

$$a^n R(n) \longleftrightarrow a^{-n} R(-n)$$

**D.I Flipping2**  
**Base Inverting**  
**Shifted Range Flipping**

$$a^n \longleftrightarrow a^{-n}$$

$$a^n R(n) \longleftrightarrow a^{sh2(n)} R(-n)$$

$$a^n R(n) \longleftrightarrow a^{-sh2(n)} R(-n)$$

**B.I Range Shifting**  
**Range Flipping**  
**Range Complementing**

$$R(n) \longleftrightarrow R(-n)$$

$$R(n) \longleftrightarrow \overline{R(n)}$$

$$R(n) \longleftrightarrow \overline{R(-n)}$$

**E.I Shifting2**  
**Shifted Range Flipping**  
**Range Complementing**

$$a^n R(n) \longleftrightarrow a^{sh2(n)} R(-n)$$

$$R(n) \longleftrightarrow \overline{R(n)}$$

$$a^n R(n) \longleftrightarrow a^{sh2(n)} \overline{R(-n)}$$

**C.I Complementary Inverting**  
**Base Inverting**  
**Range Complementing**

$$a^n \longleftrightarrow a^{-n}$$

$$R(n) \longleftrightarrow \overline{R(n)}$$

$$a^n R(n) \longleftrightarrow a^{-n} \overline{R(n)}$$

**F.I Complementary Inverting**  
**Base Inverting**  
**Range Complementing**

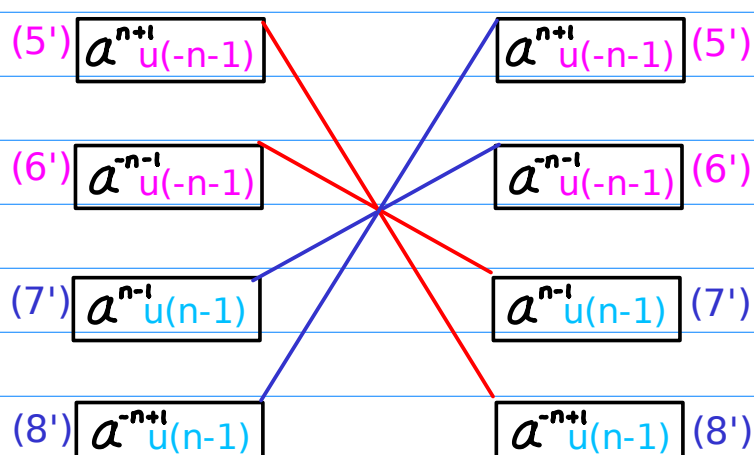
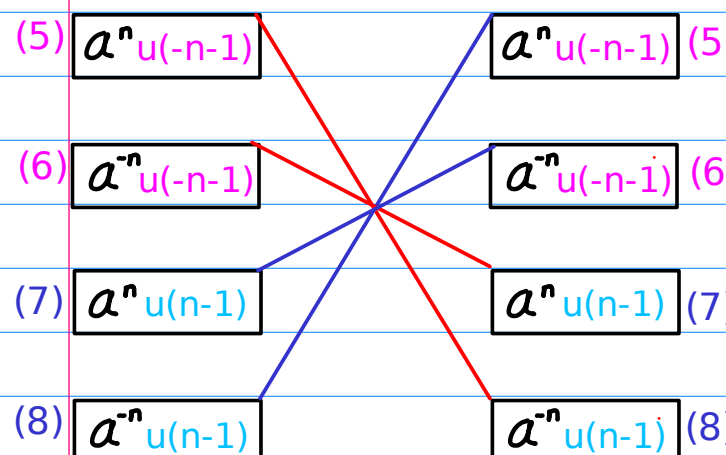
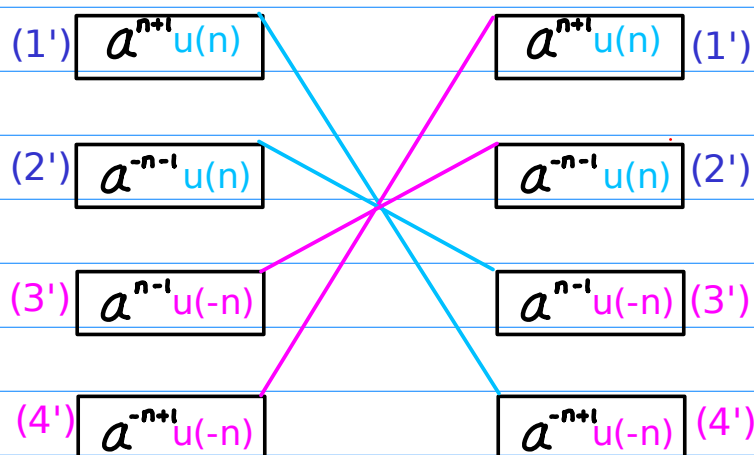
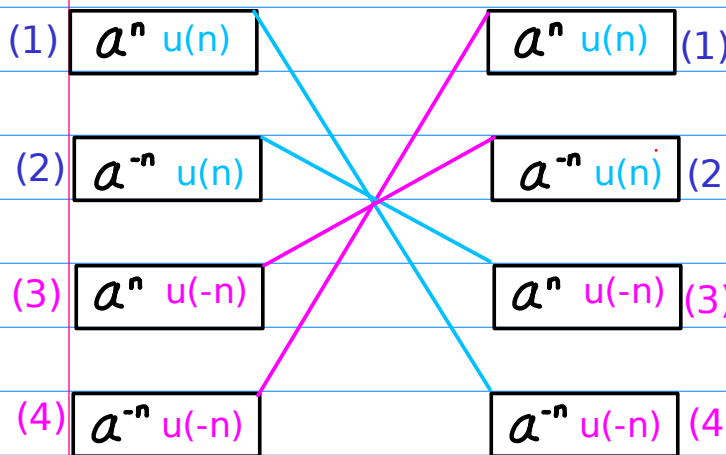
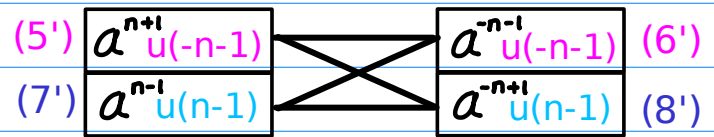
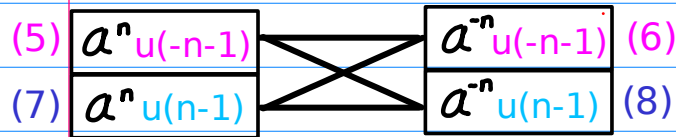
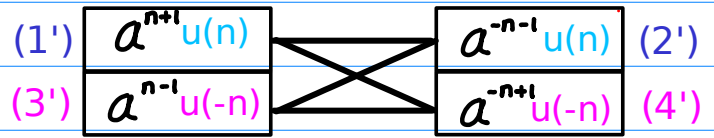
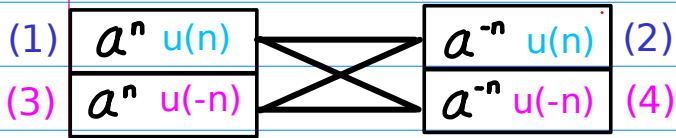
$$a^n \longleftrightarrow a^{-n}$$

