Laurent Series and z-Transform - Geometric Series Applications

A

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Unshifted Geometric Sequences

Causal

$\frac{1}{1-\square z} \Rightarrow u(n\ \ )$

Anti-causal

$\frac{1}{1-\square z^{-1}} \Rightarrow u(-n\ \ )$

Positive Exponent

$az,\ a^{-1}z^{-1} \Rightarrow a^{n}$

Negative Exponent

$a^{-1}z,\ az^{-1} \Rightarrow a^{-n}$

Positive Exponent

unshifted (1)

$\frac{1}{1-az^{-1}}$ $|z|<a^{-1}$ $a^{n}\ u(n)$

complementary (7)

$\frac{az^{-1}}{1-az^{-1}}$ $|z|<a^{-1}$ $a^{n}\ u(n-1)$

unshifted (5)

$-\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}}$ $|z|>a^{-1}$ $a^{n}\ u(-n-1)$

complementary (3)

$-\frac{1}{1-a^{-1}z^{-1}}$ $|z|>a^{-1}$ $a^{n}\ u(-n)$

Negative Exponent

unshifted (2)

$\frac{1}{1-a^{-1}z^{-1}}$ $|z|<a^{-1}$ $a^{-n}\ u(n)$

complementary (8)

$\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}}$ $|z|<a^{-1}$ $a^{-n}\ u(n-1)$

unshifted (6)

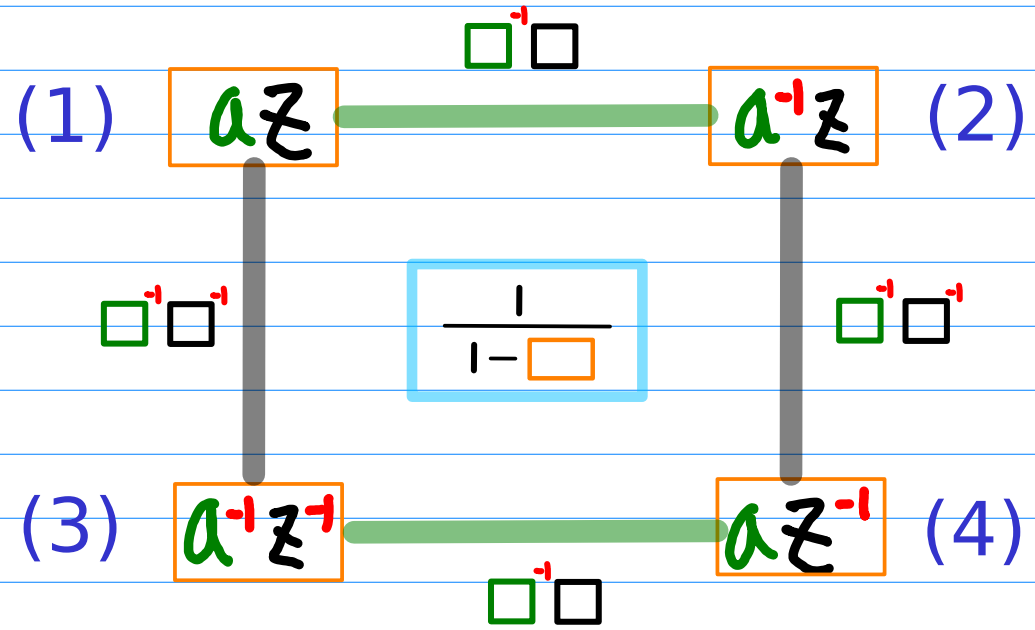
$-\frac{az^{-1}}{1-az^{-1}}$ $|z|>a^{-1}$ $a^{-n}\ u(-n-1)$

complementary (4)

