Hybrid CORDIC 2.A Sine/Cosine Generator

20171014

The details moved to

https://en.wikiversity.org/wiki/Butterfly_Hardware_Implementations

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Wilson ROM based Sine/Cosine Generation

[24] Fu & Willson Sine / Cosine Generation

ROM-based

For high resolution, ROM size grows exponentially

quater-wave symmetry

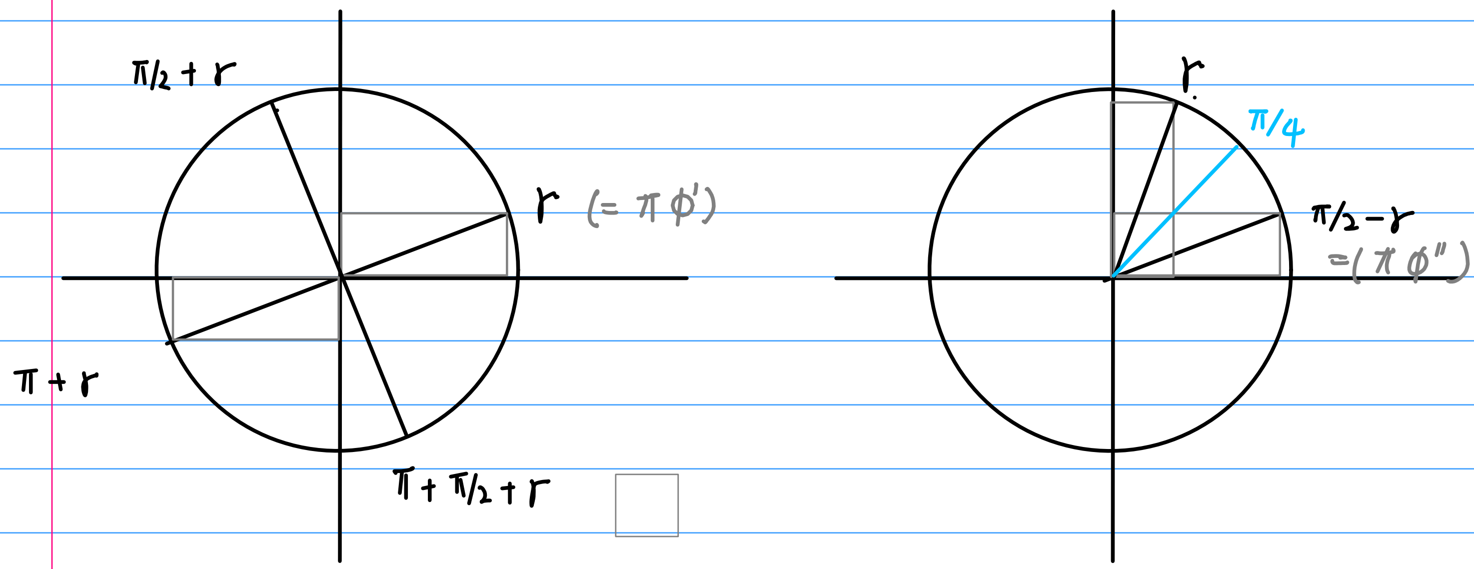
$$\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$$

$$\phi \ [0, 2\pi] \longrightarrow [0, \frac{\pi}{4}]$$

conditionally interchanging inputs X_0 & Y_0
Conditionally interchanging and negating outputs X & Y

$$X = X_0\cos\phi - Y_0\sin\phi$$
$$Y = Y_0\cos\phi + X_0\sin\phi$$

Madisetti VLSI arch



for frequency synthesis

argument: signed normalized by π angle [-1, 1]

binary representation of a radian angle required

$[-1, 1] \longrightarrow [0, \pi/4] \longrightarrow$ sine/cosine generator

ϕ

$\theta = \pi\phi$

① a phase accumulator ϕ $[-1, 1]$

② a radian converter $\phi \rightarrow \theta$

③ a sine/cosine generator $\sin\theta$, $\cos\theta$

④ an output stage $\sin\theta$, $\cos\theta$

$\sin\theta \rightarrow \sin\pi\phi$, $\cos\theta \rightarrow \cos\pi\phi$

output stage $\quad \sin\theta \to \sin\pi\phi$ $\quad [-\pi, +\pi]$

$\cos\theta \to \cos\pi\phi$

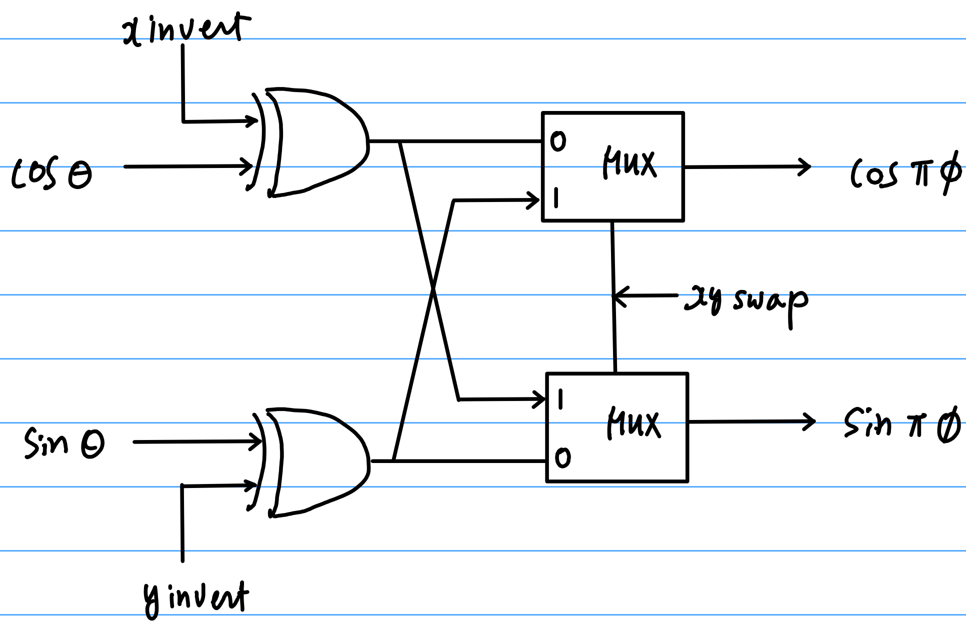
negation / interchange

MSB_1, MSB_2, MSB_3 → x invert, y invert, xy swap

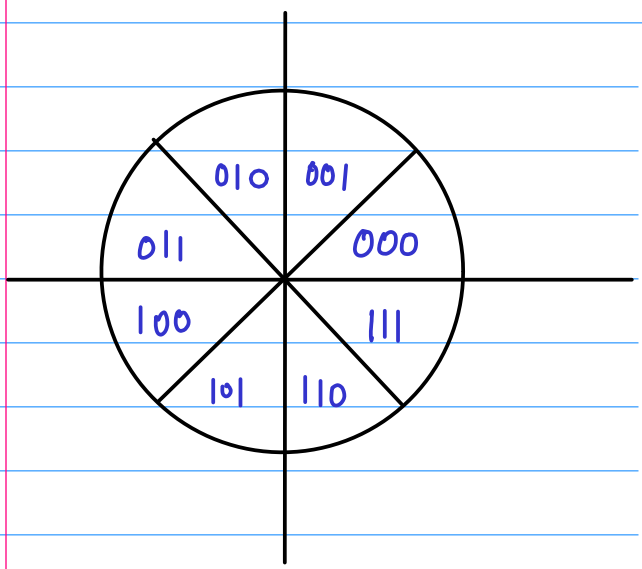
x invert, y invert: the negation of $\cos\theta = X_{N+1}$, $\sin\theta = Y_{N+1}$