$$R(n) \longleftrightarrow \overline{R(n)}$$

$$a^n R(n) \longleftrightarrow a^{-n} \overline{R(n)}$$

**A.I Flipping**  
**Base Inverting**  
**Range Flipping**

**D.I Flipping2**  
**Base Inverting**  
**Shifted Range Flipping**

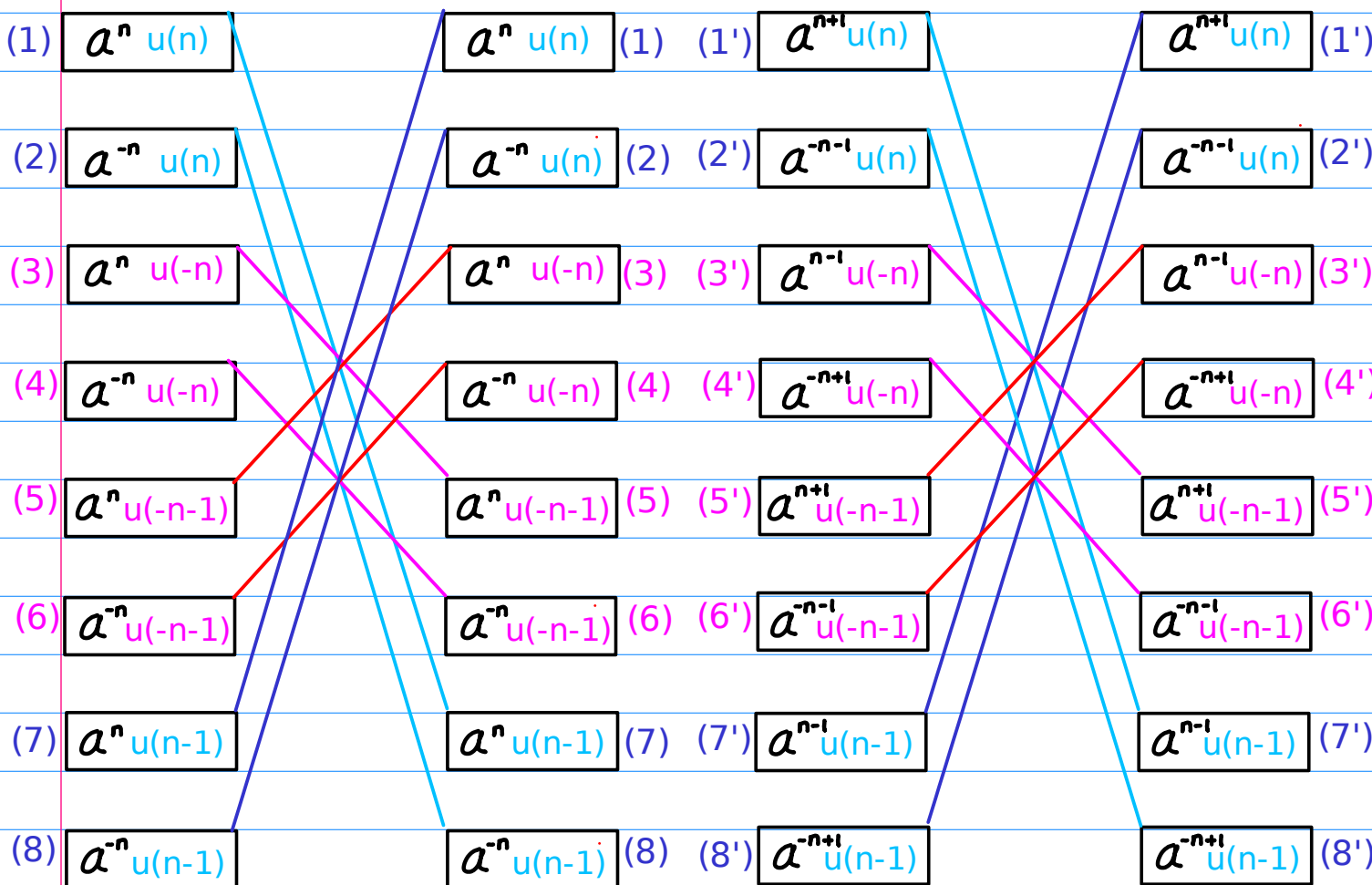
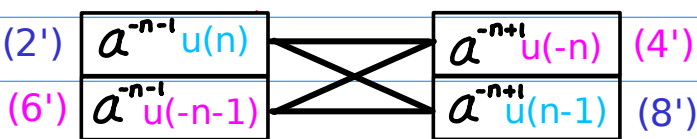
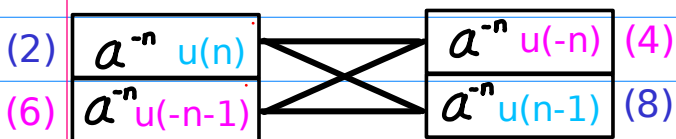
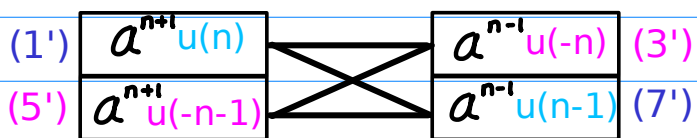
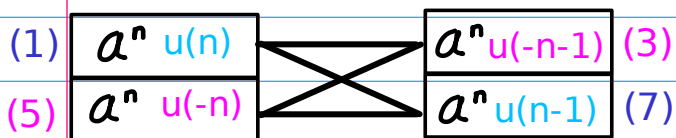


- (1) - (4)
- (2) - (3)
- (3) - (2)
- (4) - (1)
- (5) - (8)
- (6) - (7)
- (7) - (6)
- (8) - (5)

- (1') - (4')
- (2') - (3')
- (3') - (2')
- (4') - (1')
- (5') - (8')
- (6') - (7')
- (7') - (6')
- (8') - (5')