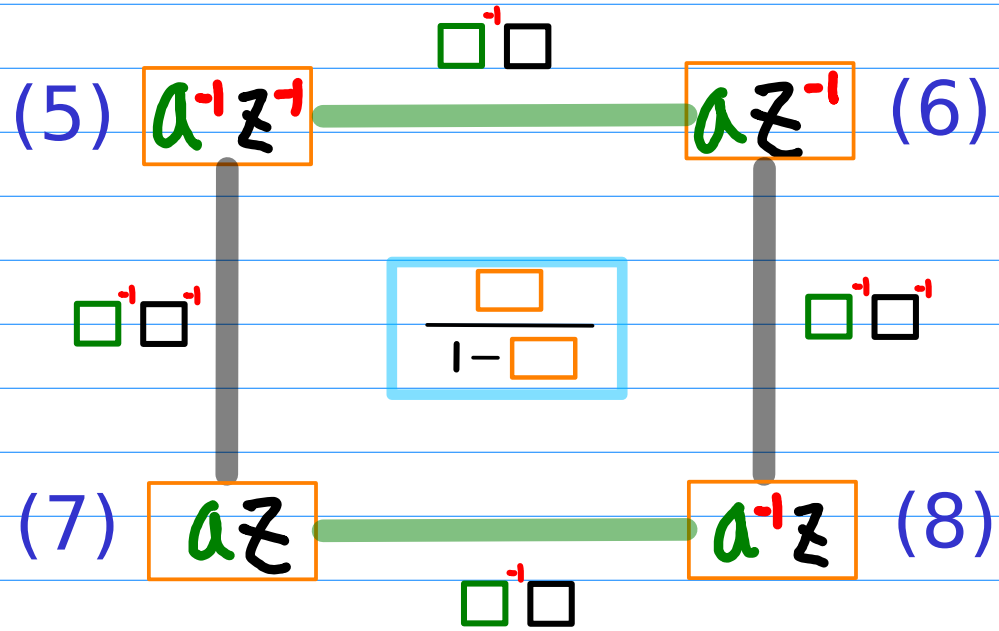
$-\frac{1}{1-az^{-1}}$ $|z|>a$ $a^{-n}\ u(-n)$