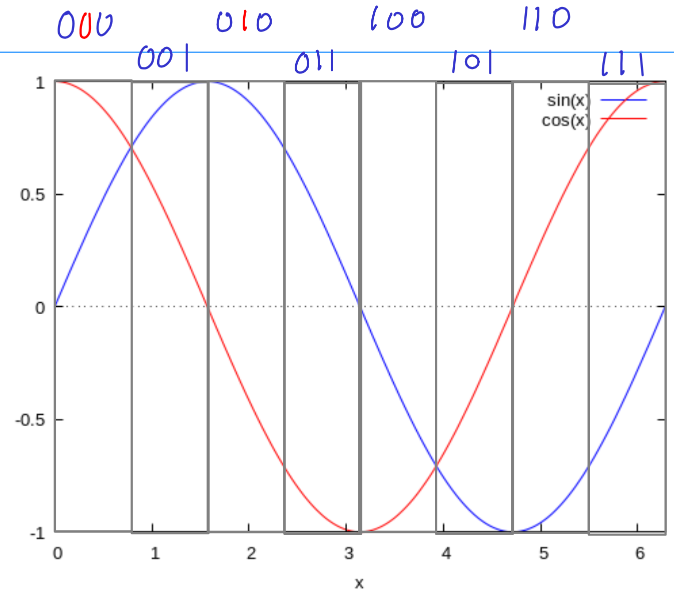
xy swap: Interchange

negate before swap



Madisetti & Willson, DDS Freq synthesizer





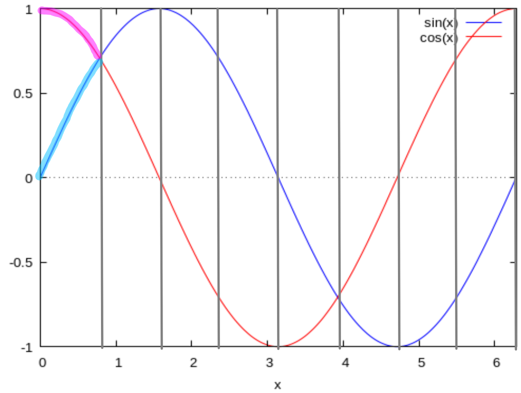
	Xinv (cos)	Yinv (sin)	swap	$\cos\pi\theta$	$\sin\pi\phi$
0 0 0	0	0	0	$\cos\theta$	$\sin\theta$
0 0 1	0	0	1	$\sin\theta$	$\cos\theta$
0 1 0	0	1	1	$-\sin\theta$	$\cos\theta$
0 1 1	1	0	0	$-\cos\theta$	$\sin\theta$
1 0 0	1	1	0	$-\cos\theta$	$-\sin\theta$
1 0 1	1	1	1	$-\sin\theta$	$-\cos\theta$
1 1 0	1	0	1	$\sin\theta$	$-\cos\theta$
1 1 1	0	1	0	$\cos\theta$	$-\sin\theta$

