Power Spectral Density - Continuous Time

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February 6, 2021

Based on
Probability, Random Variables and Random Signal Principles,
P.Z. Peebles,Jr. and B. Shi

Energy and average power in time domain
power spectral density for continuous time signals

Energy, Average Power – deterministic, time domain

a deterministic signal $x(t)$

$$x_T(t) = \begin{cases} x(t) & -T < t < T \\ 0 & otherwise \end{cases}$$

the **energy**

$$E(T) = \int_{-T}^{+T} x^2(t)dt = \int_{-\infty}^{+\infty} x_T^2(t)dt$$

the **average power**

$$P(T) = \frac{1}{2T}\int_{-T}^{+T} x^2(t)dt = \frac{1}{2T}\int_{-\infty}^{+\infty} x_T^2(t)dt$$

Fourier Transform Pair $x(t) \Longleftrightarrow X(\omega)$

Fourier transform

$$X(\omega) = \int\limits_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

a deterministic signal $x(t)$

$$x(t) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} x(\omega)e^{j\omega t}d\omega$$

Fourier transform of $x_T(t)$
for continuous time signals

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bounded duration, bounded variation

for a finite T, $x_T(t)$ is assumed to have bounded variation

$$\int_{-T}^{+T} |x(t)| dt < \infty$$

the Fourier transform of $x_T(t)$

$$X_T(\omega) = \int_{-\infty}^{+\infty} x_T(t) e^{-j\omega t} dt$$

$$= \int_{-T}^{+T} x(t) e^{-j\omega t} dt$$

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deterministic $X_T(\omega)$ v.s. random $X_T(\omega)$

a **deterministic** sample signal $x_T(t)$

$$x_T(t) \Longleftrightarrow X_T(\omega)$$

a **random process** signal $X_T(t)$

$$X_T(t) \Longleftrightarrow X_T(\omega)$$

for a deterministic $x_T(t)$

a **deterministic** sample signal $x_T(t)$

$$\int_{-\infty}^{+\infty} x_T(\tau) x_T^*(\tau) d\tau = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X_T(\omega) X_T^*(\omega) d\omega$$

$$\int_{-\infty}^{+\infty} |x_T(\tau)|^2 d\tau = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |X_T(\omega)|^2 d\omega$$

Parseval's theorem (II)
power spectral density for continuous time signals

> **for a deterministic $x_T(t)$ v.s. a random $X_T(t)$**
>
> - a **deterministic** signal $x_T(t) \Longleftrightarrow X_T(\omega)$
>
> $$\int_{-\infty}^{+\infty} |x_T(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |X_T(\omega)|^2 d\omega$$
>
> - a **random** signal $X_T(t) \Longleftrightarrow X_T(\omega)$
>
> $$\int_{-\infty}^{+\infty} E\left[|X_T(t)|^2\right] dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} E\left[|X_T(\omega)|^2\right] d\omega$$

Energy, Average Power – Parseval's theorem applied

a deterministic signal $x_T(t)$

$$x_T(t) = \begin{cases} x(t) & -T < t < T \\ 0 & \text{otherwise} \end{cases} \qquad x_T(t) \Longleftrightarrow X_T(\omega)$$

the **energy** by Parseval's theorem

$$E(T) = \int_{-T}^{+T} x^2(t)dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |X_T(\omega)|^2 d\omega$$

the **average power** by Parseval's theorem

$$P(T) = \frac{1}{2T} \int_{-T}^{+T} x^2(t)dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{|X_T(\omega)|^2}{2T} d\omega$$

deterministic $x_T(t) \Longleftrightarrow X_T(\omega)$

the **energy** for the **deterministic** $X_T(\omega)$ in $x_T(t) \Longleftrightarrow X_T(\omega)$

$$E(T) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |X_T(\omega)|^2 d\omega$$

the **average power** for the **deterministic** $X_T(\omega)$

$$P(T) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{|X_T(\omega)|^2}{2T} d\omega$$