**B.I Range Shifting  
Range Flipping  
Range Complementing**

**E.I Shifting2  
Shifted Range Flipping  
Range Complementing**

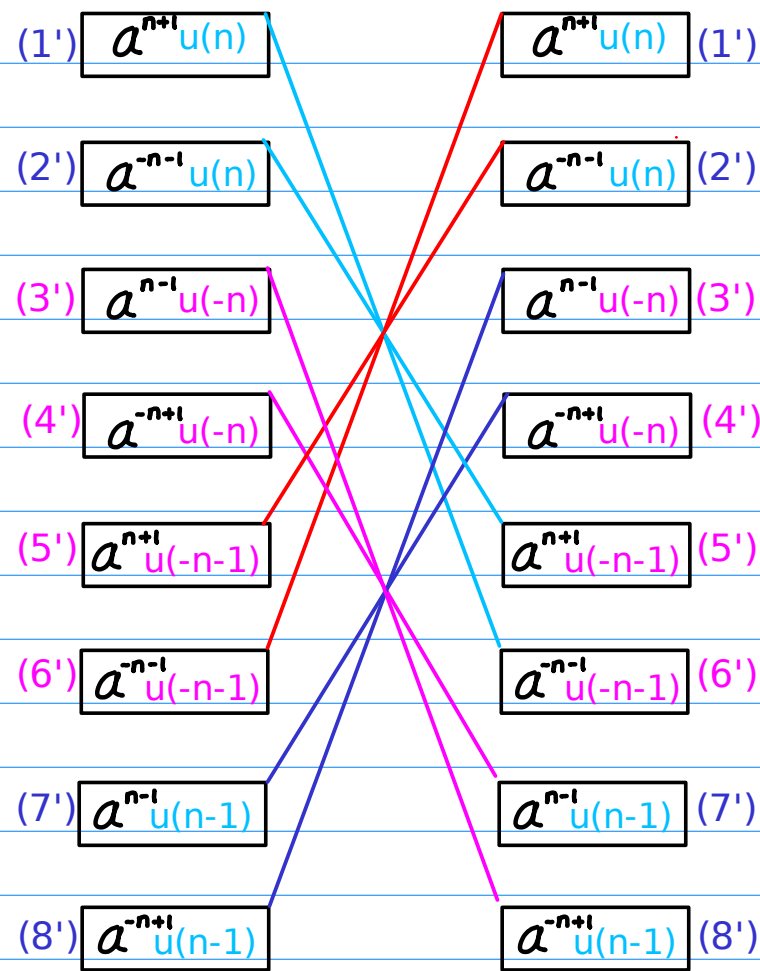
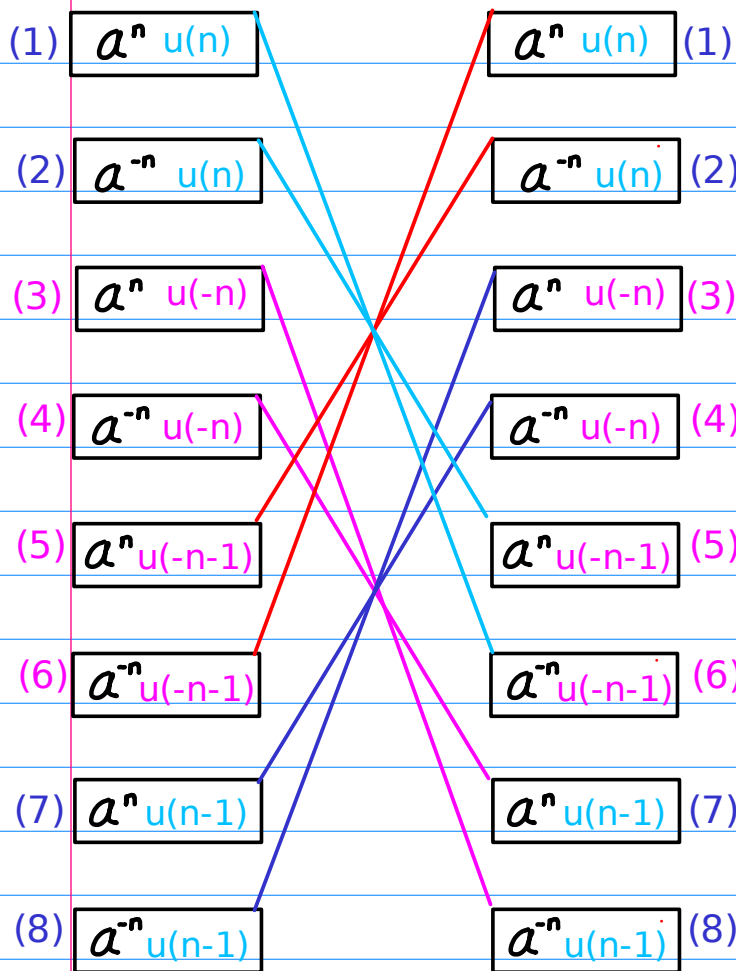
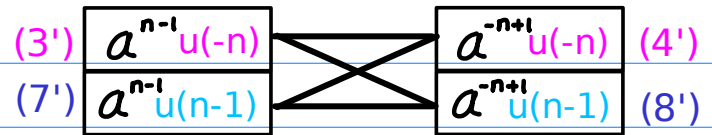
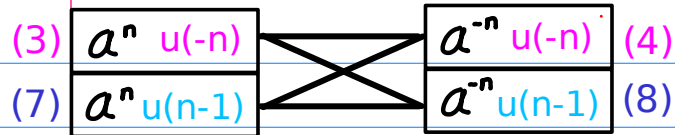
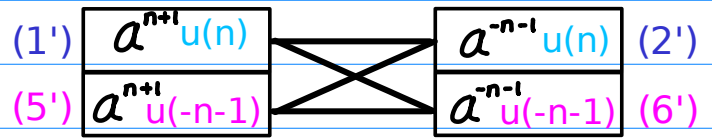
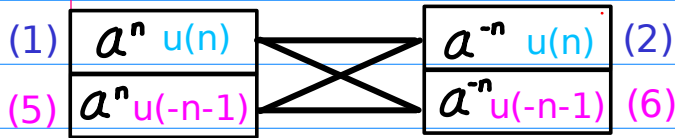


(1) - (7)      (5) - (3)  
 (2) - (8)      (6) - (4)  
 (3) - (5)      (7) - (1)  
 (4) - (6)      (8) - (2)

(1') - (7')      (5') - (3')  
 (2') - (8')      (6') - (4')  
 (3') - (5')      (7') - (1')  
 (4') - (6')      (8') - (2')

### C.I Complementary Inverting Base Inverting Range Complementing

### F.I Complementary Inverting Base Inverting Range Complementing



- (1) - (6)
- (6) - (1)
- (2) - (5)
- (5) - (2)
- (5) - (2)
- (8) - (3)

- (1') - (6')
- (6') - (1')
- (2') - (5')
- (5') - (2')
- (5') - (2')
- (8') - (3')

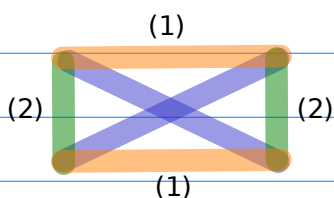


## A.I Flipping

- (1) Base Inverting
- (2) Range Flipping

$$\begin{array}{|c|} \hline (1) \ a^n u(n) \\ \hline (3) \ a^n u(-n) \\ \hline \end{array} \begin{array}{c} \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array} \begin{array}{|c|} \hline a^{-n} u(n) \quad (2) \\ \hline a^{-n} u(-n) \quad (4) \\ \hline \end{array}$$

$$\begin{array}{|c|} \hline (5) \ a^n u(-n-1) \\ \hline (7) \ a^n u(n-1) \\ \hline \end{array} \begin{array}{c} \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array} \begin{array}{|c|} \hline a^{-n} u(-n-1) \quad (6) \\ \hline a^{-n} u(n-1) \quad (8) \\ \hline \end{array}$$

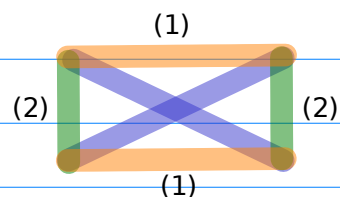


## D.I Flipping2

- (1) Base Inverting
- (2) Shifted Range Flipping

$$\begin{array}{|c|} \hline (1') \ a^{n+1} u(n) \\ \hline (3') \ a^{n-1} u(-n) \\ \hline \end{array} \begin{array}{c} \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array} \begin{array}{|c|} \hline a^{-n-1} u(n) \quad (2') \\ \hline a^{-n+1} u(-n) \quad (4') \\ \hline \end{array}$$

$$\begin{array}{|c|} \hline (5') \ a^{n+1} u(-n-1) \\ \hline (7') \ a^{n-1} u(n-1) \\ \hline \end{array} \begin{array}{c} \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array} \begin{array}{|c|} \hline a^{-n-1} u(-n-1) \quad (6') \\ \hline a^{-n+1} u(n-1) \quad (8') \\ \hline \end{array}$$



$$\begin{array}{ccc} a^n & \xleftrightarrow{(1)} & a^{-n} \\ R(n) & \xleftrightarrow{(2)} & R(-n) \\ a^n R(n) & \longleftrightarrow & a^{-n} R(-n) \end{array}$$

$$\begin{array}{ccc} a^n & \xleftrightarrow{(1)} & a^{-n} \\ a^n R(n) & \xleftrightarrow{(2)} & a^{sh2(n)} R(-n) \\ a^n R(n) & \longleftrightarrow & a^{-sh2(n)} R(-n) \end{array}$$

$b^n$	$b^{-n}$
$a^n$	$a^{-n}$
$a^{-n}$	$a^n$

$b^n$	$b^{-sh2(n)}$
$a^{(n+1)}$	$a^{-(n-1)}$
$a^{-(n+1)}$	$a^{(n-1)}$
$a^{(n-1)}$	$a^{-(n+1)}$
$a^{-(n-1)}$	$a^{(n+1)}$

$R(n)$	$R(-n)$
$u(n)$	$u(-n)$
$u(n-1)$	$u(-n-1)$
$u(-n)$	$u(n)$
$u(-n-1)$	$u(n-1)$

$R(n)$	$R(-n)$
$u(n)$	$u(-n)$
$u(n-1)$	$u(-n-1)$
$u(-n)$	$u(n)$
$u(-n-1)$	$u(n-1)$