Madisetti & Willson, DDS Freq synthesizer

0

$\cos\theta$

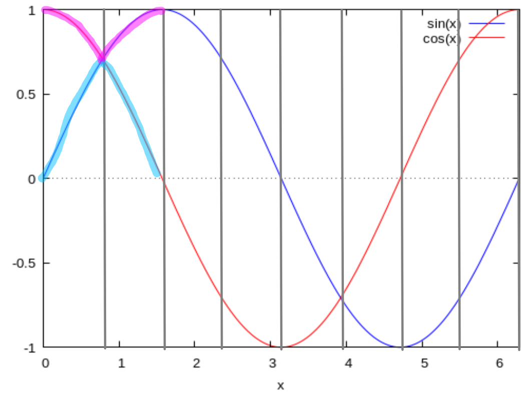
$\sin\theta$



1

$\sin\theta$

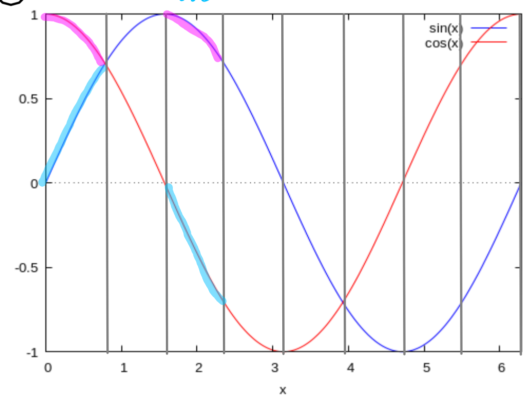
$\cos\theta$



2

$-\sin\theta$

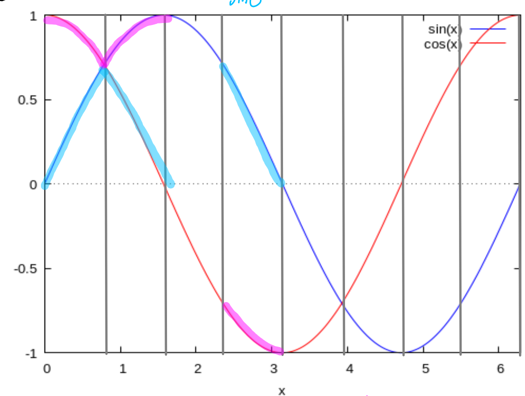
$\cos\theta$



3

$-\cos\theta$

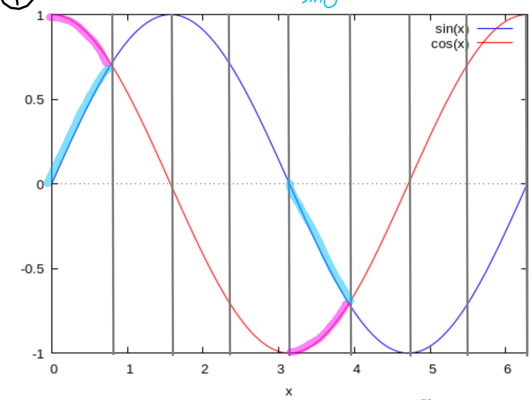
$\sin\theta$



4

$-\cos\theta$

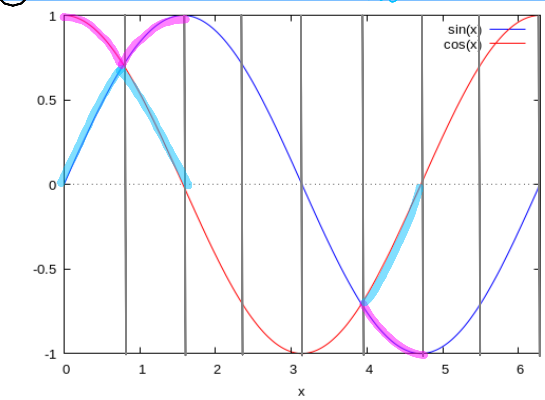
$-\sin\theta$



5

$-\sin\theta$

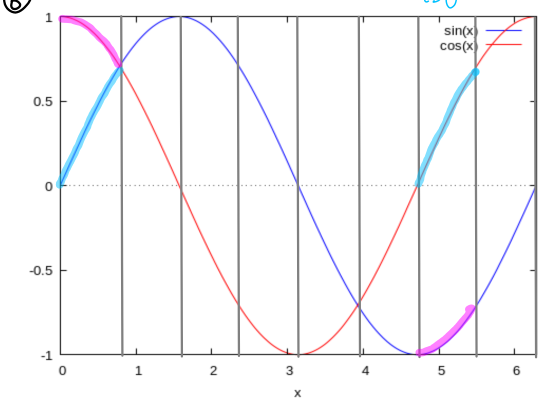
$-\cos\theta$



6

$\sin\theta$

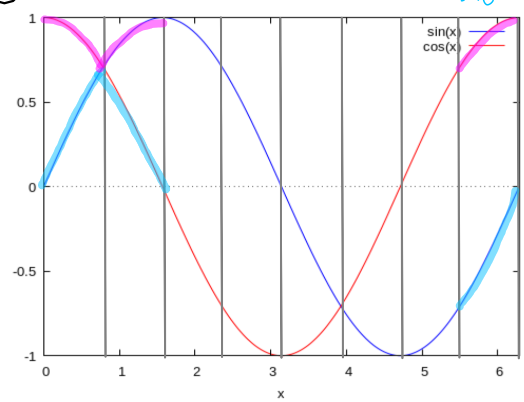
$-\cos\theta$



7

$\cos\theta$

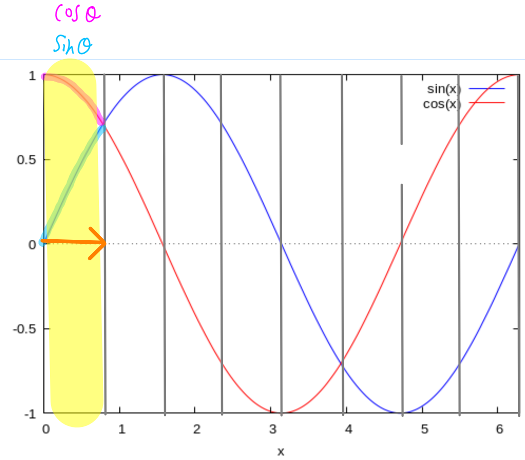
$-\sin\theta$



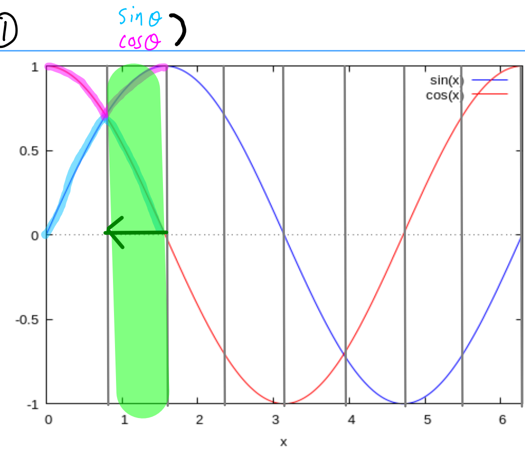
Madisetti & Willson, DDS Freq synthesizer

$$\begin{cases} \cos\phi \\ \sin\phi \end{cases}$$

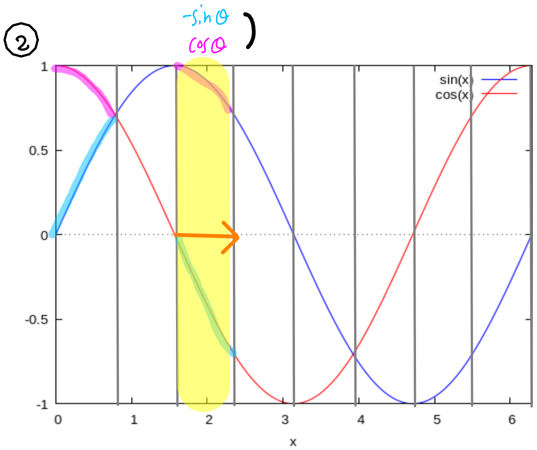
0



1



2



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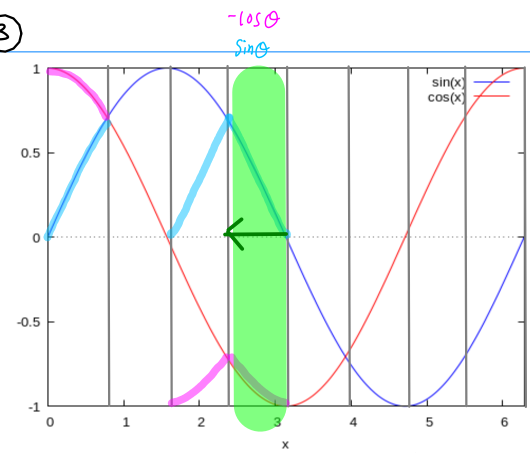

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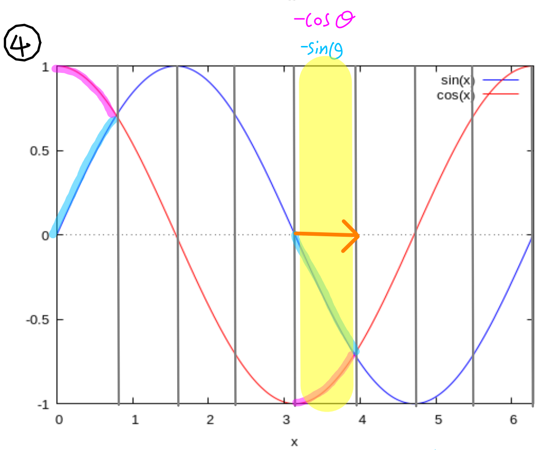

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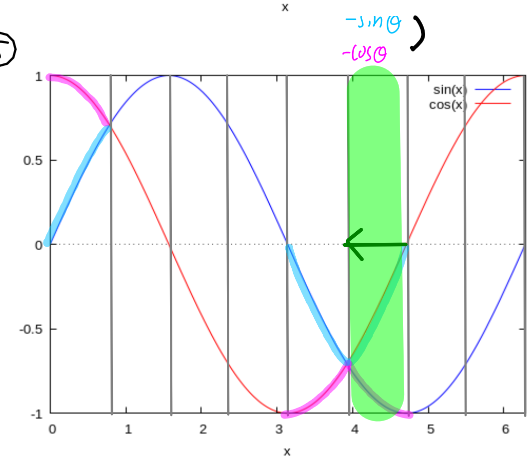

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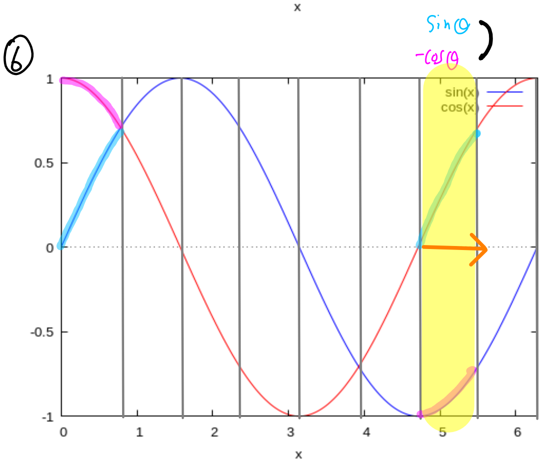
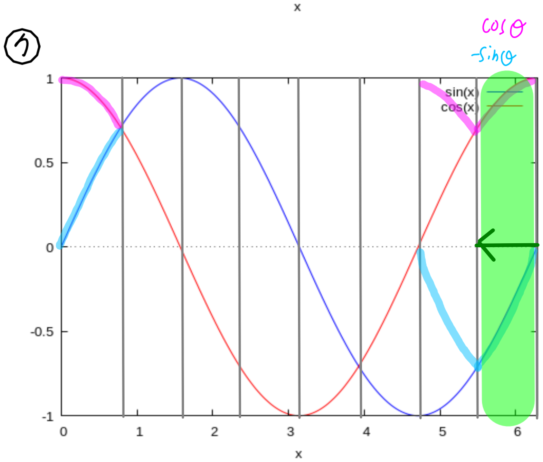
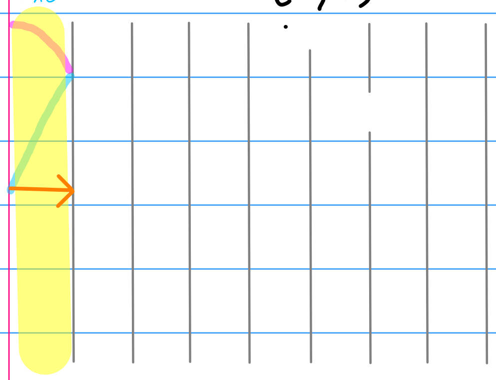