the **power spectral density** for the **deterministic** $X_T(\omega)$

$$\lim_{T \to \infty} \frac{|X_T(\omega)|^2}{2T}$$

random $X_T(t) \Longleftrightarrow X_T(\omega)$

the **energy** for the **random** $X_T(\omega)$ in $X_T(t) \Longleftrightarrow X_T(\omega)$

$$E(T) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} E\big[|X_T(\omega)|^2\big] d\omega$$

the **average power** for the **random** $X_T(\omega)$

$$P(T) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{E\big[|X_T(\omega)|^2\big]}{2T} d\omega$$

the **power spectral density** for the **random** $X_T(\omega)$

$$\lim_{T \to \infty} \frac{E\big[|X_T(\omega)|^2\big]}{2T}$$

$$P(T) = \frac{1}{2T} \int_{-T}^{+T} x^2(t) dt$$

- not the average power in a **random process**
  only the power in <u>one</u> **sample function**

  - to obtain the **average power** over <u>all possible</u> **realizations**,
    replace $x(t)$ by $X(t)$
    take the **expected value** of $x^2(t)$, that is $TE\left[X^2(t)\right]$
  - then, the **average power** is a **random variable**
    with respect to the **random process** $X(t)$

- not the average power in an <u>entire</u> **sample function**
  - take $T \rightarrow \infty$ to include all power in the **ensemble** member

$$P(T) = \frac{1}{2T} \int_{-T}^{+T} x^2(t) dt$$

- replace $x(t)$ by the **random variable** $X(t)$
- take the **expected value** of $x^2(t)$, that is $E\left[X^2(t)\right]$

$$P(T) = \frac{1}{2T} \int_{-T}^{+T} E\left[X^2(t)\right] dt$$

- take $T \to \infty$ to include all power

$$\boxed{P_{XX} = \lim_{T \to \infty} P(T)} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{+T} E\left[X^2(t)\right] dt$$

The time average

$$A_T[\bullet] = \frac{1}{2T} \int\limits_{-T}^{T} [\bullet] dt \qquad A[\bullet] = \lim_{T \to \infty} \frac{1}{2T} \int\limits_{-T}^{T} [\bullet] dt$$

time average and sample average operations

$$\boxed{P_{XX} = \lim_{T \to \infty} P(T)} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{+T} E\left[X^2(t)\right] dt$$

$$= \lim_{T \to \infty} A_T \left[E\left[X^2(t)\right]\right]$$

$$= A\left[E\left[X^2(t)\right]\right]$$

for deterministic and random signals

the **average power** $P(T)$ for a **deterministic** signal $x(t)$

$$P(T) = \frac{1}{2T} \int_{-T}^{+T} x^2(t) dt$$

the **average power** P_{XX} for a **random process** $X(t)$

$$P_{XX} = \lim_{T \to \infty} P(T)$$
$$= \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{+T} E\left[X^2(t)\right] dt$$
$$= A\left[E\left[X^2(t)\right]\right]$$

Power Spectral Density $S_{XX}(\omega)$
power spectral density for continuous time signals

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the average power via power density

the average power P_{XX} for the **random process** $X_T(\omega)$

$$P_{XX} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \boxed{\lim_{T \to \infty} \frac{E\left[|X_T(\omega)|^2\right]}{2T}} d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \boxed{S_{XX}(\omega)} d\omega$$

the **Power Spectral Density (PSD)** $S_{XX}(\omega)$

$$\boxed{S_{XX}(\omega)} = \boxed{\lim_{T \to \infty} \frac{E\left[|X_T(\omega)|^2\right]}{2T}}$$

- $S_{XX}(\omega) \geq 0$

- $S_{XX}(-\omega) = S_{XX}(\omega)$ $X(t)$ real

- $S_{XX}(\omega)$ real

- $\frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{XX}(\omega) d\omega = A\left[E\left[X^2(t)\right]\right]$