Numbering the basic elements - (1) CR

unshifted geometric sequences

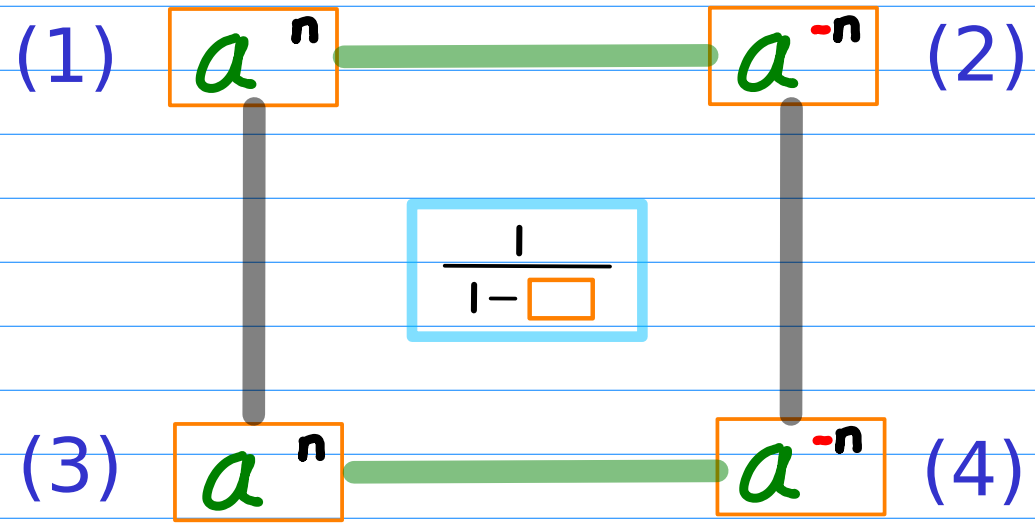


complementary geometric sequences

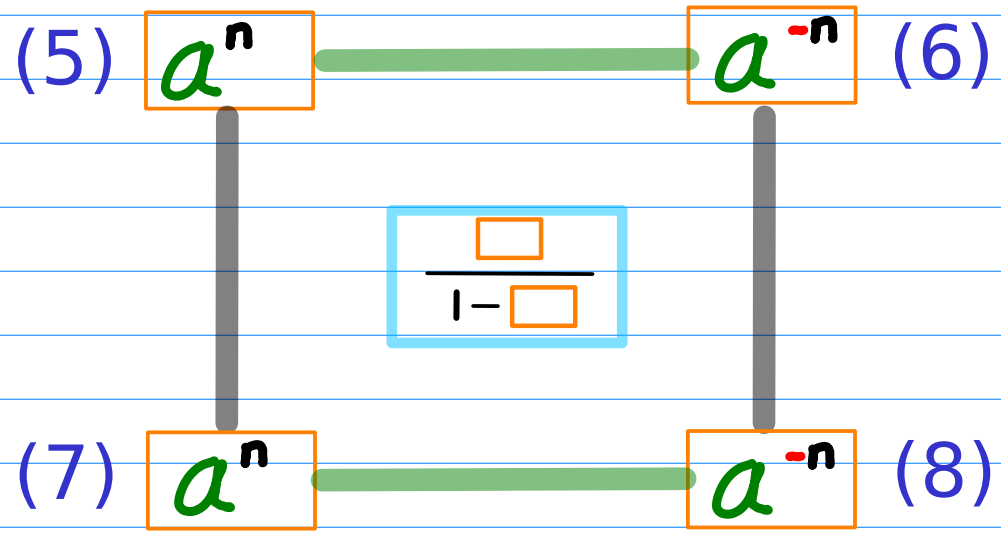


Numbering the basic elements - (2) Power

unshifted geometric sequences

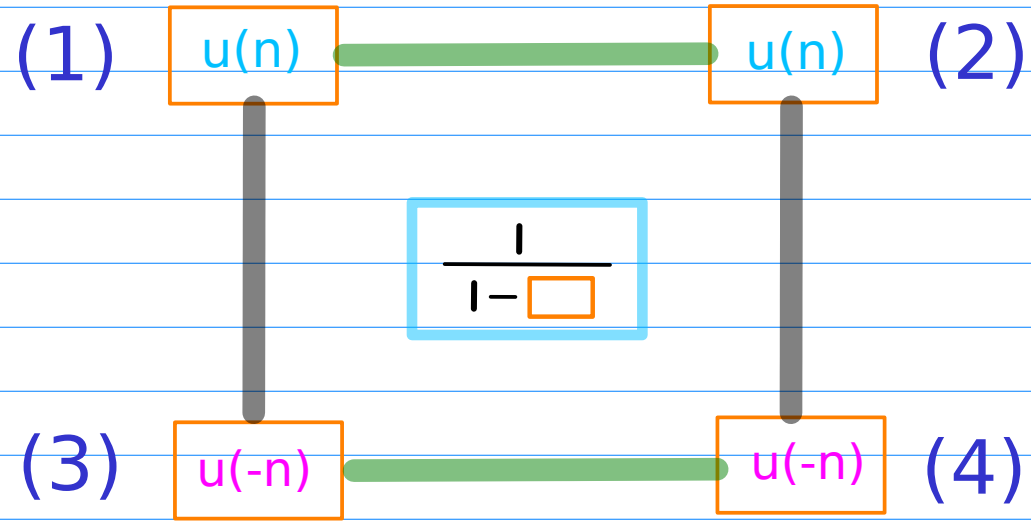


complementary geometric sequences

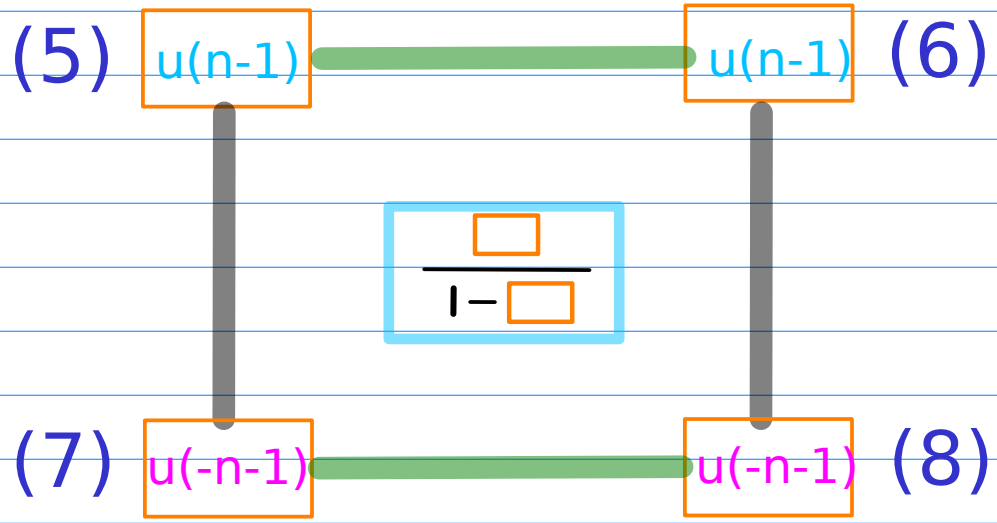


Numbering the basic elements - (3) Range

unshifted geometric sequences

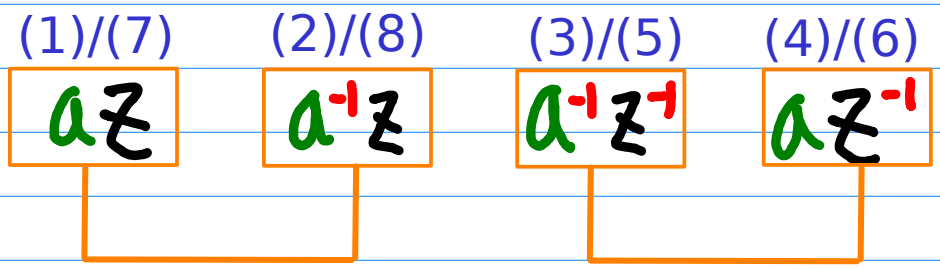


complementary geometric sequences

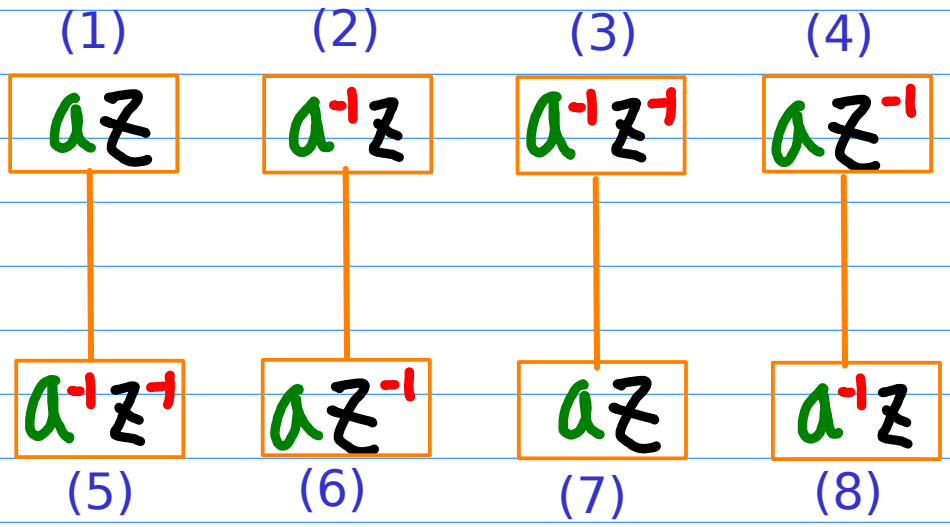


Inverse Relations

inverse power $\square^{-1}\square$



complementary $\square^{-1}\square^{-1}$