7

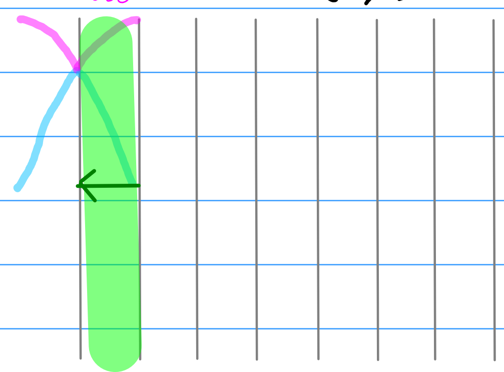


$\sin \phi$

0 $\cos\theta$ $\sin\theta$ $+\ +$ $(0, 0)$

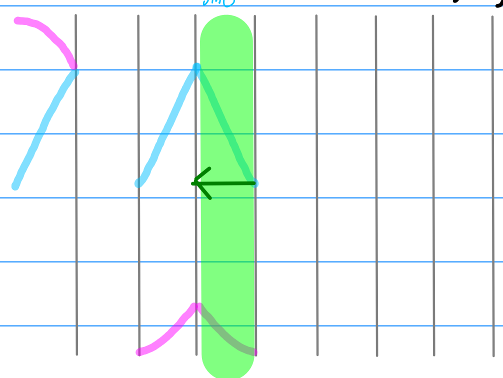


1 $\sin\theta$ $\cos\theta$) $+\ +$ $(0, 0)$

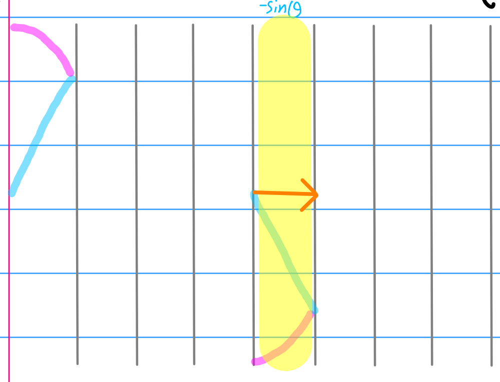


2 $-\sin\theta$ $\cos\theta$) $+\ -$ $(0, 1)$

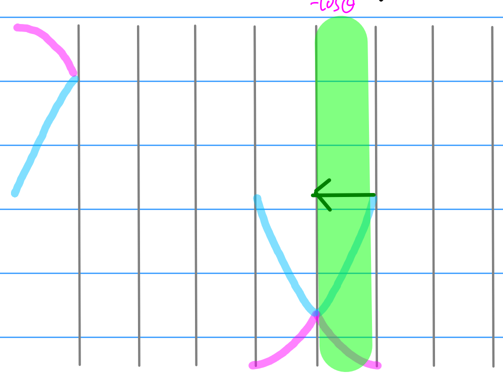
3 $-\cos\theta$ $\sin\theta$ $-\ +$ $(1, 0)$



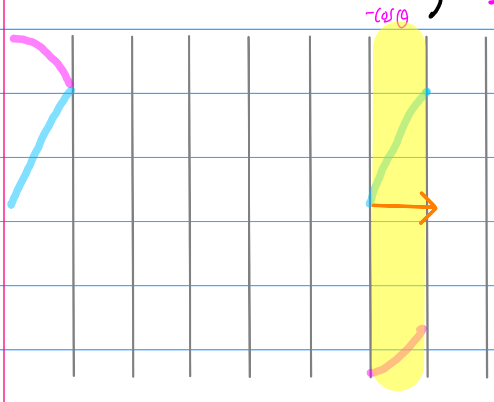
4 $-\cos\theta$ $-\sin\theta$ $-\ -$ $(1, 1)$



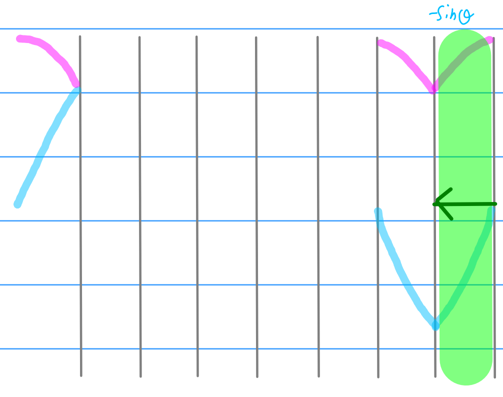
5 $-\sin\theta$ $-\cos\theta$) $-\ -$ $(1, 1)$



6 $\sin\theta$ $-\cos\theta$) $-\ +$ $(1, 0)$



7 $\cos\theta$ $-\sin\theta$ $+\ -$ $(0, 1)$



Madisetti & Willson, DDS Freq synthesizer

	Xinv	Yinv	swap	$\cos\pi\phi$	$\sin\pi\phi$
0 0 0	0	0	0	$\cos\theta$	$\sin\theta$
0 0 1	0	0	1	$\sin\theta$	$\cos\theta$
0 1 0	0	1	1	$-\sin\theta$	$\cos\theta$
0 1 1	1	0	0	$-\cos\theta$	$\sin\theta$
1 0 0	1	1	0	$-\cos\theta$	$-\sin\theta$
1 0 1	1	1	1	$-\sin\theta$	$-\cos\theta$
1 1 0	1	0	1	$\sin\theta$	$-\cos\theta$
1 1 1	0	1	0	$\cos\theta$	$-\sin\theta$

0 0	0 0
0 1	1 0
1 1	1 1
1 0	0 1

0 0 0 0

0 1 1 0

1 1 1 1

1 0 0 1

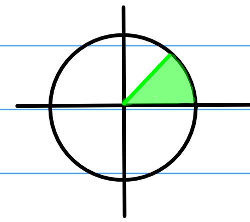
Madisetti & Willson, DDS Freq synthesizer

given angle θ (in radian)

$$0 \le \theta \le \pi/4 < 1$$

$$\Downarrow$$

0.785398163



$$\theta = \sum_{k=1}^{N} b_k \theta_k$$ Binary Representation

$$b_k \in \{0, 1\}$$

$$\theta_k = 2^{-k}$$

(N+1) bit fractional binary

Sign + N bit $\Rightarrow$ | S | b_1 b_2 .... b_N |

θ is constrained to be positive $\boxed{b_0 = 0}$ $\boxed{S = 0}$

$$\boxed{\theta = \sum_{k=1}^{N} b_k 2^{-k} = \phi_0 + \sum_{k=2}^{N+1} r_k 2^{-k}}$$

$r_k \in \{-1, +1\}$ signed digits

ϕ_0 constant

⊕ subrotation by 2^{-k}

2 equal ⊕ half rotations by 2^{-k-1}

⊖ subrotation

2 equal opposite half rotations by $\pm 2^{-k-1}$

Binary Representation

$b_k = 1$: rotation by 2^{-k}

$b_k = 0$: zero rotation

k-th rotation

- fixed rotation by 2^{-k-1}
- pos rotation $\leftarrow b_k = 1$; neg rotation $\leftarrow b_k = 0$

Combining all the fixed rotations

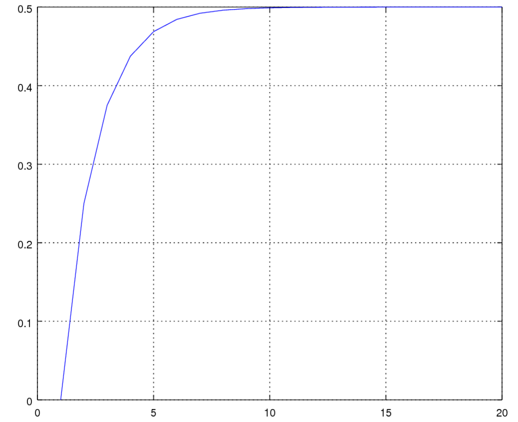
$\rightarrow$ initial fixed rotation

	b_1	b_2	b_3		b_N
	2^{-1}	2^{-2}	2^{-3}		2^{-N}
fixed $\Rightarrow$	$+2^{-2}$	$+2^{-3}$	$+2^{-4}$		$+2^{-N-1}$
	$(b_1=1)$ $+2^{-2}$	$(b_2=1)$ $+2^{-3}$	$(b_3=1)$ $+2^{-4}$		$(b_N=1)$ $+2^{-N-1}$
	$(b_1=0)$ -2^{-2}	$(b_2=0)$ -2^{-3}	$(b_3=0)$ -2^{-4}		$(b_N=0)$ -2^{-N-1}

initial fixed rotation

$$\phi_0 = \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{N+1}}$$

$$= \frac{\frac{1}{2^2}\left(1-\frac{1}{2^N}\right)}{\left(1-\frac{1}{2}\right)} = \frac{1}{2}\left(1-\frac{1}{2^N}\right) = \frac{1}{2} - \frac{1}{2^{N+1}}$$



