

Stationary Random Processes

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Based on
Probability, Random Variables and Random Signal Principles,
P.Z. Peebles,Jr. and B. Shi

Outline

- 1 First-Order Stationary Processes
- 2 Correlation and Covariance Functions

First Order Stationary

 $f_X(x; t)$

if $X(t)$ is to be a **first-order stationary**

$$f_X(x_1; t_1) = f_X(x_1; t_1 + \Delta)$$

must be true for any time t_1 and any real number Δ

the **first order density function**

does not change with a shift in time origin

Consequences of stationarity

$f_X(x; t)$

- $f_X(x, t_1)$ is independent of t_1
the **first order density function**
does not change with a shift in time origin

- the **process mean** value is a **constant**

$$m_X(t) = \bar{X} = \text{constant}$$

the process mean value

$$m_X(t) = \bar{X} = \text{constant}$$

$$m_X(t_1) = \int_{-\infty}^{\infty} x f_X(x; t_1) dx$$

$$m_X(t_2) = \int_{-\infty}^{\infty} x f_X(x; t_2) dx$$

let $t_2 = t_1 + \Delta$

$$m_X(t_1) = m_X(t_1 + \Delta)$$

Second-Order Stationary Process

$$f_X(x_1, x_2; t_1, t_2)$$

if $X(t)$ is to be a **second-order stationary**

$$f_X(x_1, x_2; t_1, t_2) = f_X(x_1, x_2; t_1 + \Delta, t_2 + \Delta)$$

must be true for any time t_1, t_2 and any real number Δ

the **second order density function**

does not change with a shift in time origin

Second-Order Stationary Process

$$f_X(x_1, x_2; t_1, t_2)$$

- $f_X(x_1, x_2; t_1, t_2)$ is independent of t_1 and t_2
 the **second order density function**
 does not change with a shift in time origin

- the **autocorrelation function**

$$R_{XX}(t, t + \tau) = E[X(t)X(t + \tau)] = R_{XX}(\tau)$$

N^{th} -order Stationary Processes

$$f_X(x_1, \dots, x_N; t_1, \dots, t_N)$$

if $X(t)$ is to be a N^{th} -order stationary

$$f_X(x_1, \dots, x_N; t_1, \dots, t_N) = f_X(x_1, \dots, x_N; t_1 + \Delta, \dots, t_N + \Delta)$$

must be true for any time t_1, \dots, t_N
and any real number Δ

the N^{th} order density function
does not change with a shift in time origin

Stationary Process

joint probability distribution

a **stationary process** is a stochastic process whose unconditional joint probability distribution does not change when shifted in time.

Consequently, parameters such as **mean** and **variance** also do not change over time.

https://en.wikipedia.org/wiki/Stationary_process

Stationary Process - nomenclature

nomenclature

- stationary process
- strictly stationary process
- strongly stationary process
- strict sense stationary (SSS) process

https://en.wikipedia.org/wiki/Stationary_process

Strict Sense Stationary Process

for all natural number N

if $X(t)$ is to be a **strict sense stationary (SSS)** process

$$f_X(x_1, \dots, x_N; t_1, \dots, t_N) = f_X(x_1, \dots, x_N; t_1 + \Delta, \dots, t_N + \Delta)$$

must be true for any time t_1, \dots, t_N

and any real number Δ

and for all natural number N

- **white noise** is the simplest example of a **strictly stationary process**.

https://en.wikipedia.org/wiki/Stationary_process

Wide Sense Stationary Process

1st and 2nd moments

Wide Sense Stationary (WSS) random processes only require that

1st moment (i.e. the **mean**) and **autocovariance** do not vary with respect to time and that the **2nd moment** is finite for all times.