## B.I Range Shifting

(1) Range Complementing  
(2) Range Flipping

## E.I Shifting2

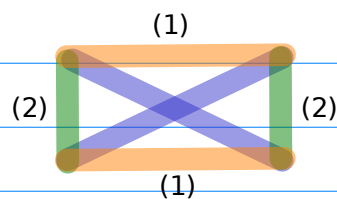
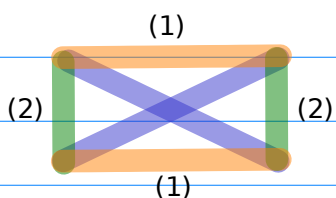
(1) Shifted Range Flipping  
(2) Range Complementing

$$\begin{array}{l} (1) \\ (5) \end{array} \begin{array}{|c|} \hline a^n u(n) \\ \hline a^n u(-n) \\ \hline \end{array} \begin{array}{|c|} \hline \diagdown \\ \diagup \\ \hline \end{array} \begin{array}{|c|} \hline a^n u(-n-1) \\ \hline a^n u(n-1) \\ \hline \end{array} \begin{array}{l} (3) \\ (7) \end{array}$$

$$\begin{array}{l} (1') \\ (5') \end{array} \begin{array}{|c|} \hline a^{n+1} u(n) \\ \hline a^{n+1} u(-n-1) \\ \hline \end{array} \begin{array}{|c|} \hline \diagdown \\ \diagup \\ \hline \end{array} \begin{array}{|c|} \hline a^{n+1} u(-n) \\ \hline a^{n+1} u(n-1) \\ \hline \end{array} \begin{array}{l} (3') \\ (7') \end{array}$$

$$\begin{array}{l} (2) \\ (6) \end{array} \begin{array}{|c|} \hline a^{-n} u(n) \\ \hline a^{-n} u(-n-1) \\ \hline \end{array} \begin{array}{|c|} \hline \diagdown \\ \diagup \\ \hline \end{array} \begin{array}{|c|} \hline a^{-n} u(-n) \\ \hline a^{-n} u(n-1) \\ \hline \end{array} \begin{array}{l} (4) \\ (8) \end{array}$$

$$\begin{array}{l} (2') \\ (6') \end{array} \begin{array}{|c|} \hline a^{-n-1} u(n) \\ \hline a^{-n-1} u(-n-1) \\ \hline \end{array} \begin{array}{|c|} \hline \diagdown \\ \diagup \\ \hline \end{array} \begin{array}{|c|} \hline a^{-n-1} u(-n) \\ \hline a^{-n-1} u(n-1) \\ \hline \end{array} \begin{array}{l} (4') \\ (8') \end{array}$$



$$\begin{array}{l} R(n) \xleftrightarrow{(1)} \overline{R(-n)} \\ R(n) \xleftrightarrow{(2)} \overline{R(n)} \\ R(n) \xleftrightarrow{\quad} \overline{R(-n)} \end{array}$$

$$\begin{array}{l} a^n R(n) \xleftrightarrow{(1)} \overline{a^{sh2(n)} R(-n)} \\ R(n) \xleftrightarrow{(2)} \overline{R(n)} \\ a^n R(n) \xleftrightarrow{\quad} \overline{a^{sh2(n)} R(-n)} \end{array}$$

$b^n$	$b^{sh2(n)}$
$a^{(n+1)}$	$a^{(n-1)}$
$a^{-(n+1)}$	$a^{-(n-1)}$
$a^{(n-1)}$	$a^{(n+1)}$
$a^{-(n-1)}$	$a^{-(n+1)}$

$R(n)$	$\overline{R(-n)}$
$u(n)$	$u(n-1)$
$u(n-1)$	$u(n)$
$u(-n)$	$u(-n-1)$
$u(-n-1)$	$u(-n)$

$R(n)$	$\overline{R(-n)}$
$u(n)$	$u(n-1)$
$u(n-1)$	$u(n)$
$u(-n)$	$u(-n-1)$
$u(-n-1)$	$u(-n)$



## C.I Complementary Inverting

(1) Base Inverting  
(2) Range Complementing

## F.I Complementary Inverting

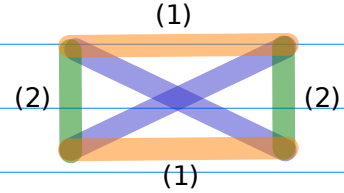
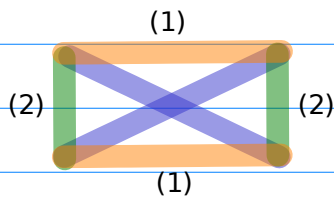
(1) Base Inverting  
(2) Range Complementing

$$\begin{array}{|c|} \hline (1) \quad a^n u(n) \\ \hline (5) \quad a^n u(-n-1) \\ \hline \end{array} \begin{array}{c} \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array} \begin{array}{|c|} \hline a^{-n} u(n) \quad (2) \\ \hline a^{-n} u(-n-1) \quad (6) \\ \hline \end{array}$$

$$\begin{array}{|c|} \hline (1') \quad a^{n+1} u(n) \\ \hline (5') \quad a^{n+1} u(-n-1) \\ \hline \end{array} \begin{array}{c} \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array} \begin{array}{|c|} \hline a^{-n-1} u(n) \quad (2') \\ \hline a^{-n-1} u(-n-1) \quad (6') \\ \hline \end{array}$$

$$\begin{array}{|c|} \hline (3) \quad a^n u(-n) \\ \hline (7) \quad a^n u(n-1) \\ \hline \end{array} \begin{array}{c} \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array} \begin{array}{|c|} \hline a^{-n} u(-n) \quad (4) \\ \hline a^{-n} u(n-1) \quad (8) \\ \hline \end{array}$$

$$\begin{array}{|c|} \hline (3') \quad a^{n-1} u(-n) \\ \hline (7') \quad a^{n-1} u(n-1) \\ \hline \end{array} \begin{array}{c} \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array} \begin{array}{|c|} \hline a^{-n+1} u(-n) \quad (4') \\ \hline a^{-n+1} u(n-1) \quad (8') \\ \hline \end{array}$$



$$\begin{array}{ccc} a^n & \xleftrightarrow{(1)} & a^{-n} \\ R(n) & \xleftrightarrow{(2)} & \overline{R(n)} \\ a^n R(n) & \longleftrightarrow & a^{-n} \overline{R(n)} \end{array}$$

$$\begin{array}{ccc} a^n & \xleftrightarrow{(1)} & a^{-n} \\ R(n) & \xleftrightarrow{(2)} & \overline{R(n)} \\ a^n R(n) & \longleftrightarrow & a^{-n} \overline{R(n)} \end{array}$$

$b^n$	$b^{-n}$
$a^n$	$a^{-n}$
$a^{-n}$	$a^n$

$b^n$	$b^{-n}$
$a^{(n+1)}$	$a^{-(n+1)}$
$a^{-(n+1)}$	$a^{(n+1)}$
$a^{(n-1)}$	$a^{-(n-1)}$
$a^{-(n-1)}$	$a^{(n-1)}$

$R(n)$	$\overline{R(n)}$
$u(n)$	$u(-n-1)$
$u(n-1)$	$u(-n)$
$u(-n)$	$u(n)$
$u(-n-1)$	$u(n-1)$

$R(n)$	$\overline{R(n)}$
$u(n)$	$u(-n-1)$
$u(n-1)$	$u(-n)$
$u(-n)$	$u(n)$
$u(-n-1)$	$u(n-1)$

## A.I Flipping

- (1) Base Inverting
- (2) Range Flipping

$$\begin{array}{ccc}
 a^n & \xleftrightarrow{(1)} & a^{-n} \\
 R(n) & \xleftrightarrow{(2)} & R(-n) \\
 a^n R(n) & \longleftrightarrow & a^{-n} R(-n)
 \end{array}$$