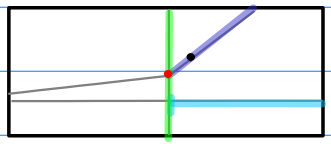
Geometric Series Form Combinations with a unit start term **unshifted**

(1) $+\dfrac{1}{1-az}$ a^n u(n)

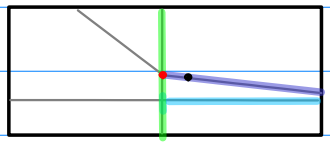
(2) $+\dfrac{1}{1-a^{-1}z}$ a^{-n} u(n)

(3) $-\dfrac{1}{1-a^{-1}z^{-1}}$ $-a^n$ u(-n)

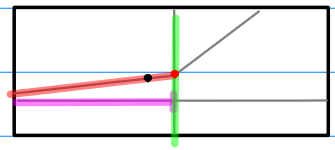
(4) $-\dfrac{1}{1-az^{-1}}$ $-a^{-n}$ u(-n)



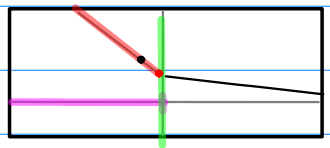
a^n u(n)



a^{-n} u(n)



a^n u(-n)



a^{-n} u(-n)

Geometric Series Form Combinations with a common-ratio start term

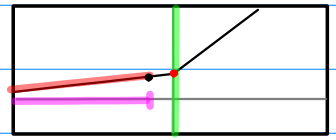
unshifted, complementary

(5) $$-\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}} \quad -a^n\,u(-n-1)$$