Signed Digit Recoding

the rotation after recoding

— a fixed initial rotation ϕ_0

a sequence of ⊕/⊖ rotations

$b_k = 1 \quad + 2^{-k-1}$ rotation

$b_k = 0 \quad - 2^{-k-1}$ rotation

$r_k = (2b_{k-1} - 1)$

$2 \cdot 1 - 1 = +1$ $\qquad b_{k-1} = 1 \longrightarrow r_k = +1$

$2 \cdot 0 - 1 = -1$ $\qquad b_{k-1} = 0 \longrightarrow r_k = -1$

The recoding need not be explicitly performed

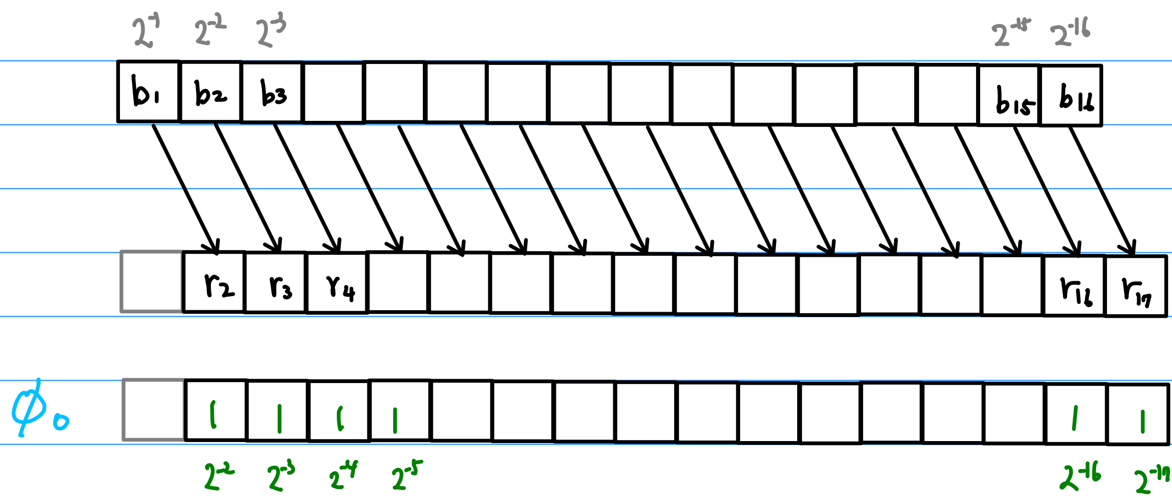
Simply replacing $b_k = 0$ with -1

This recoding maintains

a constant scaling factor K

$$\theta = \sum_{k=1}^{N} b_k 2^{-k} = \phi_0 + \sum_{k=2}^{N+1} r_k 2^{-k}$$

Binary Representation $\{ b_k \}$



Signed Digit Recoding $\{ r_k \}$

The Scaling K.
The initial rotation ϕ_0

rotation Starting point
$(X_0, Y_0) = (K\cos\phi_0,\ K\sin\phi)$

— fixed
— no error buildup

— rotation direction
immediately obtained from the binary representation
→ no need for comparison

the Subangles $\boxed{\theta_k = 2^{-k}}$ used in recoding
the Subangles $\boxed{\theta_k = \tan^{-1}(2^{-k})}$ used in CORDIC

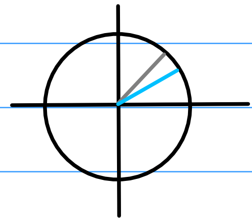
tan θ_k multipliers used
in the first few subrotation stages
Cannot be implemented
as a Simple Shift-and-add operations

→ ROM implementation

reduced Chip area
higher operating Speed.

Architecture

1. phase accumulator $\phi \in [-1, +1]$
2. radian converter $\phi \rightarrow \theta \in [0, \frac{\pi}{4}]$
3. Sine/cosine generator $\sin(\theta)$ $\cos(\theta)$
4. Output stage $\sin(\pi\phi)$ $\cos(\pi\phi)$



$\phi \in [-1, +1]$ normalized angle

$\phi \in [-\pi, +\pi] \rightarrow \theta \in [0, \frac{\pi}{4}]$ 1st half quadrant

$\sin(\theta)$ $\cos(\theta)$

$\sin(\pi\phi)$ $\cos(\pi\phi)$

Overflowing 2's complement accumulator

normalized by π angle ϕ

need radian angle $\theta \in [0, \frac{\pi}{4}]$

$0 < \theta < 1$ rad

N-bit binary representation of θ

controls the direction of subrotation

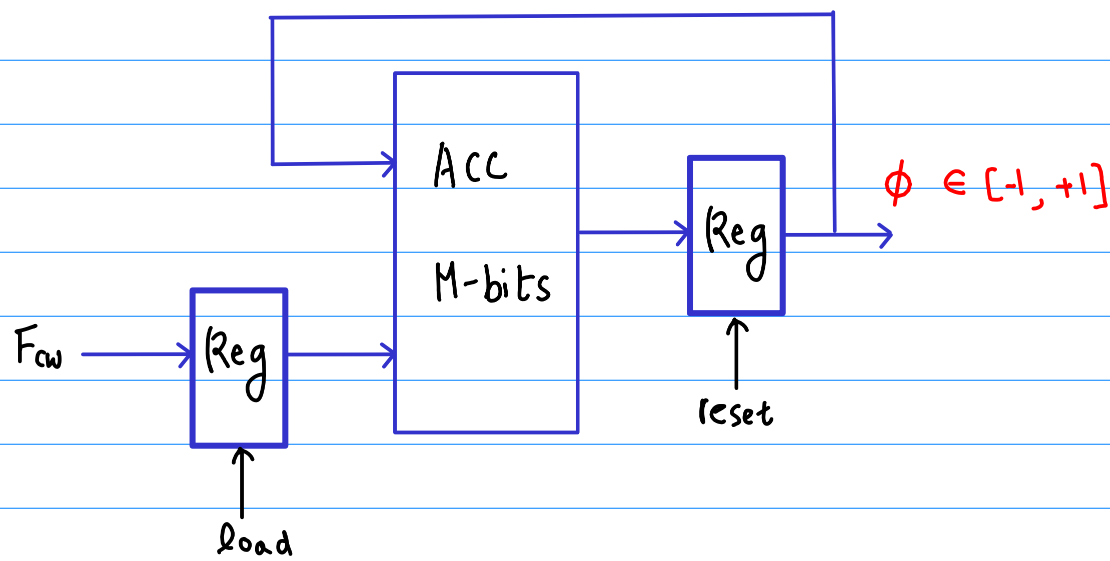
N-bit precision of $\cos\theta$ & $\sin\theta$

Output stage $\theta \rightarrow \pi\phi$

$\sin\theta \rightarrow \sin\pi\phi$

$\cos\theta \rightarrow \cos\pi\phi$

① phase accumulator



M-bit adder

repeatedly increments the phase angle

by F_{cw} at each clock cycle

frequency control word

at time n, $\boxed{\phi = n F_{cw} / 2^M}$

$$\cos \phi = \cos (n F_{cw} / 2^M)$$
$$\sin \phi = \sin (n F_{cw} / 2^M)$$

$-1 < \phi < +1$ normalized angle

② Radian Converter

normalized angle ϕ

ϕ

MSB1	MSB2	MSB3				...	

