Systems of Linear Equations

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Based on
A First Course in Linear Algebra, R. A. Beezer
http://linear.ups.edu/fcla/front-matter.html

Outline

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System of a Linear Equations

A System of Linear Equations

is a collection of m equations in the variable quantities $x_1, x_2, x_3, \ldots, x_n$ of the form,

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= b_3 \\
\vdots \qquad\qquad &\quad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= b_m
\end{aligned}$$

where the values of a_{ij}, x_j , and b_i, $(1 \leq i \leq m,\ 1 \leq j \leq n)$, are from the set of complex numbers, $\mathbb{C}$.

Solution of a System of a Linear Equations

A Solution of a System of Linear Equations

is an ordered list of n complex numbers, $s_1, s_2, s_3, \ldots, s_n$
for n variables, $x_1, x_2, x_3, \ldots, x_n$, such that

if we substitute
s_1 for x_1,
s_2 for x_2,
s_3 for x_3,
$\ldots$,
s_3 for x_n,
then all m equations are true simultaneously, i.e,

for every equation of the system
the left side will equal to the right side

Solution Set of a System of a Linear Equations

The solution set of a System of Linear Equations

is the set which contains every solution to the system,
and nothing more.

Three types of a solution set

- $$\begin{aligned} 2x_1 &+3x_2 &= 3 \\ x_1 &-x_2 &= 4 \end{aligned}$$ a single solution
- $$\begin{aligned} 2x_1 &+3x_2 &= 3 \\ 4x_1 &+6x_2 &= 6 \end{aligned}$$ inifintely many solution
- $$\begin{aligned} 2x_1 &+3x_2 &= 3 \\ 4x_1 &+6x_2 &= 10 \end{aligned}$$ no soution

Equivalent Systems

Equivalent Systems

Two systems of linear equations are **equivalent**
if their solution sets are equal.

Equation Operations

Equation Operations

Given a system of linear equations, the following three operations will transform the system into a different one, and each operation is known as an **equation operation**.

1. swap the locations of two equations in the list of equations.
2. multiply each term of an equation by a nonzero quantity.
3. multiply each term of one equation by some quantity, and add these terms to a second equation, on both sides of the equality. leave the first equation the same after this operation, but replace the second equation by the new one.

Equation Operations Preserve Solution Sets

Equation Operations

If we apply one of the three equation operations
to a system of linear equations,
then the original system and the transformed system
are equivalent.

Three Equations and One Solution (1)

solve the following by a sequence of equation operations

$$\begin{aligned} x_1 &+2x_2 &+2x_3 &= 4 \\ x_1 &+3x_2 &+3x_3 &= 5 \\ 2x_1 &+6x_2 &+5x_3 &= 6 \end{aligned}$$

1. $-1 \cdot eq1 + eq2 \to eq2$
   $-1 \cdot (1,2,2,4) + (1,3,3,5) \to (0,1,1,1)$

$$\begin{aligned} x_1 &+2x_2 &+2x_3 &= 4 \\ 0x_1 &+1x_2 &+1x_3 &= 1 \\ 2x_1 &+6x_2 &+5x_3 &= 6 \end{aligned}$$

2. $-2 \cdot eq1 + eq3 \to eq3$
   $-2 \cdot (1,2,2,4) + (2,6,5,6) \to (0,2,1,-2)$

$$\begin{aligned} x_1 &+2x_2 &+2x_3 &= 4 \\ 0x_1 &+1x_2 &+1x_3 &= 1 \\ 0x_1 &+2x_2 &+1x_3 &= -2 \end{aligned}$$

Three Equations and One Solution (2)

3. $-2 \cdot eq2 + eq3 \rightarrow eq3$
   $-2 \cdot (0,1,1,1) + (0,2,1,-2) \rightarrow (0,0,-1,-4)$

$$\begin{array}{rrrr} x_1 & +2x_2 & +2x_3 & = 4 \\ 0x_1 & +1x_2 & +1x_3 & = 1 \\ 0x_1 & +0x_2 & -1x_3 & = -4 \end{array}$$

4. $-1 \cdot eq3 \rightarrow eq3$
   $-1 \cdot (0,0,-1,-4) \rightarrow (0,0,1,4)$

$$\begin{array}{rrrr} x_1 & +2x_2 & +2x_3 & = 4 \\ 0x_1 & +1x_2 & +1x_3 & = 1 \\ 0x_1 & +2x_2 & +1x_3 & = 4 \end{array}$$

Three Equations and One Solution (3)

which can be written more clearly

$$\begin{aligned} x_1 \quad +2x_2 \quad +2x_3 &= 4 \\ x_2 \quad +x_3 &= 1 \\ x_3 &= 4 \end{aligned}$$

thus, the solution is $(x_1, x_2, x_3) = (2, -3, 4)$

Three Equations and Infinitely Many Solutions (1)

solve the following by a sequence of equation operations

$$\begin{array}{rrrrcr} 1x_1 & +2x_2 & +0x_3 & +1x_4 & = & 7 \\ 1x_1 & +1x_2 & +1x_3 & -1x_4 & = & 3 \\ 3x_1 & +1x_2 & +5x_3 & -7x_4 & = & 1 \end{array}$$

1. $-1 \cdot eq1 + eq2 \rightarrow eq2$
   $-1 \cdot (1,2,0,1,7) + (1,1,1,-1,3) \rightarrow (0,-1,1,-2,-4)$

$$\begin{array}{rrrrcr} 1x_1 & +2x_2 & +0x_3 & +1x_4 & = & 7 \\ 0x_1 & -1x_2 & +1x_3 & -2x_4 & = & -4 \\ 3x_1 & +1x_2 & +5x_3 & -7x_4 & = & 1 \end{array}$$