## D.I Flipping2

- (1) Base Inverting
- (2) Shifted Range Flipping

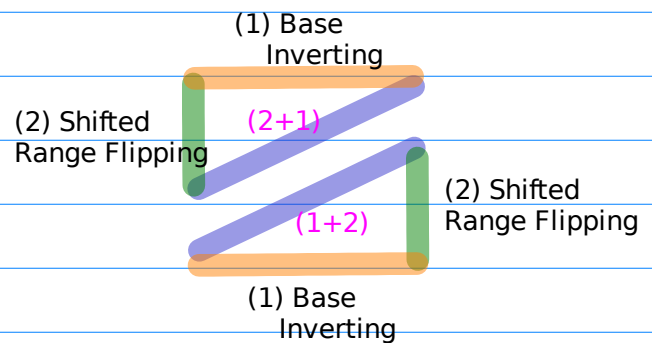
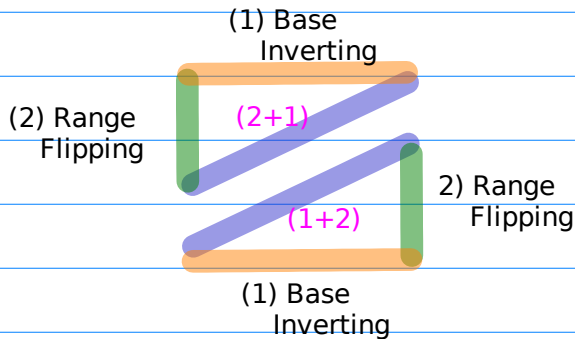
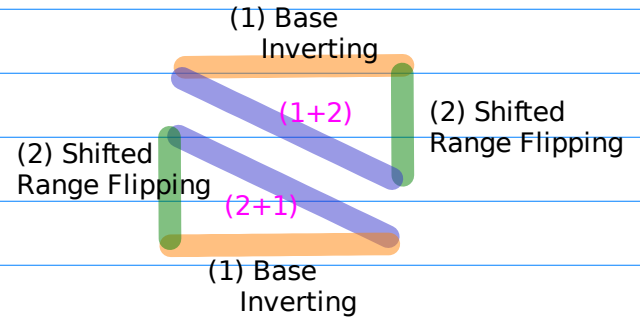
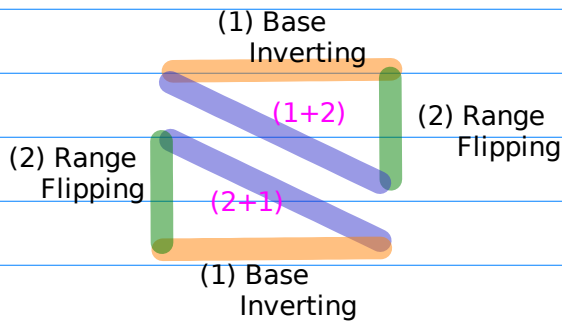
$$\begin{array}{ccc}
 a^n & \xleftrightarrow{(1)} & a^{-n} \\
 a^n R(n) & \xleftrightarrow{(2)} & a^{sh2(n)} R(-n) \\
 a^n R(n) & \longleftrightarrow & a^{-sh2(n)} R(-n)
 \end{array}$$

$$\begin{array}{ccc}
 a^n R(n) & \xrightarrow{(1)} & a^{-n} R(n) \\
 & \xrightarrow{(2)} & a^{-n} R(-n)
 \end{array}$$

$$\begin{array}{ccc}
 a^n R(n) & \xrightarrow{(1)} & a^{-n} R(n) \\
 & \xrightarrow{(2)} & a^{-sh2(n)} R(-n)
 \end{array}$$

$$\begin{array}{ccc}
 a^n R(n) & \xrightarrow{(2)} & a^n R(-n) \\
 & \xrightarrow{(1)} & a^{-n} R(-n)
 \end{array}$$

$$\begin{array}{ccc}
 a^n R(n) & \xrightarrow{(2)} & a^{sh2(n)} R(-n) \\
 & \xrightarrow{(1)} & a^{-sh2(n)} R(-n)
 \end{array}$$



**B.I Range Shifting**  
**(1) Range Complementing**  
**(2) Range Flipping**

**E.I Shifting2**  
**(1) Shifted Range Flipping**  
**(2) Range Complementing**

$$R(n) \xleftrightarrow{(1)} R(-n)$$

$$R(n) \xleftrightarrow{(2)} \overline{R(n)}$$

$$R(n) \longleftrightarrow \overline{R(-n)}$$

$$a^n R(n) \xleftrightarrow{(1)} a^{sh2(n)} R(-n)$$

$$R(n) \xleftrightarrow{(2)} \overline{R(n)}$$

$$a^n R(n) \longleftrightarrow a^{sh2(n)} \overline{R(-n)}$$

$$a^n R(n) \xrightarrow{(1)} a^{-n} R(n)$$

$$\xrightarrow{(2)} a^{-n} \overline{R(n)}$$

$$a^n R(n) \xrightarrow{(1)} a^{-n} R(n)$$

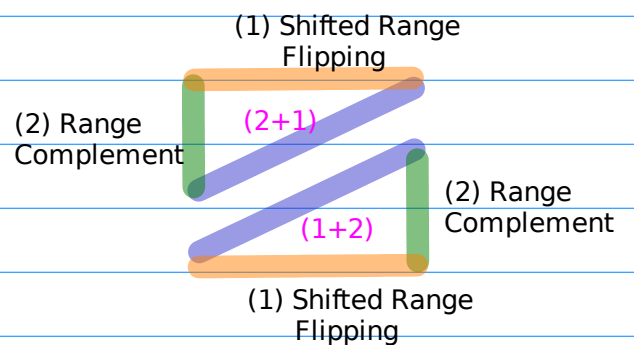
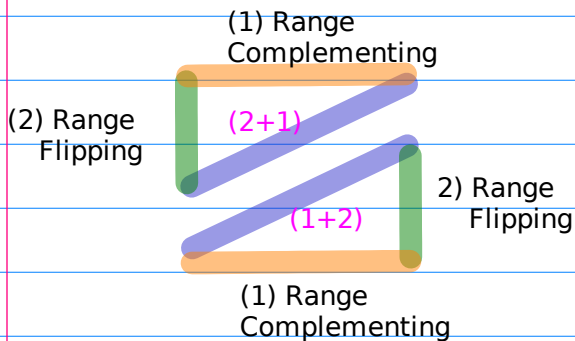
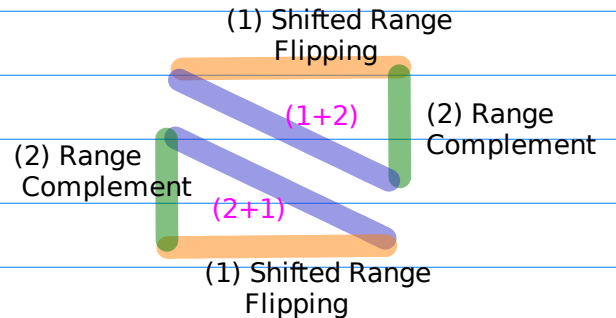
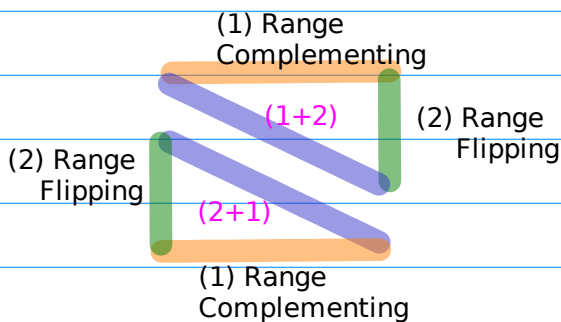
$$\xrightarrow{(2)} a^{-sh2(n)} R(-n)$$

$$a^n rng(n) \xrightarrow{(2)} a^n \overline{R(n)}$$

$$\xrightarrow{(1)} a^{-n} \overline{R(n)}$$

$$a^n R(n) \xrightarrow{(2)} a^{sh2(n)} R(-n)$$

$$\xrightarrow{(1)} a^{-sh2(n)} R(-n)$$



## C.I Complementary Inverting

(1) Base Inverting  
(2) Range Complementing

## F.I Complementary Inverting

(1) Base Inverting  
(2) Range Complementing

$$\begin{array}{ccc}
 a^n & \xleftrightarrow{(1)} & a^{-n} \\
 R(n) & \xleftrightarrow{(2)} & \overline{R(n)} \\
 a^n R(n) & \longleftrightarrow & a^{-n} \overline{R(n)}
 \end{array}$$