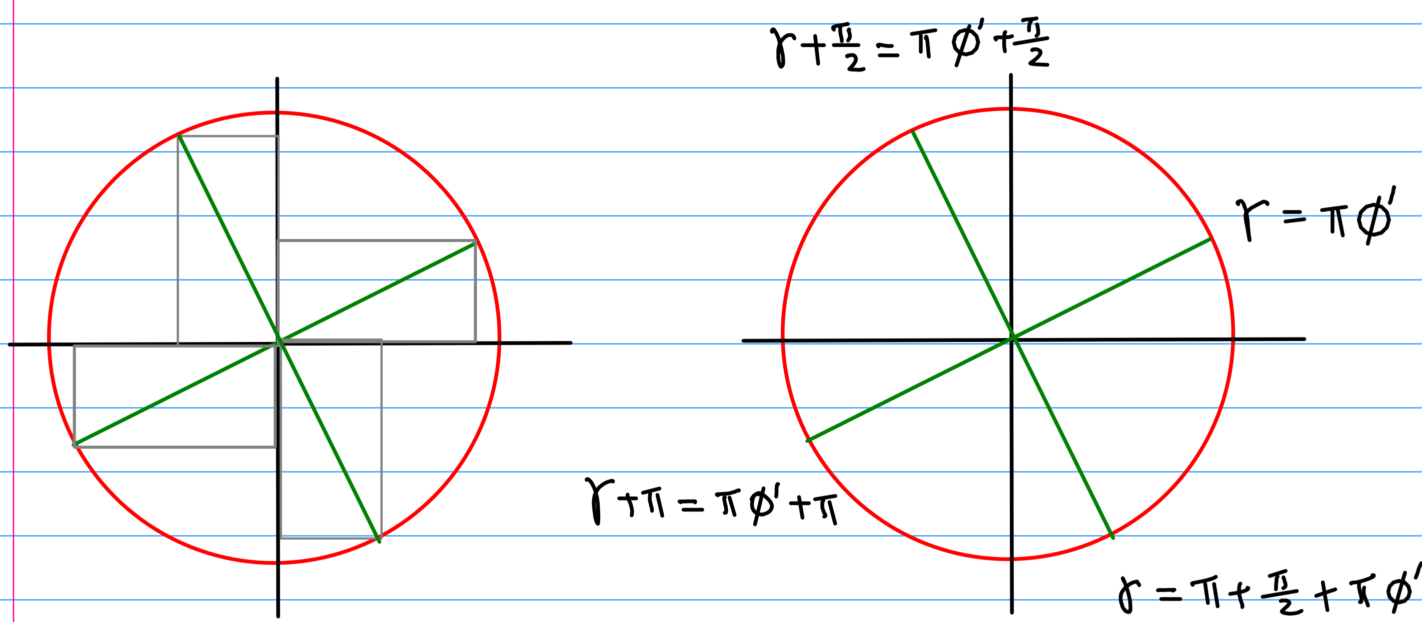
Quadrant of $\pi\phi$

U. half
L. half

Stored to interchange / negate

ϕ'

0	0	MSB3				...	

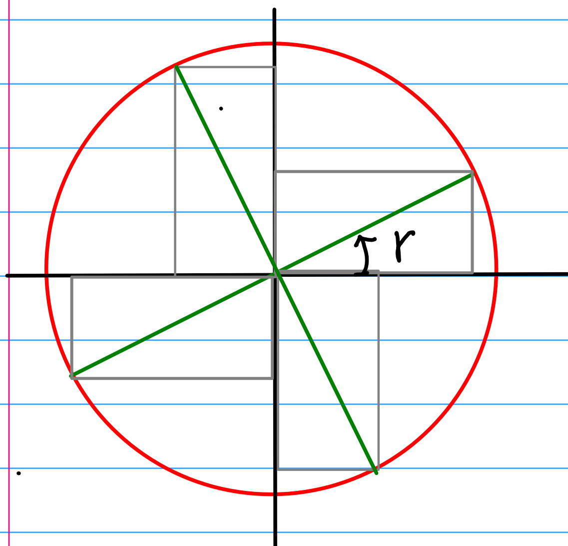


$\phi \rightarrow \phi' \rightarrow$

$\pi\phi'$	$+\ 0 \cdot \frac{\pi}{2}$	00
$\pi\phi'$	$+\ 1 \cdot \frac{\pi}{2}$	01
$\pi\phi'$	$+\ 2 \cdot \frac{\pi}{2}$	10
$\pi\phi'$	$+\ 3 \cdot \frac{\pi}{2}$	11

1st Quad

Quadrant Symmetry



Quadrant	
$\boxed{0\,1}$: $\gamma + \frac{\pi}{2} = \pi\phi' + \frac{\pi}{2}$	$\boxed{0\,0}$: $\gamma = \pi\phi'$
$\boxed{1\,0}$: $\gamma + \pi = \pi\phi' + \pi$	$\boxed{1\,1}$: $\gamma + \frac{3\pi}{2} = \pi\phi' + \frac{3\pi}{2}$

0	1	+1
0	0	0
1	1	-1
1	0	-2

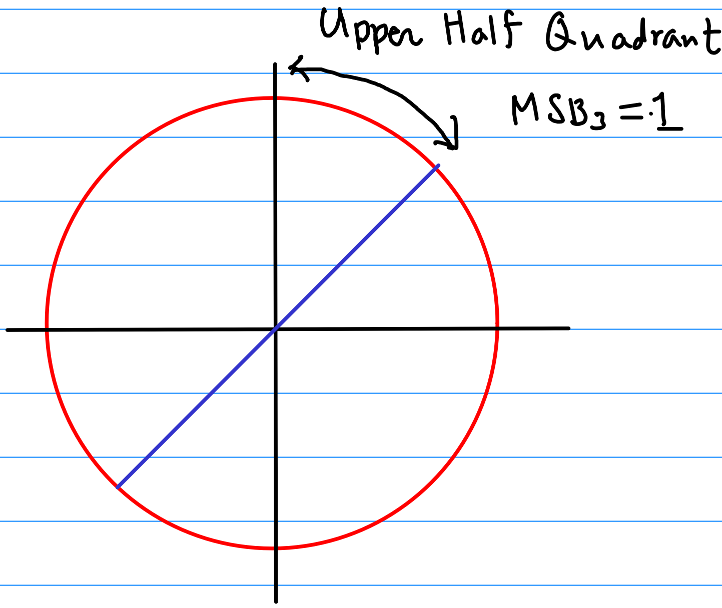
$$\theta = \pi\phi \Rightarrow \theta' = \pi\phi'$$

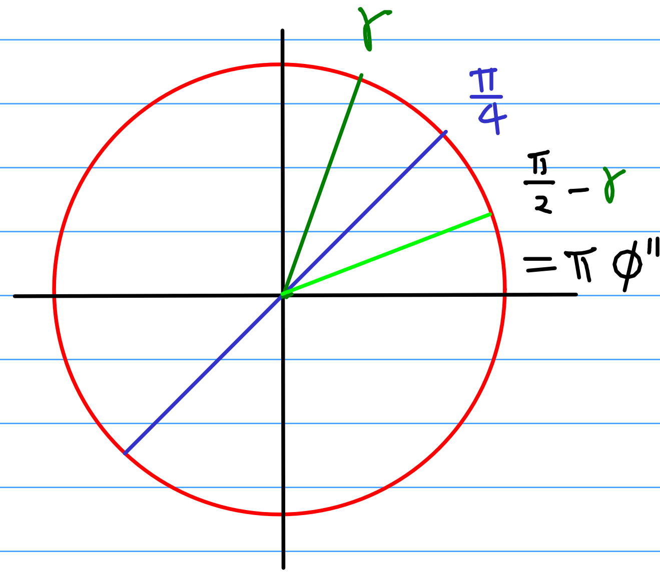
$$\theta \in [-\pi, +\pi] \Rightarrow \theta = [0, \tfrac{\pi}{2}]$$

$$\phi \in [-1, +1] \Rightarrow \phi' = [0, 0.5]$$

ϕ'									
	0	0	MSB3					. . .	