- $E[X(t_1)] = E[X(t_2)] = \bar{X} = \text{constant}$ for all t_1 and t_2
- $C_{XX}(t_1, t_2) = C_{XX}(t_1 - t_2, 0) \triangleq C_{XX}(\tau)$ for all t_1 and t_2
- $E[|X(t)|^2] < \infty$ for all t

https://en.wikipedia.org/wiki/Stationary_process

Wide Sense Stationary Process - nomenclature

nomenclature

- weak sense stationary (WSS) process
- wide sense stationary (WSS) process

https://en.wikipedia.org/wiki/Stationary_process

WSS - auto-covariance & auto-correlation

mean, auto-covariance, auto-correlation

$$m_X(t) = \bar{X} = \text{constant}$$

$$C_{XX}(t_1, t_2) = E[\{X(t_1) - m_X(t_1)\} \{X(t_2) - m_X(t_2)\}]$$

$$= E[\{X(t_1) - \bar{X}\} \{X(t_2) - \bar{X}\}]$$

$$= E[X(t_1)X(t_2)] - \bar{X}^2$$

$$\triangleq C_{XX}(\tau)$$

$$\triangleq R_{XX}(\tau) - \bar{X}^2$$

$$R_{XX}(t_1, t_2) \triangleq R_{XX}(\tau)$$

Wide Sense Stationary Process

$$m_X(t), R_{XX}(\tau)$$

WSS random processes only require that
1st moment (i.e. the **mean**) and **autocorrelation**
do not vary with respect to time

$$E[X(t)] = m_X(t) = \bar{X} = \text{constant}$$

$$E[X(t)X(t+\tau)] = R_{XX}(\tau)$$

Wide Sense Stationary Process

$m_X(t), R_{XX}(\tau)$

- the 2nd order stationarity is sufficient for wide sense stationarity
- if $f_X(x_1; t_1)$ is independent of t_1 then $E[X(t)] = \text{constant}$
- if $f_X(x_1, x_2; t_1, t_2)$ is independent of t_1 and t_2 then $E[X(t)X(t+\tau)] = R_{XX}(\tau)$

The properties of autocorrelation functions (1)

$$|R_{XX}(\tau)|, R_{XX}(-\tau), R_{XX}(0)$$

$$|R_{XX}(\tau)| \leq R_{XX}(0)$$

$$R_{XX}(-\tau) = R_{XX}(\tau)$$

$$R_{XX}(0) = E[X^2(t)]$$

$$P[|X(t+\tau) - X(t)| > \varepsilon] = \frac{2}{\varepsilon^2} (R_{XX}(0) - R_{XX}(\tau))$$

The properties of autocorrelation functions (2)

$$R_{NN}(\tau), R_{XX}(\tau)$$

if $X(t) = \bar{X} + N(t)$

where $N(t)$ is WSS, is **zero-mean**, and

has autocorrelation function $R_{NN}(\tau) \rightarrow 0$ as $|\tau| \rightarrow \infty$, then

$$\lim_{|\tau| \rightarrow \infty} R_{XX}(\tau) = \bar{X}^2$$

The properties of autocorrelation functions (3)

$$R_{NN}(\tau), R_{XX}(\tau)$$

if $X(t)$ is **mean square periodic**, i.e.,

there exists a $T \neq 0$

such that $E[\{X(t+T) - X(t)\}^2] = 0$ for all t ,

then $R_{XX}(t)$ will have a **periodic** component
with the same period

$$R_{XX}(\tau + T) = R_{XX}(\tau)$$

$$R_{XX}(T) = R_{XX}(0)$$

$$E[X(t)X(t+T)] = E[X(t)X(t)]$$

$$E[X(t)X(t+T)] = E[X(t+T)X(t+T)]$$

$$2E[X(t)X(t+T)] = E[X(t)X(t)] + E[X(t+T)X(t+T)]$$

The properties of autocorrelation functions (4)

$R_{NN}(\tau), R_{XX}(\tau)$

$R_{XX}(\tau)$ cannot have an arbitrary shape
any arbitrary function cannot be an autocorrelation function.
power density spectrum

$R_{XX}(\tau)$ is related to the power density spectrum through the
Fourier transform and the form of the spectrum is not arbitrary

Crosscorrelation functions (1)

$$R_{XY}(t_1, t_2), R_{XY}(t, t + \tau)$$

$$R_{XY}(t_1, t_2) = E[X(t_1)Y(t_2)]$$

$$R_{XY}(t, t + \tau) = E[X(t)Y(t + \tau)] = R_{XY}(\tau)$$

if

$$R_{XY}(t, t + \tau) = 0$$

then $X(t)$ and $Y(t)$ are called **orthogonal processes**

Crosscorrelation functions (2)

$$R_{XY}(t, t + \tau), R_{XY}(\tau)$$

if $X(t)$ and $Y(t)$ are statistically independent

$$R_{XY}(t, t + \tau) = E[X(t)Y(t + \tau)] = m_X(t)m_Y(t + \tau)$$

if $X(t)$ and $Y(t)$ are statistically independent and are at least WSS,

$$R_{XY}(\tau) = \overline{XY}$$

which is constant

The properties of crosscorrelation functions (1)