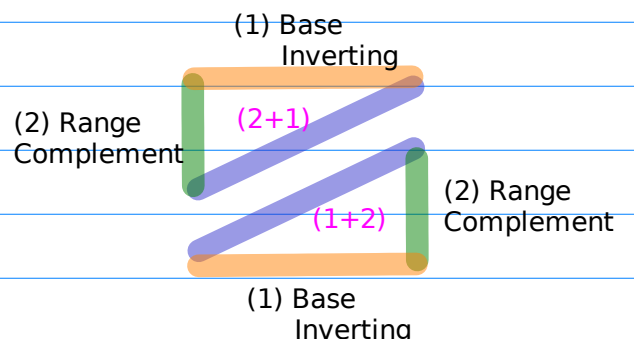
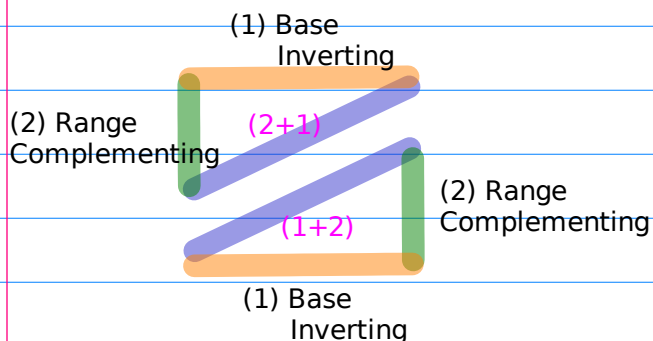
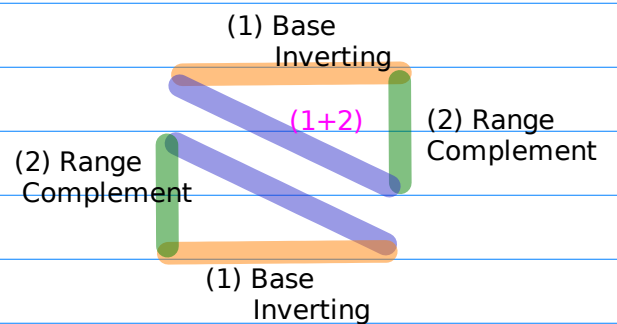
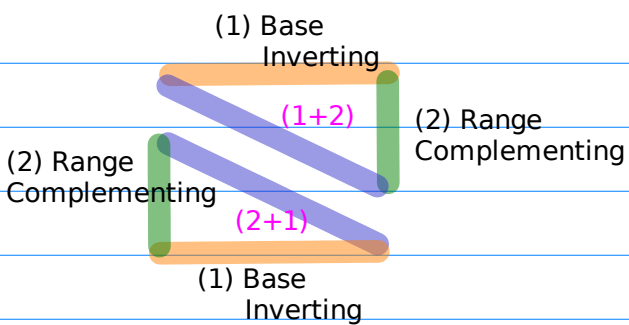
$$\begin{array}{ccc}
 a^n & \xleftrightarrow{(1)} & a^{-n} \\
 R(n) & \xleftrightarrow{(2)} & \overline{R(n)} \\
 a^n R(n) & \longleftrightarrow & a^{-n} \overline{R(n)}
 \end{array}$$

$$\begin{array}{ccc}
 a^n R(n) & \xrightarrow{(1)} & a^{-n} R(n) \\
 & \xrightarrow{(2)} & a^{-n} \overline{R(n)}
 \end{array}$$

$$\begin{array}{ccc}
 a^n R(n) & \xrightarrow{(1)} & a^{-n} R(n) \\
 & \xrightarrow{(2)} & a^{-n} \overline{R(n)}
 \end{array}$$

$$\begin{array}{ccc}
 a^n R(n) & \xrightarrow{(2)} & a^n \overline{R(n)} \\
 & \xrightarrow{(1)} & a^{-n} \overline{R(n)}
 \end{array}$$

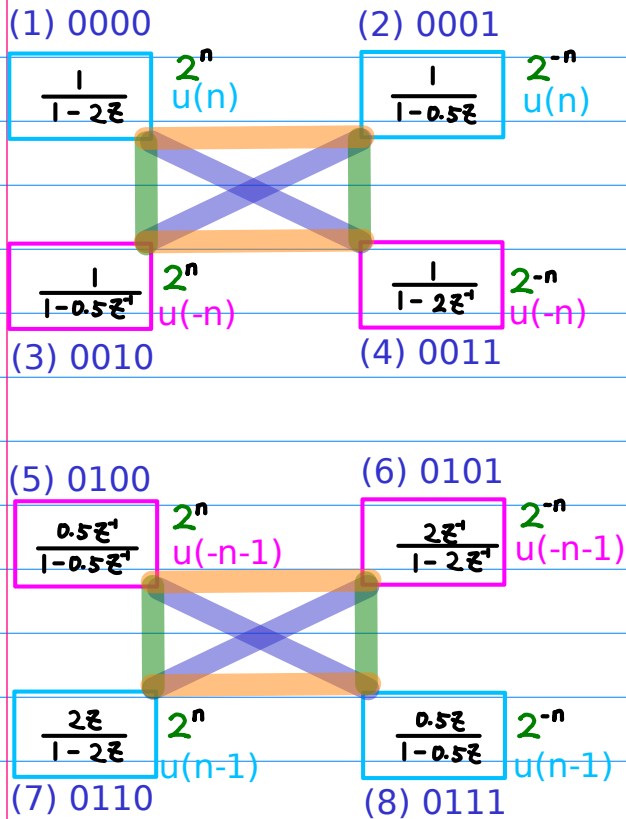
$$\begin{array}{ccc}
 a^n R(n) & \xrightarrow{(2)} & a^n \overline{R(n)} \\
 & \xrightarrow{(1)} & a^{-n} \overline{R(n)}
 \end{array}$$



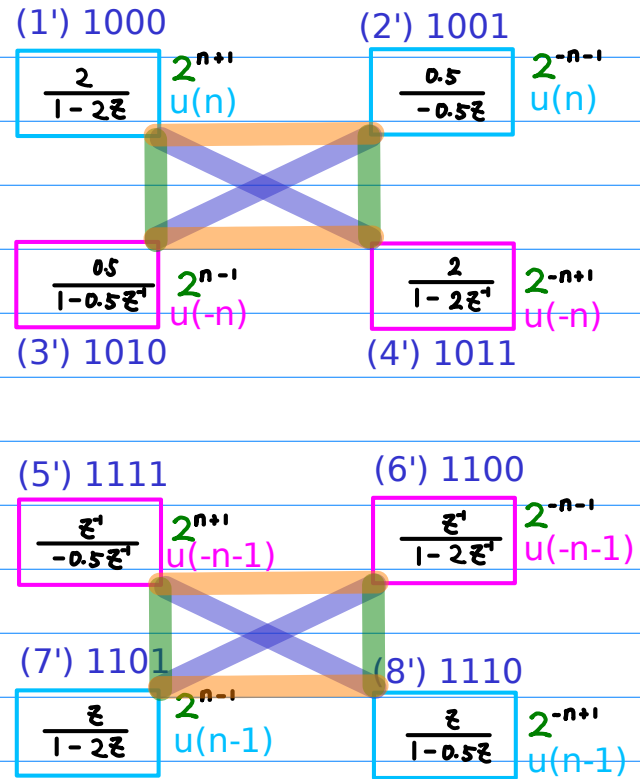




## A.I Flipping Base Inverting Range Flipping



## D.I Flipping2 Base Inverting Shifted Range Flipping



Shifted Range Flipping  
= Exponent Shifting2  
+ Range Flipping

$$a^n \longleftrightarrow a^{-n}$$

$$R(n) \longleftrightarrow R(-n)$$

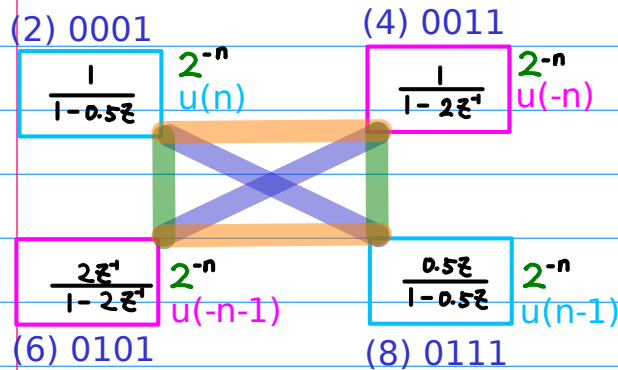
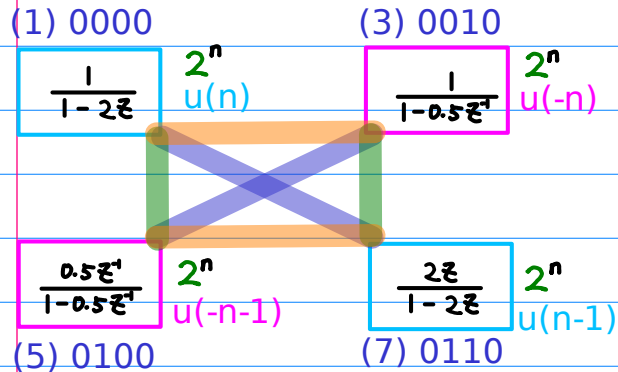
$$a^n \longleftrightarrow a^{-n}$$