1st Quadrant



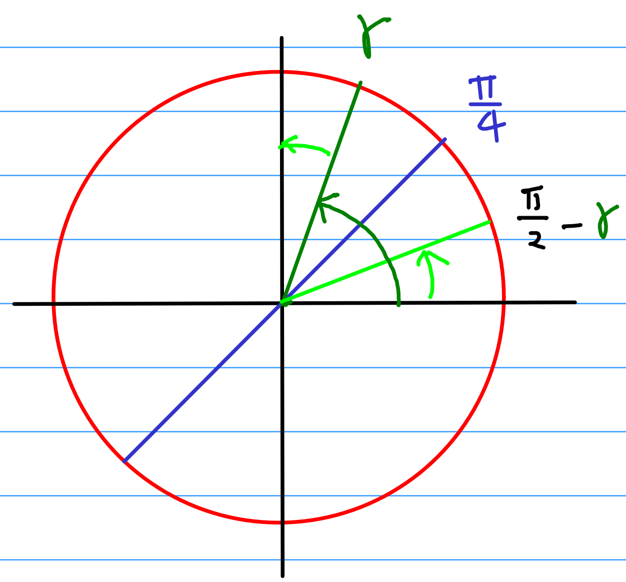


$\gamma > \frac{\pi}{4}$: Upper Half $(MSB_3 = 1)$

$\gamma < \frac{\pi}{4}$: Lower Half $(MSB_3 = 0)$

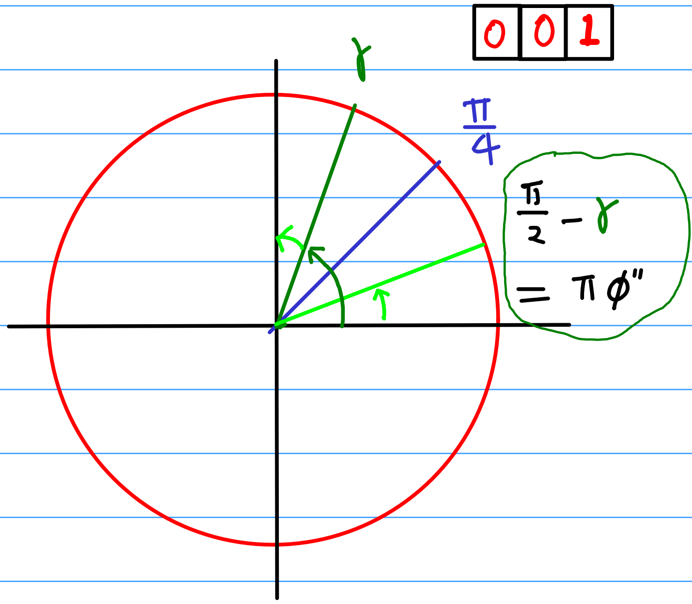
$$\cos\gamma = \sin\left(\frac{\pi}{2} - \gamma\right)$$

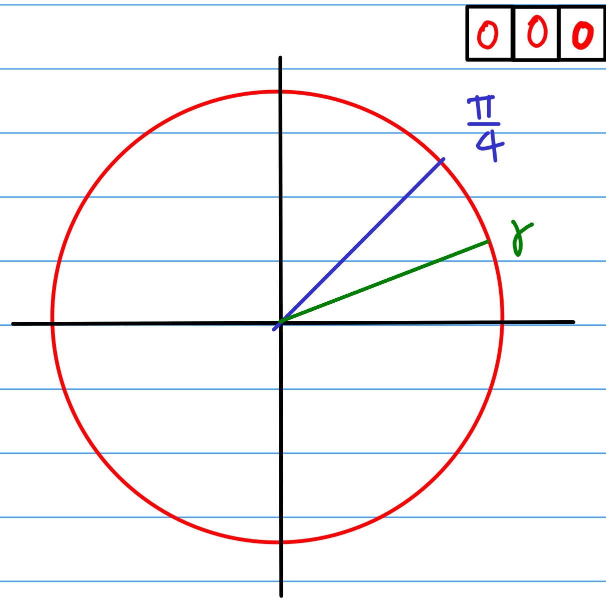
$$\sin\gamma = \cos\left(\frac{\pi}{2} - \gamma\right)$$



$\gamma > \frac{\pi}{4} \qquad \frac{\pi}{2} - \gamma < \frac{\pi}{4}.$

$\pi/4$ mirror





$\frac{\pi}{4} < \gamma < \frac{\pi}{2}$

$\phi'' = 0.5 - \phi'$

$0 < \gamma < \frac{\pi}{4}$

$\phi'' = \phi'$

$\theta = \pi\phi \rightarrow \theta' = \pi\phi' \rightarrow \theta'' = \pi\phi''$

$\theta \in [-\pi, +\pi] \rightarrow \theta' = [0, \pi/2] \rightarrow \theta'' = [0, \pi/4]$

$\phi \in [-1, +1] \rightarrow \phi' = [0, 0.5] \rightarrow \phi'' = [0, 0.25]$

ϕ'	0	0	MSB3					...	

$MSB3 = 1 \qquad \phi' > \frac{\pi}{4}$

$\phi'' = \frac{\pi}{2} - \phi'$

0.5	1	0	0					...	
$\ominus\ \phi'$	0	0	MSB3					...	

$$\begin{cases} MSB3 = 0 & \phi'' = \phi' \\ MSB_3 = 1 & \phi'' = 0.5 - \phi' \end{cases}$$

$\theta = \pi \phi''$ (Handwired Multiplier)

$0 < \theta < \frac{\pi}{4}$

$\phi \longrightarrow \phi' \longrightarrow \phi''$

1st Quad Lower Half

$$\theta = \pi\phi \longrightarrow \theta' = \pi\phi' \longrightarrow \theta'' = \pi\phi''$$

$$\theta \in [-\pi, +\pi] \longrightarrow \theta' = [0, \pi/2] \longrightarrow \theta'' = [0, \pi/4]$$

$$\phi \in [-1, +1] \longrightarrow \phi' = [0, 0.5] \longrightarrow \phi'' = [0, 0.25]$$

$\theta \in [-\pi, +\pi]$ $\quad$ $\boxed{\phi = \theta / \pi}$

$\downarrow$

$$\phi \in [-1, +1] \longrightarrow \phi' = [0, 0.5] \longrightarrow \phi'' = [0, 0.25]$$

$\downarrow$

$\boxed{\text{Compute}}$

$\downarrow$

$$\phi \in [-1, +1] \longleftarrow \phi' = [0, 0.5] \longleftarrow \phi'' = [0, 0.25]$$

$\downarrow$

$\theta \in [-\pi, +\pi]$ $\quad$ $\boxed{\theta = \pi\phi}$

* radian converter

$\boxed{\theta''} = \pi \boxed{\phi''}$

radian angle — normalized angle

$0 < \theta'' < \pi/4$ — $0 < \phi'' < 0.25$

The multiplication by π

→ could have used a hardwired multiplier
→ but don't have to use a multiplier at all

① in table lookup DDFS architecture
→ here, the multiplication by π is implicit

② in CORDIC architecture
the elementary angle are divided by π

$$\theta_k = \tan^{-1}(2^{-k}) / 2\pi$$

the direction of sub rotations are determined by the sign of angle difference.

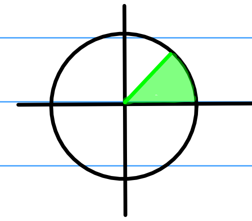
therefore the multiplication by π is <u>not</u> <u>necessary</u>

③ Sine / Cosine Generator

given angle θ (in radian)

$0 \leq \theta \leq \pi/4 < 1$

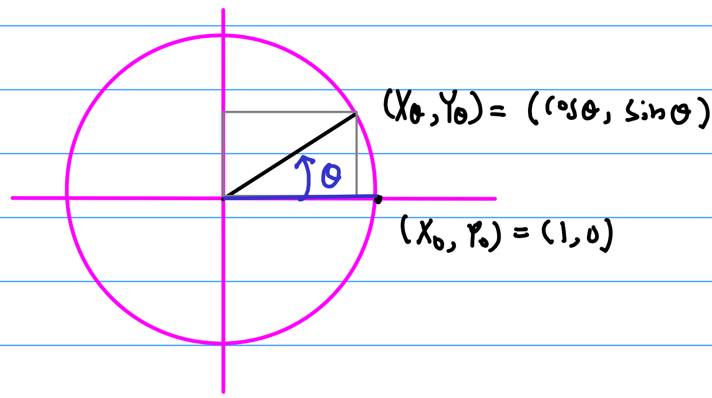
$\pi/4 \Downarrow 0.785398163$



compute $\boxed{\cos\theta}$, $\boxed{\sin\theta}$?