- $S_{\dot{X}\dot{X}}(\omega) = \omega^2 S_{XX}(\omega)$

- $\frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{XX}(\omega) e^{j\omega t} d\omega = A[R_{XX}(t, t+\tau)]$

- $S_{XX}(\omega) = \int_{-\infty}^{+\infty} A[R_{XX}(t, t+\tau)] e^{-j\omega\tau} d\tau$

the average power P_{xx} and the inverse Fourier transform of $S_{XX}(\omega)$

the **average power** related equation

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} S_{XX}(\omega)d\omega = A\left[E\left[X^2(t)\right]\right]$$

the **autocorrelation** related equation

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} S_{XX}(\omega)e^{j\omega t}d\omega = A[R_{XX}(t, t+\tau)]$$

the average power P_{xx}

the average power related equation

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{XX}(\omega)d\omega = A\left[E\left[X^2(t)\right]\right]$$

- a **random process** $X(t)$ in time domain
- a **random process** $X(\omega)$ in frequency domain

$$X(t) = \lim_{T \to \infty} X_T(t) \qquad X(\omega) = \lim_{T \to \infty} X_T(\omega)$$

- Parseval's theorem over $X_T(t) \Longleftrightarrow X_T(\omega)$

Average power P_{XX} in time / frequency domain

power spectral density for continuous time signals

Average power P_{XX} using $X_T(t)$ and $X_T(\omega)$

- Using a **random process** $X_T(t)$ in time domain

$$P_{XX} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{+T} E\left[X^2(t)\right] dt$$

$$= \lim_{T \to \infty} \frac{1}{2T} \int_{-\infty}^{+\infty} E\left[X_T^2(t)\right] dt$$

$$= \lim_{T \to \infty} A_T \left[E\left[X^2(t)\right]\right] = \boxed{A\left[E\left[X^2(t)\right]\right]}$$

- Using a **random process** $X_T(\omega)$ in frequency domain

$$P_{XX} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \boxed{\lim_{T \to \infty} \frac{E\left[|X_T(\omega)|^2\right]}{2T}} d\omega$$

$$= \boxed{\frac{1}{2\pi} \int_{-\infty}^{+\infty} \boxed{S_{XX}(\omega)} d\omega}$$

the Inverse Fourier transform of $S_{XX}(\omega)$

the average power related equation

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{XX}(\omega)e^{j\omega t}d\omega = A[R_{XX}(t, t+\tau)]$$

- auto-correlation function

$$R_{XX}(t, t+\tau) = E[X(t)X(t+\tau)] \qquad \Rightarrow R_{XX}(\tau)$$

- a **random process** $X(t)$ in time domain
- a **random process** $X(\omega)$ in frequency domain

Fourier transform pairs

- $A[R_{XX}(t, t+\tau)] \Longleftrightarrow S_{XX}(\omega)$

$$S_{XX}(\omega) = \int_{-\infty}^{+\infty} A[R_{XX}(t, t+\tau)] e^{-j\omega\tau} d\tau$$

$$A[R_{XX}(t, t+\tau)] = \frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{XX}(\omega) e^{j\omega t} d\omega$$

- $R_{XX}(\tau) \Longleftrightarrow S_{XX}(\omega)$

$$S_{XX}(\omega) = \int_{-\infty}^{+\infty} R_{XX}(\tau) e^{-j\omega\tau} d\tau$$

$$R_{XX}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{XX}(\omega) e^{+j\omega\tau} d\omega$$

for a WSS $X(t)$, $A[R_{XX}(t, t+\tau)] = R_{XX}(\tau)$

$S_{XX}(\omega)$ and $R_{XX}(\tau)$

the power spectral density

$$S_{XX}(\omega) = \int_{-\infty}^{+\infty} R_{XX}(\tau) e^{-j\omega\tau} d\tau$$

the auto-correlation function

$$R_{XX}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{XX}(\omega) e^{+j\omega\tau} d\omega$$