 $R_{XY}(\tau), |R_{XY}(\tau)|$

$$R_{XY}(\tau) = R_{XY}(-\tau)$$

$$|R_{XY}(\tau)| = \sqrt{R_{XX}(0)R_{YY}(0)}$$

$$|R_{XY}(\tau)| \leq \frac{1}{2} [R_{XX}(0) + R_{YY}(0)]$$

The properties of crosscorrelation functions (2)

$$R_{YX}(-\tau)$$

$$R_{YX}(-\tau) = E[Y(t)X(t-\tau)] = E[Y(s+\tau)X(s)] = R_{XY}(\tau)$$

$$E\left[\{Y(t+\tau) + \alpha X(t)\}^2\right] \geq 0$$

the **geometric mean** of two positive numbers
cannot exceed their **arithmetic mean**

The properties of crosscorrelation functions (3)

$$|R_{XY}(\tau)|$$

$$|R_{XY}(\tau)| \leq \frac{1}{2} [R_{XX}(0) + R_{YY}(0)]$$

$$\sqrt{R_{XX}(0)R_{YY}(0)} \leq \frac{1}{2} [R_{XX}(0) + R_{YY}(0)]$$

Covariance Functions

$$C_{XX}(t, t + \tau), C_{XY}(t, t + \tau)$$

$$C_{XX}(t, t + \tau) = E[\{X(t) - m_X(t)\} \{X(t + \tau) - m_X(t + \tau)\}]$$

$$C_{XY}(t, t + \tau) = E[\{X(t) - m_X(t)\} \{Y(t + \tau) - m_Y(t + \tau)\}]$$

$$C_{XX}(t, t + \tau) = R_{XX}(t, t + \tau) - m_X(t)m_X(t + \tau)$$

$$C_{XY}(t, t + \tau) = R_{XY}(t, t + \tau) - m_X(t)m_Y(t + \tau)$$

at least jointly WSS

$$C_{XX}(\tau) = R_{XX}(\tau) - \bar{X}^2$$

$$C_{XY}(\tau) = R_{XY}(\tau) - \bar{X}\bar{Y}$$

The properties of covariance functions

$$C_{XX}(0)$$

For a **WSS process**, variance does not depend on time and if $\tau = 0$

$$C_{XX}(0) = R_{XX}(0) - \bar{X}^2$$

$$\sigma_X^2 = E \left[\{X(t) - E[X(t)]\}^2 \right] = C_{XX}(0)$$

if the two random processes **uncorrelated**

$$C_{XY}(t, t + \tau) = R_{XY}(t, t + \tau) - m_X(t)m_Y(t + \tau) = 0$$

$$R_{XY}(t, t + \tau) = m_X(t)m_Y(t + \tau)$$

Discrete-Time Processes and Sequences (1)

$$R_{XX}[n, n+k], R_{YY}[n, n+k], C_{XX}[n, n+k], C_{YY}[n, n+k]$$

$$m_X[n] = \bar{X}, m_Y[n] = \bar{Y}$$

$$R_{XX}[n, n+k] = R_{XX}[k]$$

$$R_{YY}[n, n+k] = R_{YY}[k]$$

$$C_{XX}[n, n+k] = R_{XX}[k] - \bar{X}^2$$

$$C_{YY}[n, n+k] = R_{YY}[k] - \bar{Y}^2$$

Discrete-Time Processes and Sequences (2)

$$R_{XY}[n, n+k], R_{YX}[n, n+k], C_{XY}[n, n+k], C_{YX}[n, n+k]$$

$$m_X[n] = \bar{X}, m_Y[n] = \bar{Y}$$

$$R_{XY}[n, n+k] = R_{XY}[k]$$

$$R_{YX}[n, n+k] = R_{YX}[k]$$

$$C_{XY}[n, n+k] = R_{XY}[k] - \bar{X}\bar{Y}$$

$$C_{YX}[n, n+k] = R_{YX}[k] - \bar{Y}\bar{X}$$