$$a^n R(n) \longleftrightarrow a^{sh2(n)} R(-n)$$

$$a^n R(n) \longleftrightarrow a^{-n} R(-n)$$

$$a^n R(n) \longleftrightarrow a^{-sh2(n)} R(-n)$$

## B.I Range Shifting Range Flipping Range Complementing



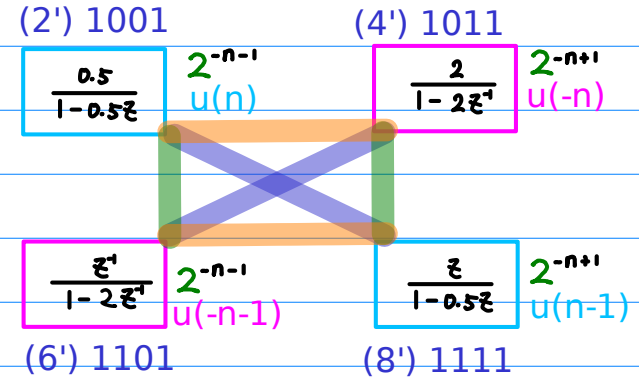
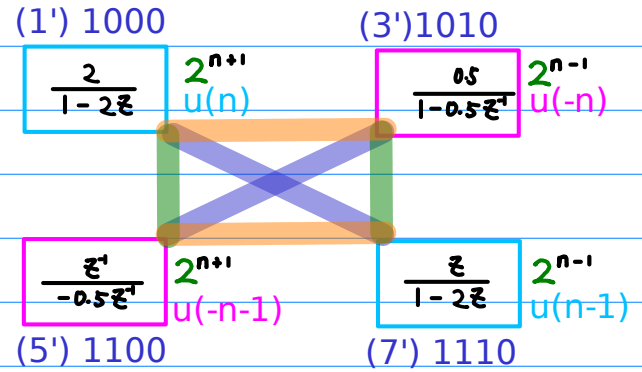
Range Shifting  
= Range Flipping  
+ Range Complementing

$$R(n) \longleftrightarrow R(-n)$$

$$R(n) \longleftrightarrow \overline{R(n)}$$

$$R(n) \longleftrightarrow \overline{R(-n)}$$

## E.I Shifting2 Shifted Range Flipping Range Complementing



Shifted Range Flipping  
= Exponent Shifting2  
+ Range Flipping

$$a^n R(n) \longleftrightarrow a^{sh 2(n)} R(-n)$$

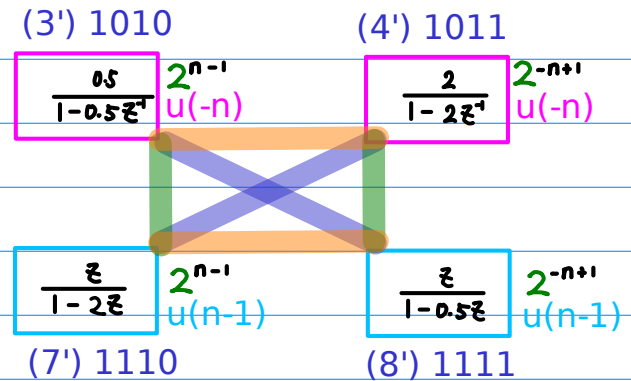
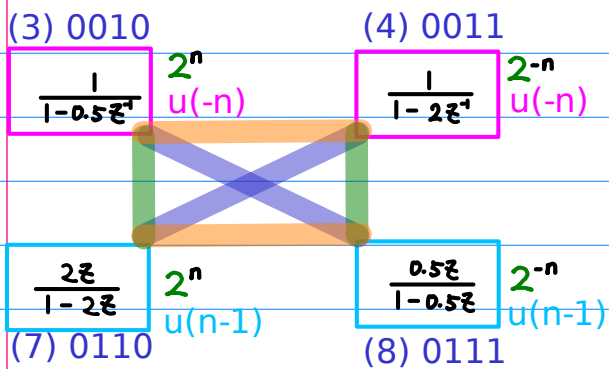
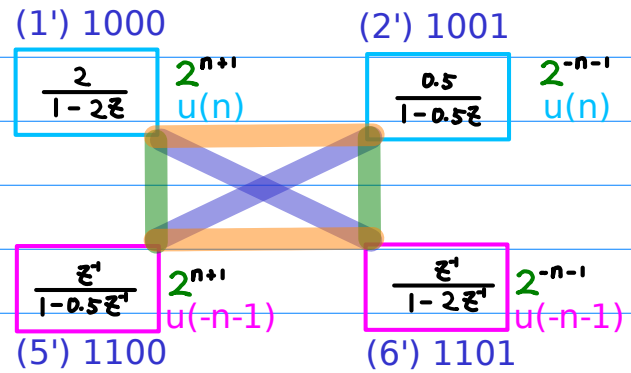
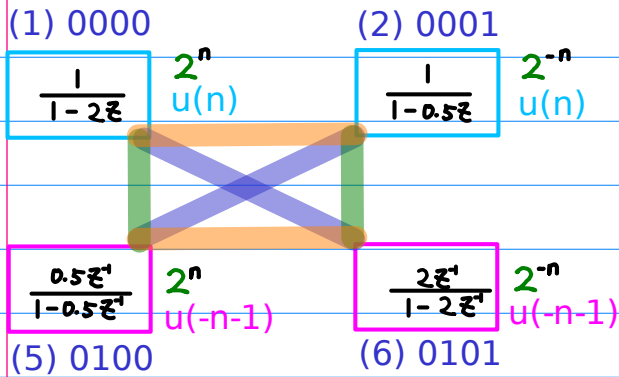
$$R(n) \longleftrightarrow \overline{R(n)}$$

$$a^n R(n) \longleftrightarrow a^{sh 2(n)} \overline{R(-n)}$$



### C.I Complementary Inverting Base Inverting Range Complementing

### F.I Complementary Inverting Base Inverting Range Complementing



$$a^n \longleftrightarrow a^{-n}$$

$$R(n) \longleftrightarrow \overline{R(n)}$$

$$a^n \longleftrightarrow a^{-n}$$

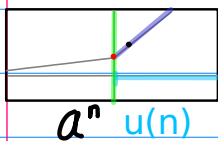
$$R(n) \longleftrightarrow \overline{R(n)}$$

$$a^n R(n) \longleftrightarrow a^{-n} \overline{R(n)}$$

$$a^n R(n) \longleftrightarrow a^{-n} \overline{R(n)}$$

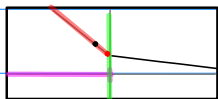
## A.II Flipping Base Inverting Range Flipping

