Differentiation of Continuous Functions

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Sep 02, 2024

Based on
Introduction to Matrix Algebra, Autar Kaw
https://ma.mathforcollege.com

Outline

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1. Approximations of a first derivative
   - Forward Difference Approximation
   - Backward Difference Approximation
   - Taylor Series
   - Central Divided Difference

Forward Difference Approximation (1)

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

for a finite $\Delta x > 0$

$$f'(x) \approx \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Forward Difference Approximation (2)

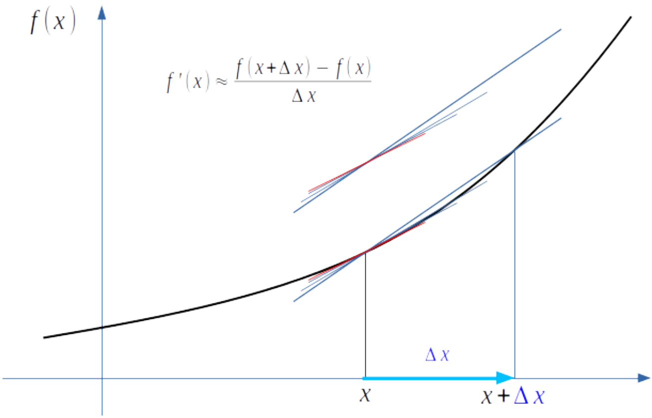

Figure: forward difference approximation

Forward Difference Approximation (3)

a forward difference approximation
as you are taking a point forward from x.

To find the value of $f'(x)$ at $x = x_i$,
we may choose another point Δx forwad as $x = x_{i+1}$.

$$f'(x) \approx \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$\begin{aligned} f'(x_i) &\approx \frac{f(x_{i+1}) - f(x_i)}{\Delta x} \\ &= \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i} \end{aligned}$$

Outline

Backward Difference Approximation (1a)

forward difference approximation

for a finite $\Delta x > 0$,

$$f'(x) \approx \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

backward difference approximation

for a finite $\Delta x < 0$, then $-\Delta x > 0$,

$$f'(x) \approx \frac{f(x-\Delta x) - f(x)}{-\Delta x}$$

$$= \frac{f(x) - f(x-\Delta x)}{\Delta x}$$

Backward Difference Approximation (1b)

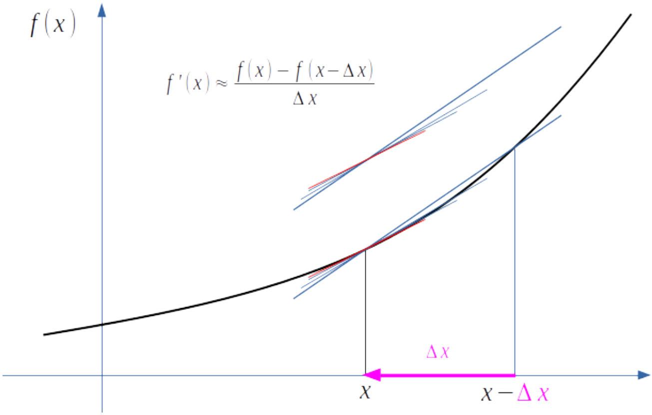

Figure: backward difference approximation (a)

Backward Difference Approximation (2a)

forward difference approximation

for a finite $\Delta x > 0$,

$$f'(x) \approx \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

backward difference approximation

for a finite $\Delta x > 0$, then $-\Delta x < 0$,

$$f'(x) \approx \frac{f(x) - f(x-\Delta x)}{x - (x - \Delta x)} = \frac{f(x) - f(x-\Delta x)}{\Delta x}$$

Backward Difference Approximation (2b)