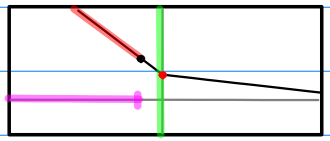
(6) $$-\frac{az^{-1}}{1-az^{-1}} \quad -a^{-n}u(-n-1)$$

(7) $$+\frac{az}{1-az} \quad a^n\,u(n-1)$$

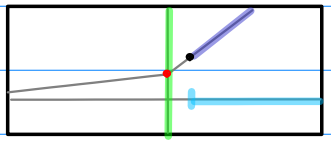
(8) $$+\frac{a^{-1}z}{1-a^{-1}z} \quad a^{-n}\,u(n-1)$$



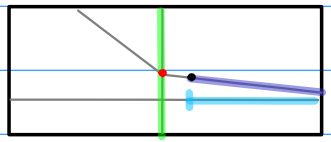
$a^n\,u(-n-1)$



$a^{-n}\,u(-n-1)$

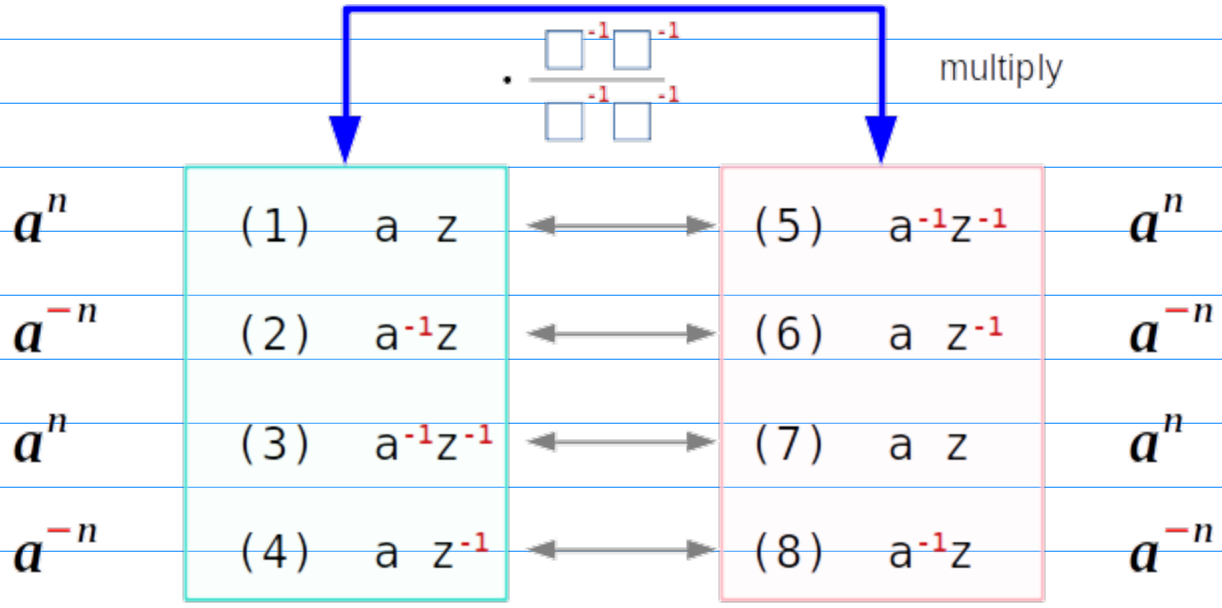


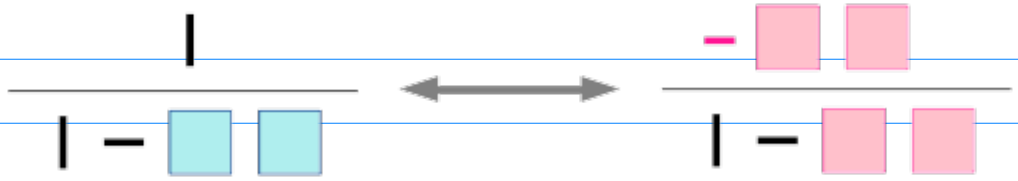
$a^n\,u(n-1)$



$a^{-n}\,u(n-1)$

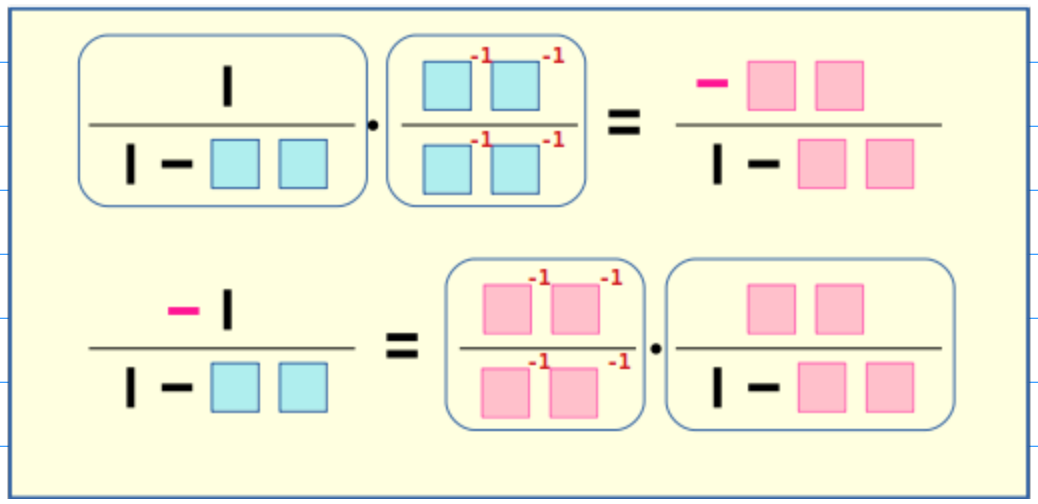
Complementary Range Relations - CR only



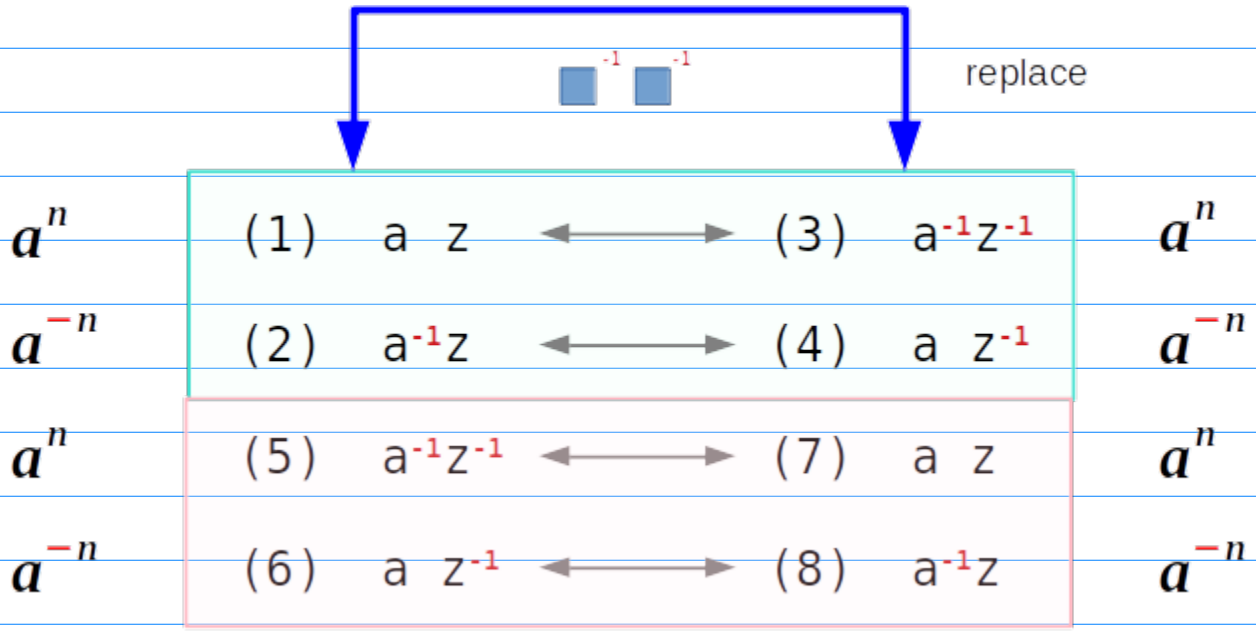