Fourier transform of $\frac{d^n}{dt^n}x(t)$

$$x(t) \Longleftrightarrow X(\omega)$$
$$\frac{d^n}{dt^n}x(t) \Longleftrightarrow (j\omega)^n X(\omega)$$

$$X_T(t) \Longleftrightarrow X_T(\omega)$$
$$Y(t) = \quad \frac{d}{dt}X_T(t) \Longleftrightarrow (j\omega)X_T(\omega) \qquad = Y(\omega)$$

PSD and Auto-Correlation of a Derivative Function
power spectral density for continuous time signals

$S_{\dot{X}\dot{X}}(\omega)$ and $S_{XX}(\omega)$

$$S_{XX}(\omega) = \lim_{T \to \infty} \frac{E\left[|X_T(\omega)|^2\right]}{2T}$$

$$S_{\dot{X}\dot{X}}(\omega) = \lim_{T \to \infty} \frac{E\left[|\dot{X}_T(\omega)|^2\right]}{2T}$$

$$= \lim_{T \to \infty} \frac{\omega^2 E\left[|X_T(\omega)|^2\right]}{2T}$$

$$= \omega^2 S_{XX}(\omega)$$

$S_{\dot{X}\dot{X}}(\omega)$ and $S_{XX}(\omega)$

$$S_{XX}(\omega) = \lim_{T\to\infty} \frac{E\left[|X_T(\omega)|^2\right]}{2T}$$

$$Y(t) = \dot{X}_T(t) = \frac{d}{dt} X_T(t)$$

$$|\dot{X}(\omega)|^2 = \dot{X}_T(\omega)\dot{X}_T^*(\omega)$$

$$S_{YY}(\omega) = \lim_{T\to\infty} \frac{E\left[|Y(\omega)|^2\right]}{2T}$$

$$= \lim_{T\to\infty} \frac{\omega^2 E\left[|X(\omega)|^2\right]}{2T}$$

$$= \omega^2 S_{XX}(\omega)$$

Definition

Fourier transform of an autocorrelation functions

$$S_{\dot{X}\dot{X}}(\omega) = \int_{-\infty}^{+\infty} R_{\dot{X}\dot{X}}(\tau)e^{-j\omega\tau}d\tau$$

$$\omega^2 S_{XX}(\omega) = \int_{-\infty}^{+\infty} \omega^2 R_{XX}(\tau)e^{-j\omega\tau}d\tau$$

- auto-correlation function

$$R_{XX}(t, t+\tau) = E[X(t)X(t+\tau)] \Rightarrow R_{XX}(\tau)$$

$$R_{\dot{X}\dot{X}}(t, t+\tau) = E\left[\dot{X}(t)\dot{X}(t+\tau)\right] \Rightarrow R_{\dot{X}\dot{X}}(\tau)$$

- a **random process** $X(t)$ in time domain
- $\dot{X}(t) = \frac{d}{dt}X(t)$: the derivative of $X(t)$

Definition

the standard deviation is
a measure of the spread in a density function.
the analogous quantity for the normalized power spectral density is
a measure of its spread that we call the rms bandwidth
(root-mean-square)

$$W_{rms}^2 = \frac{\int_{-\infty}^{+\infty} \omega^2 S_{XX}(\omega) d\omega}{\int_{-\infty}^{+\infty} S_{XX}(\omega) d\omega}$$

Definition

the mean frequence $\bar{\omega}_0$

$$\bar{\omega}_0 = \frac{\int_{-\infty}^{+\infty} \omega S_{XX}(\omega) d\omega}{\int_{-\infty}^{+\infty} S_{XX}(\omega) d\omega}$$

the rms bandwidth

$$W_{rms}^2 = \frac{4 \int_{-\infty}^{+\infty} (\omega - \bar{\omega}_0)^2 S_{XX}(\omega) d\omega}{\int_{-\infty}^{+\infty} S_{XX}(\omega) d\omega}$$