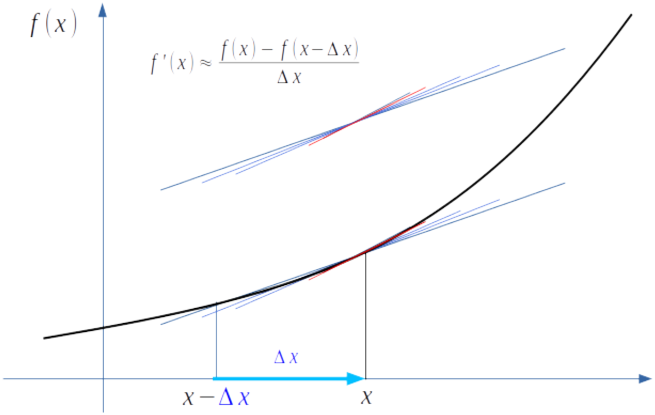

Figure: backward difference approximation (b)

Backward Difference Approximation (3)

a backward difference approximation
as you are taking a point backward from x.

To find the value of $f'(x)$ at $x = x_i$,
we may choose another point Δx backwad as $x = x_{i-1}$.

$$f'(x) \approx \frac{f(x) - f(x - \Delta x)}{\Delta x}$$

$$f'(x_i) \approx \frac{f(x_i) - f(x_{i-1})}{\Delta x}$$
$$= \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}$$

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Taylor Series (1)

the Taylor series of a function $f(x)$,
that is infinitely differentiable at a point a is the power series

$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$

Taylor Series (2)

If $f(x)$ is given by a convergent power series
in an open disk centred at a,
it is said to be *analytic* in this region.

Thus for x in this region,
f is given by a convergent power series

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$

Deriving Forward Difference Approximation (1)

A Taylor expansion approximate $f(x)$ using $f(a), f'(a), f''(a), \cdots,$

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$

Let $x_i = a$ and $x_{i+1} = x$

$$f(x_{i+1}) = f(x_i) + f'(x_i)(x_{i+1} - x_i) + \frac{f''(x_i)}{2!}(x_{i+1} - x_i)^2 + \cdots$$

Substituting for convenience $\Delta x = x_{i+1} - x_i$

$$f(x_{i+1}) = f(x_i) + f'(x_i)(\Delta x) + \frac{f''(x_i)}{2!}(\Delta x)^2 + \cdots$$

Deriving Forward Difference Approximation (2)

$$f(x_{i+1}) = f(x_i) + f'(x_i)(\Delta x) + \frac{f''(x_i)}{2!}(\Delta x)^2 + \cdots$$

$$f(x_{i+1}) - f(x_i) - \frac{f''(x_i)}{2!}(\Delta x)^2 - \cdots = f'(x_i)(\Delta x)$$

$$\frac{f(x_{i+1}) - f(x_i)}{\Delta x} - \frac{f''(x_i)}{2!}(\Delta x) - \cdots = f'(x_i)$$

$$\frac{f(x_{i+1}) - f(x_i)}{\Delta x} + O(\Delta x) = f'(x_i)$$

Deriving Forward Difference Approximation (3)

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{\Delta x} + O(\Delta x)$$

the $O(\Delta x)$ term shows that
the error in the approximation is of the order of Δx

now derive from Taylor series the formula
for backward divided difference approximation of the first derivative

both forward and backward divided difference approximation
of the first derivative are accurate on the order of $O(\Delta x)$

to get better approximations?
another method to approximate the first derivative is called
the Central divided difference approximation of the first derivative.

Deriving Backward Difference Approximation (1)

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \cdots$$

Let $x_i = a$ and $x_{i-1} = x$

$$f(x_{i-1}) = f(x_i) + f'(x_i)(x_{i-1} - x_i) + \frac{f''(x_i)}{2!}(x_{i-1} - x_i)^2 + \cdots$$

Substituting for convenience $\Delta x = x_i - x_{i+1}$

$$f(x_{i-1}) = f(x_i) - f'(x_i)(\Delta x) + \frac{f''(x_i)}{2!}(\Delta x)^2 - \cdots$$