a unit start term

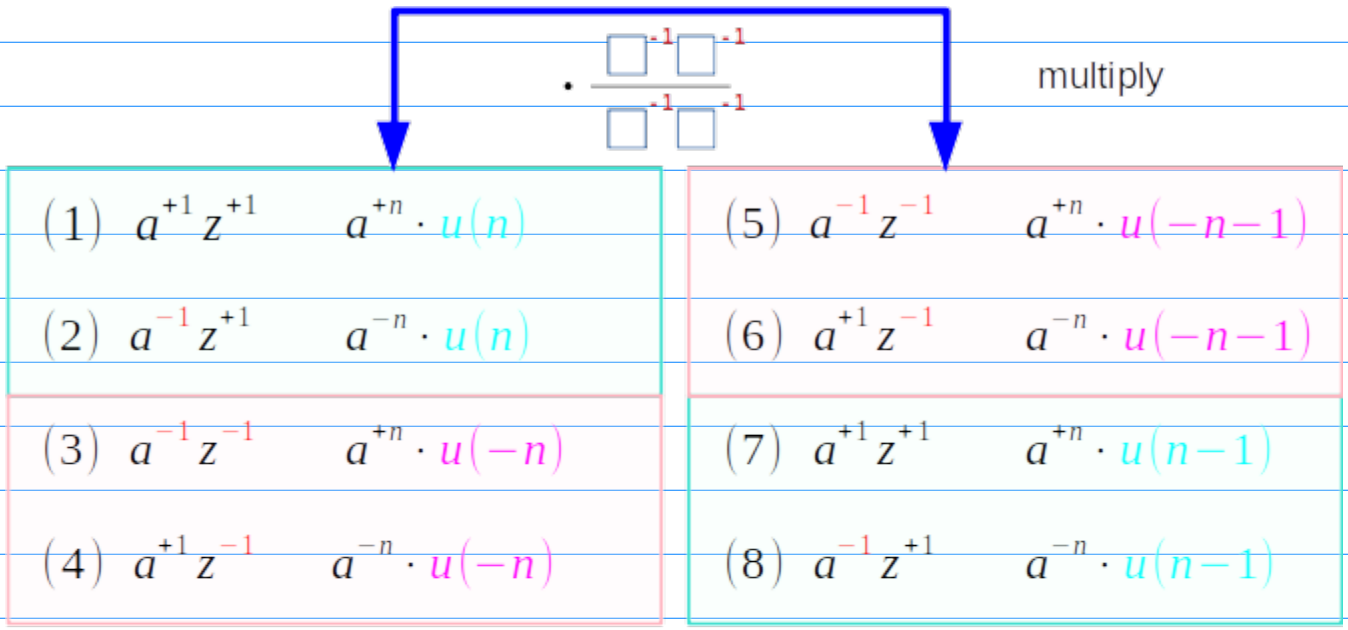
a non-unit start term

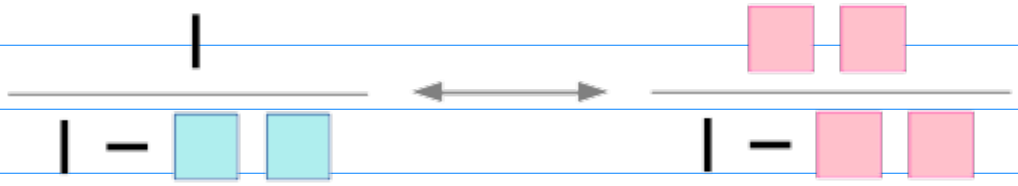


Symmetric Range Relations - CR only



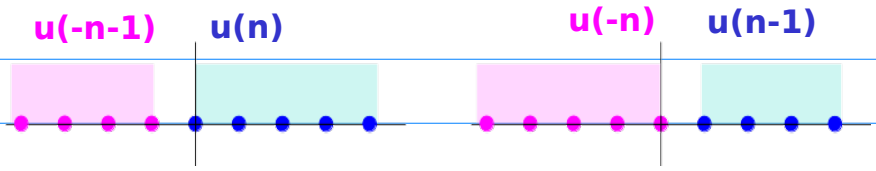
Complementary Range Relations - CR & Range