(1) 0000



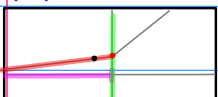
$$a^n u(n)$$

(4) 0011



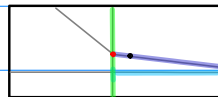
$$a^{-n} u(-n)$$

(3) 0010



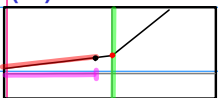
$$a^n u(-n)$$

(2) 0001



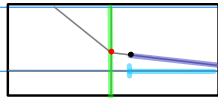
$$a^{-n} u(n)$$

(5) 0100



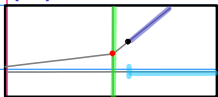
$$a^n u(-n-1)$$

(8) 0111



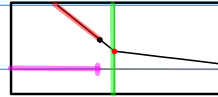
$$a^{-n} u(n-1)$$

(7) 0110



$$a^n u(n-1)$$

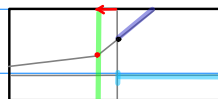
(6) 0101



$$a^{-n} u(-n-1)$$

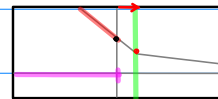
## D.II Flipping2 Base Inverting Shifted Range Flipping

(1') 1000



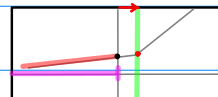
$$a^{n+1} u(n)$$

(4') 1011



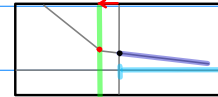
$$a^{-n+1} u(-n)$$

(3') 1010



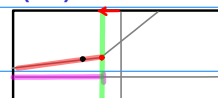
$$a^{n-1} u(-n)$$

(2') 1001



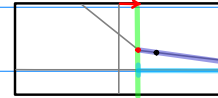
$$a^{-n-1} u(n)$$

(5') 1100



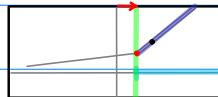
$$a^{n+1} u(-n-1)$$

(8') 1111



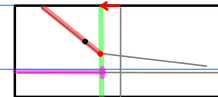
$$a^{-n+1} u(n-1)$$

(7') 1110



$$a^{n-1} u(n-1)$$

(6') 1101



$$a^{-n-1} u(-n-1)$$

Shifted Range Flipping  
= Exponent Shifting2  
+ Range Flipping

$$a^n R(n) \longleftrightarrow a^{-n} R(-n)$$

$$a^n R(n) \longleftrightarrow a^{-sh2(n)} R(-n)$$

$$a^n \longleftrightarrow a^{-n}$$

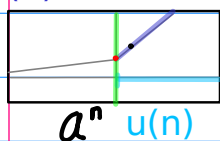
$$a^n \longleftrightarrow a^{-n}$$

$$R(n) \longleftrightarrow R(-n)$$

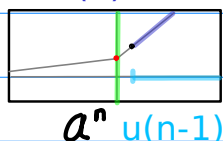
$$a^n R(n) \longleftrightarrow a^{sh2(n)} R(-n)$$

## B.II Range Shifting Range Flipping Range Complementing

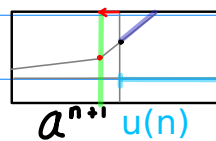
(1) 0000



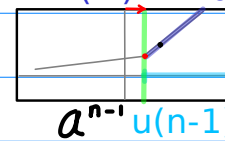
(7) 0110



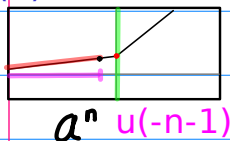
(1') 1000



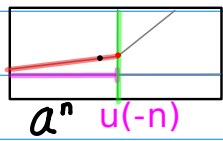
(7') 1110



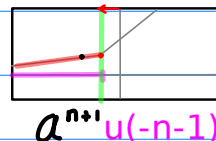
(5) 0100



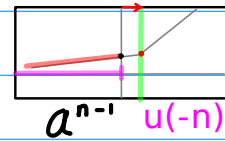
(3) 0010



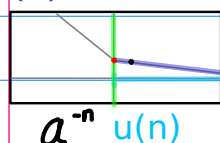
(5') 1100



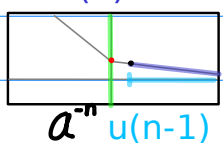
(3') 1010



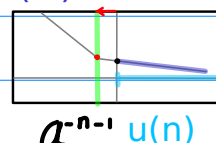
(2) 0001



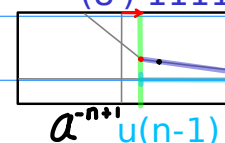
(8) 0111



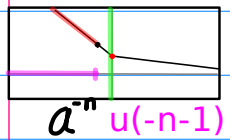
(2') 1001



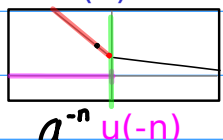
(8') 1111



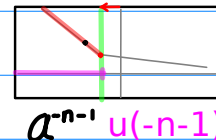
(6) 0101



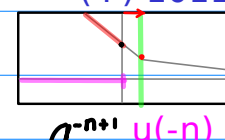
(4) 0011



(6') 1101



(4') 1011



Range Shifting  
= Range Flipping  
+ Range Complementing

Shifted Range Flipping  
= Exponent Shifting2  
+ Range Flipping

$$R(n) \longleftrightarrow \overline{R(-n)}$$

$$a^n R(n) \longleftrightarrow a^{sh2(n)} \overline{R(-n)}$$

$$R(n) \longleftrightarrow \overline{R(-n)}$$

$$a^n R(n) \longleftrightarrow a^{sh2(n)} \overline{R(-n)}$$

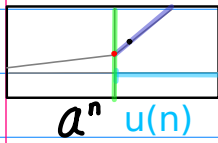
$$R(n) \longleftrightarrow \overline{R(n)}$$

$$R(n) \longleftrightarrow \overline{R(n)}$$

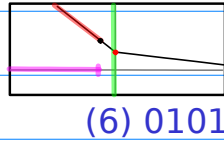
## C.II Complementary Inverting Base Inverting Range Complementing

## F.II Complementary Inverting Base Inverting Range Complementing

(1) 0000

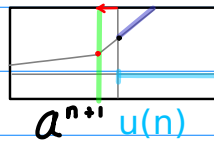


$a^{-n} u(-n-1)$

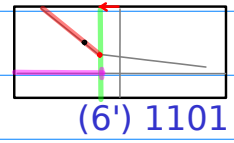


(6) 0101

(1') 1000

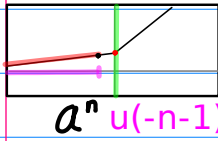


$a^{-n-1} u(-n-1)$

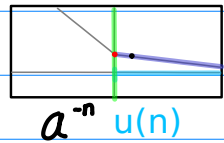


(6') 1101

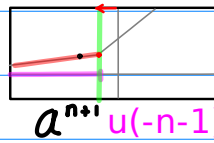
(5) 0100



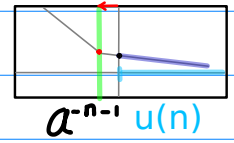
(2) 0001



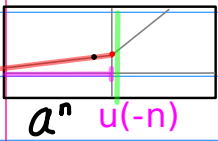
(5') 1100



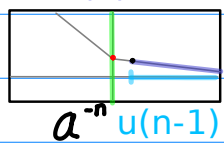
(2') 1001



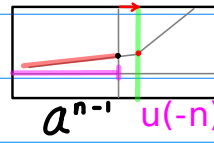
(3) 0010



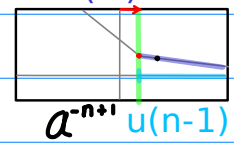
(8) 0111



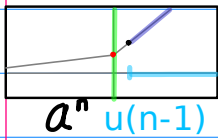
(3') 1010



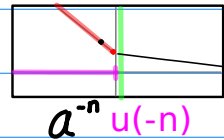
(8') 1111



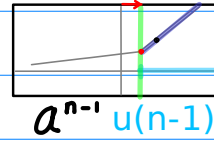
(7) 0110



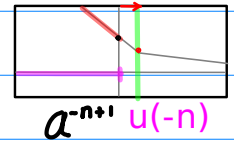
(4) 0011



(7') 1110



(4') 1011



$$a^n R(n) \longleftrightarrow a^{-n} \overline{R(n)}$$

$$a^n R(n) \longleftrightarrow a^{-n} \overline{R(n)}$$

$$a^n \longleftrightarrow a^{-n}$$

$$R(n) \longleftrightarrow \overline{R(n)}$$

$$a^n \longleftrightarrow a^{-n}$$

$$R(n) \longleftrightarrow \overline{R(n)}$$