2. $-3 \cdot eq1 + eq3 \rightarrow eq3$
   $-3 \cdot (1,2,0,1,7) + (3,1,5,-7,1) \rightarrow (0,-5,5,-10.-20)$

$$\begin{array}{rrrrcr} 1x_1 & +2x_2 & +0x_3 & +1x_4 & = & 7 \\ 0x_1 & -1x_2 & +1x_3 & -2x_4 & = & -4 \\ 0x_1 & -5x_2 & +5x_3 & -10x_4 & = & -20 \end{array}$$

Three Equations and Infinitely Many Solutions (2)

3. $-5 \cdot eq2 + eq3 \to eq3$

   $-5 \cdot (0, -1, 1, -2, -4) + (0, -5, 5, -10, -20) \to (0, 0, 0, 0.0)$

$$\begin{array}{ccccccc} 1x_1 & +2x_2 & +0x_3 & +1x_4 & = & 7 \\ 0x_1 & -1x_2 & +1x_3 & -2x_4 & = & -4 \\ 0x_1 & +0x_2 & +0x_3 & +0x_4 & = & 0 \end{array}$$

4. $-1 \cdot eq2 \to eq2$

   $-1 \cdot (0, -1, 1, -2, -4) \to (0, 1, -1, 2, 4)$

$$\begin{array}{ccccccc} 1x_1 & +2x_2 & +0x_3 & +1x_4 & = & 7 \\ 0x_1 & +1x_2 & -1x_3 & +2x_4 & = & 4 \\ 0x_1 & +0x_2 & +0x_3 & +0x_4 & = & 0 \end{array}$$

Three Equations and Infinitely Many Solutions (3)

5. $-2 \cdot eq2 + eq1 \rightarrow eq1$
   $-2 \cdot (0,1,-1,2,4) + (1,2,0,1,7) \rightarrow (1,0,2,-3.-1)$

$$\begin{array}{cccccc} 1x_1 & +0x_2 & +2x_3 & -3x_4 & = & -1 \\ 0x_1 & +1x_2 & -1x_3 & +2x_4 & = & 4 \\ 0x_1 & +0x_2 & +0x_3 & +0x_4 & = & 0 \end{array}$$

which can be written more clearly

$$\begin{array}{cccc} x_1 & +2x_3 & -3x_4 & = -1 \\ x_2 & -x_3 & +2x_4 & = 4 \\ & & 0 & = 0 \end{array}$$

Three Equations and Infinitely Many Solutions (4-1)

- the meaning of the equation $0 = 0$
  - can choose any values for x_1, x_2, x_3, x_4 and this equation $0 = 0$ will be true,
  - only need to consider further the first two equations, since $0 = 0$ is true no matter what.

Three Equations and Infinitely Many Solutions (4-2)

- We can analyze $x_1 + 2x_3 - 3x_4 = -1$
  without consideration of the variable x_1.
  - It would appear that there is considerable latitude
    in how we can choose x_2, x_3, x_4
    and make this equation $x_1 + 2x_3 - 3x_4 = -1$ true.
  - Let us choose x_3 and x_4 to be anything we please,
    say $x_3 = a$ and $x_4 = b$

Three Equations and Infinitely Many Solutions (5)

- with $x_3 = a$ and $x_4 = b$

$$\begin{aligned} x_1 \quad +2x_3 \quad -3x_4 &= -1 \\ x_2 \quad -x_3 \quad +2x_3 &= 4 \\ 0 &= 0 \end{aligned}$$

- $$\begin{aligned} x_1 + 2a - 3b &= -1 \\ x_1 &= -1 - 2a + 3b \end{aligned}.$$
- $$\begin{aligned} x_2 - a + 2b &= -4 \\ x_2 &= 4 + a - 2b \end{aligned}.$$

Three Equations and Infinitely Many Solutions (5)

- So our arbitrary choices of values for x3 and x4 (a and b) translate into specific values of x1 and x2.
  - choosing $a = 2$ and $b = 1$.
  - choosing $a = 5$ and $b = -2$.
  - Now we can easily and quickly find many more (infinitely more)
  - Suppose we choose $a = 5$ and $b = -2$,
  - then we compute

$$x_1 = -1 - 2(5) + 3(-2) = -17$$

$$x_2 = 4 + 5 - 2(-2) = 13$$

  - and you can verify that $(x1, x2, x3, x4) = (-17, 13, 5, -2)$ makes all three equations true..

Three Equations and Infinitely Many Solutions (5)

- The entire solution set is written as

$$S = (-1 - 2a + 3b, 4 + a - 2b, a, b) | a \in C, b \in C$$

  - Evaluate the three equations of the original system with these expressions in a and b and
  - verify that each equation is true, no matter what values are chosen for a and b

Non-zero scalar (1)

- If we were to allow a zero scalar to multiply an equation then that equation would be transformed to the equation $0 = 0$,
- $0 = 0$ is true for any possible values of the variables.
- Any restrictions on the solution set imposed by the original equation would be lost.
- However, in the third operation,
  it is allowed to choose a zero scalar,
  multiply an equation by this scalar
  and add the transformed equation to a second equation
  (leaving the first unchanged).
  The result - Nothing changed
  The second equation is the same as it was before.

Non-zero scalar (2)

- So the theorem is true in this case, the two systems are equivalent.
- But in practice, this would be a silly thing to actually ever do!
- We still allow it though, in order to keep our theorem as general as possible.
- Notice the location in the proof of Theorem EOPSS where the expression 1α appears — this explains the prohibition on $\alpha=0$ in the second equation operation.