$$\begin{bmatrix} X_\theta \\ Y_\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} X_0 \\ Y_0 \end{bmatrix}$$

$$= \cos\theta \begin{bmatrix} 1 & -\tan\theta \\ \tan\theta & 1 \end{bmatrix} \begin{bmatrix} X_0 \\ Y_0 \end{bmatrix}$$



$$\begin{bmatrix} X_\theta \\ Y_\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} X_0 \\ Y_0 \end{bmatrix}$$

$$= \cos\theta \begin{bmatrix} 1 & -\tan\theta \\ \tan\theta & 1 \end{bmatrix} \begin{bmatrix} X_0 \\ Y_0 \end{bmatrix}$$

$$= \cos\theta \begin{bmatrix} 1 & -\tan\theta \\ \tan\theta & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

a **sequence of subrotations** of the priori **known** angle

Suppose: θ as a sequence of sub-rotation

$\{\theta_k\}$ the subrotation angles are known a priori

then $\theta = \sigma_0 \theta_0 + \sigma_1 \theta_1 + \cdots + \sigma_N \theta_N$

$\sigma_k = \{-1, 0, +1\}$

$$\theta = \sigma_0 \theta_0 + \sigma_1 \theta_1 + \cdots + \sigma_N \theta_N$$

$$\sigma_k = \{-1, 0, +1\}$$

$$\sigma_0 \theta_0 \rightarrow \cos(\sigma_0 \theta_0) \begin{bmatrix} 1 & -\tan(\sigma_0 \theta_0) \\ \tan(\sigma_0 \theta_0) & 1 \end{bmatrix}$$

$$\sigma_1 \theta_1 \rightarrow \cos(\sigma_1 \theta_1) \begin{bmatrix} 1 & -\tan(\sigma_1 \theta_1) \\ \tan(\sigma_1 \theta_1) & 1 \end{bmatrix}$$

$$\sigma_N \theta_N \rightarrow \cos(\sigma_N \theta_N) \begin{bmatrix} 1 & -\tan(\sigma_N \theta_N) \\ \tan(\sigma_N \theta_N) & 1 \end{bmatrix}$$

$$\theta = \sigma_0 \theta_0 + \sigma_1 \theta_1 + \cdots + \sigma_N \theta_N$$

$$\sigma_k = \{-1, 0, +1\}$$

$$\begin{bmatrix} X_\theta \\ Y_\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} X_0 \\ Y_0 \end{bmatrix}$$

$$= \cos(\sigma_0\theta_0) \begin{bmatrix} 1 & -\tan(\sigma_0\theta_0) \\ \tan(\sigma_0\theta_0) & 1 \end{bmatrix} \cos(\sigma_1\theta_1) \begin{bmatrix} 1 & -\tan(\sigma_1\theta_1) \\ \tan(\sigma_1\theta_1) & 1 \end{bmatrix}$$

$$\cdots \cos(\sigma_N\theta_N) \begin{bmatrix} 1 & -\tan(\sigma_N\theta_N) \\ \tan(\sigma_N\theta_N) & 1 \end{bmatrix}$$

$$= K \begin{bmatrix} 1 & -\tan(\sigma_0\theta_0) \\ \tan(\sigma_0\theta_0) & 1 \end{bmatrix} \begin{bmatrix} 1 & -\tan(\sigma_1\theta_1) \\ \tan(\sigma_1\theta_1) & 1 \end{bmatrix} \cdots \begin{bmatrix} 1 & -\tan(\sigma_N\theta_N) \\ \tan(\sigma_N\theta_N) & 1 \end{bmatrix}$$

$K = \cos(\sigma_0\theta_0)\cos(\sigma_1\theta_1)\cdots\cos(\sigma_N\theta_N)$ scale factor

$\sigma_k = +1$ positive angle rotation

$\sigma_k = -1$ negative angle rotation

$$\theta = \sigma_0 \theta_0 + \sigma_1 \theta_1 + \cdots + \sigma_N \theta_N$$

$$\sigma_k = \{-1, 0, +1\}$$

CORDIC Algorithm

$$\theta_k = \tan^{-1} 2^{-k}$$

$$\tan \theta_k = 2^{-k} \qquad\qquad \sigma_k = \{-1, +1\}$$

$$\tan \sigma_k \theta_k = \sigma_k 2^{-k}$$

$$K \begin{bmatrix} 1 & -\tan(\sigma_0\theta_0) \\ \tan(\sigma_0\theta_0) & 1 \end{bmatrix} \begin{bmatrix} 1 & -\tan(\sigma_1\theta_1) \\ \tan(\sigma_1\theta_1) & 1 \end{bmatrix} \cdots \begin{bmatrix} 1 & -\tan(\sigma_N\theta_N) \\ \tan(\sigma_N\theta_N) & 1 \end{bmatrix}$$

$$K = \cos(\sigma_0\theta_0)\cos(\sigma_1\theta_1)\cdots\cos(\sigma_N\theta_N)$$ scale factor

$$K \underbrace{\begin{bmatrix} 1 & -\sigma_0 2^{-0} \\ \sigma_0 2^{-0} & 1 \end{bmatrix}}_{\hookrightarrow \text{ shift-and-add}} \begin{bmatrix} 1 & -\sigma_1 2^{-1} \\ \sigma_1 2^{-1} & 1 \end{bmatrix} \cdots \begin{bmatrix} 1 & -\sigma_N 2^{-N} \\ \sigma_N 2^{-N} & 1 \end{bmatrix}$$

$$K = \cos(\theta_0)\cos(\theta_1)\cdots\cos(\theta_N)$$

constant ← each rotation
+/− rotation
is actually performed

$$\theta = \sigma_0 \theta_0 + \sigma_1 \theta_1 + \cdots + \sigma_N \theta_N$$

$$\sigma_k = \{-1, 0, +1\}$$

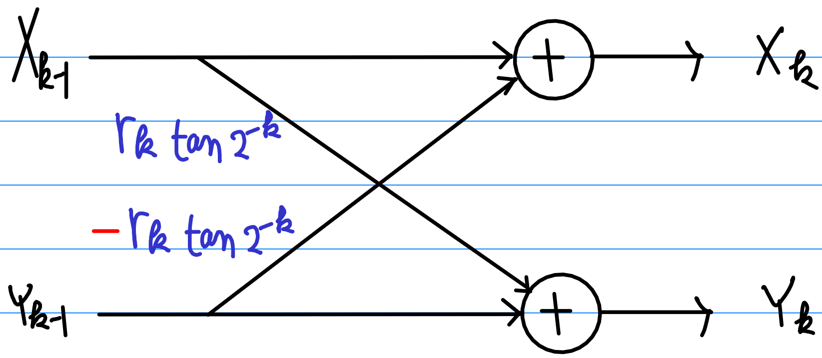
$$\begin{bmatrix} X_\theta \\ Y_\theta \end{bmatrix} = K \begin{bmatrix} 1 & -\tan \sigma_N \theta_N \\ \tan \sigma_N \theta_N & 1 \end{bmatrix} \cdots \begin{bmatrix} 1 & -\tan \sigma_0 \theta_0 \\ \tan \sigma_0 \theta_0 & 1 \end{bmatrix} \begin{bmatrix} X_0 \\ Y_0 \end{bmatrix}$$

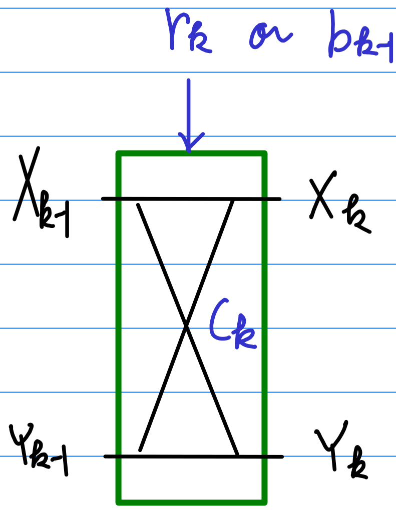
$$K = \cos \sigma_0 \theta_0 \cdot \cos \sigma_1 \theta_1 \cdots \cos \sigma_N \theta_N$$

Subrotation

$$X_{k+1} = X_k - (r_k \tan \theta_k) Y_k$$

$$Y_{k+1} = Y_k + (r_k \tan \theta_k) X_k$$





Butterfly