Deriving Backward Difference Approximation (2)

$$f(x_{i-1}) = f(x_i) - f'(x_i)(\Delta x) + \frac{f''(x_i)}{2!}(\Delta x)^2 - \cdots$$

$$f'(x_i)(\Delta x) = f(x_i) - f(x_{i-1}) + \frac{f''(x_i)}{2!}(\Delta x)^2 - \cdots$$

$$f'(x_i) = \frac{f(x_i) - f(x_{i-1})}{\Delta x} + \frac{f''(x_i)}{2!}(\Delta x) - \cdots$$

$=$

$$f'(x_i) = \frac{f(x_i) - f(x_{i-1})}{\Delta x} + O(\Delta x)$$

Deriving Central Divide Difference Approximation (1)

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$

Let $x_i = a$ and $x_{i+1} = x$, and substitute $\Delta x = x_{i+1} - x_i$

$$f(x_{i+1}) = f(x_i) + f'(x_i)(x_{i+1} - x_i) + \frac{f''(x_i)}{2!}(x_{i+1} - x_i)^2 + \cdots$$

Let $x_i = a$ and $x_{i-1} = x$, and substitute $\Delta x = x_i - x_{i-1}$

$$f(x_{i-1}) = f(x_i) + f'(x_i)(x_{i-1} - x_i) + \frac{f''(x_i)}{2!}(x_{i-1} - x_i)^2 + \cdots$$

Deriving Central Divide Approximation

$$f(x_{i+1}) = f(x_i) + f'(x_i)(\Delta x) + \frac{f''(x_i)}{2!}(\Delta x)^2 + \frac{f^{(3)}(x_i)}{3!}(\Delta x)^3 + \cdots$$
$$f(x_{i-1}) = f(x_i) - f'(x_i)(\Delta x) + \frac{f''(x_i)}{2!}(\Delta x)^2 - \frac{f^{(3)}(x_i)}{3!}(\Delta x)^3 \cdots$$

subtracting eq(2) from eq(1)

$$f(x_{i+1}) - f(x_{i-1}) = 2f'(x_i)(\Delta x) + \frac{2f^{(3)}(x_i)}{3!}(\Delta x)^3 + \cdots$$
$$2f'(x_i)(\Delta x) = f(x_{i+1}) - f(x_{i-1}) - \frac{2f^{(3)}(x_i)}{3!}(\Delta x)^3 - \cdots$$
$$f'(x_i) = \frac{f(x_{i+1}) - f(x_{i-1})}{2(\Delta x)} - \frac{f^{(3)}(x_i)}{3!}(\Delta x)^2 - \cdots$$
$$f'(x_i) = \frac{f(x_{i+1}) - f(x_{i-1})}{2\Delta x} + O\left((\Delta x)^2\right)$$

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Central Divided Approximation

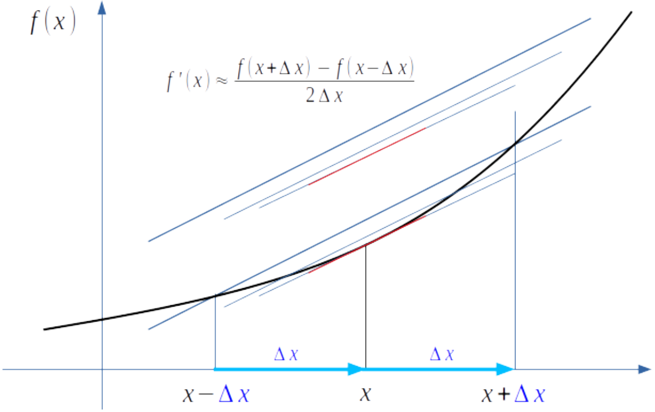

Figure: central difference approximation

Tangent Lines

- as $h \to 0$, $\mathrm{Q} \to \mathrm{P}$
  and the secant line $\to$ the tangent line
- the slope of the tangent line

$$m_{tangent} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{(a+h) - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$