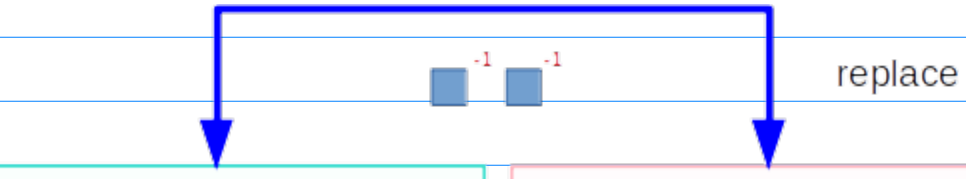


a unit start term

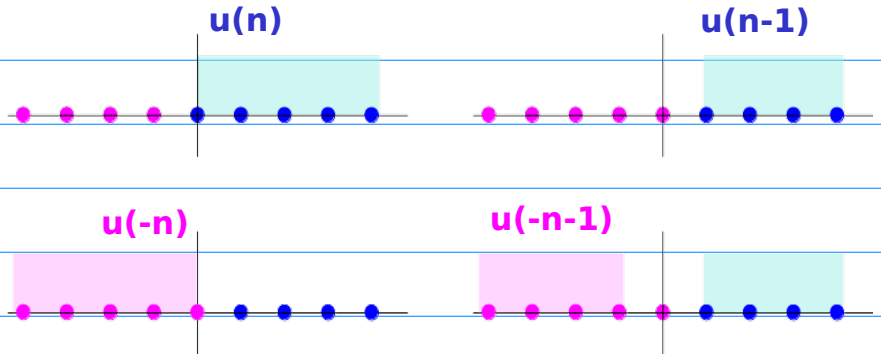
a non-unit start term



Symmetric Range Relations - CR & Range



(1)	$a^{+1}z^{+1}$	$a^{+n}\cdot u(n)$	(3)	$a^{-1}z^{-1}$	$a^{+n}\cdot u(-n)$
(2)	$a^{-1}z^{+1}$	$a^{-n}\cdot u(n)$	(4)	$a^{+1}z^{-1}$	$a^{-n}\cdot u(-n)$
(5)	$a^{-1}z^{-1}$	$a^{+n}\cdot u(-n-1)$	(7)	$a^{+1}z^{+1}$	$a^{+n}\cdot u(n-1)$
(6)	$a^{+1}z^{-1}$	$a^{-n}\cdot u(-n-1)$	(8)	$a^{-1}z^{+1}$	$a^{-n}\cdot u(n-1)$



row major ordering

Positive Exponent	Negative Exponent
(1)	(2)
(3)	(4)
(5)	(6)
(7)	(8)

u(n)	(1)	(2)	Causal
u(-n)	(3)	(4)	Anti-Causal
u(-n-1)	(5)	(6)	Anti-Causal
u(n-1)	(7)	(8)	Causal

ranges include the origin	(1)	(2)	u(n)
	(3)	(4)	u(-n)
ranges exclude the origin	(5)	(6)	u(-n-1)
	(7)	(8)	u(n-1)

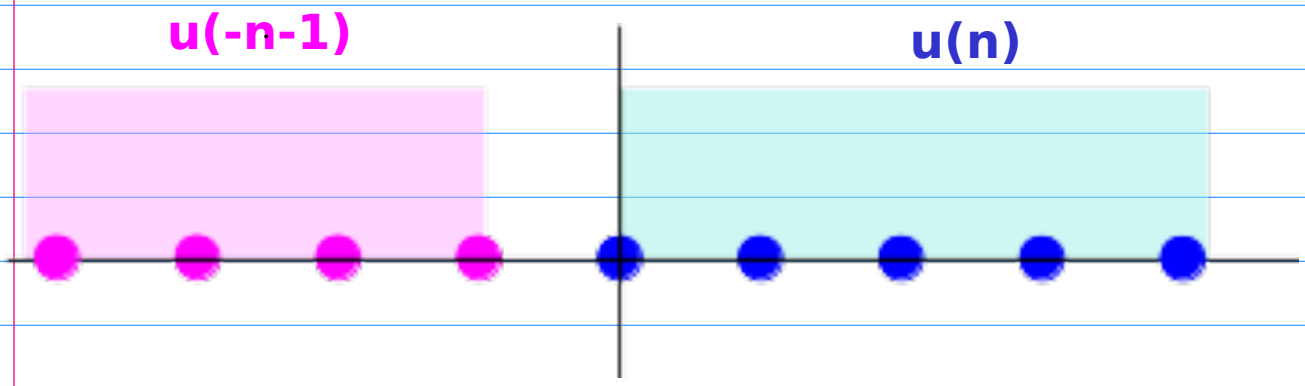
Complement Type I

u(n)	(1)	(2)	Causal
u(-n)	(3)	(4)	Anti-Causal
u(-n-1)	(5)	(6)	Anti-Causal
u(n-1)	(7)	(8)	Causal

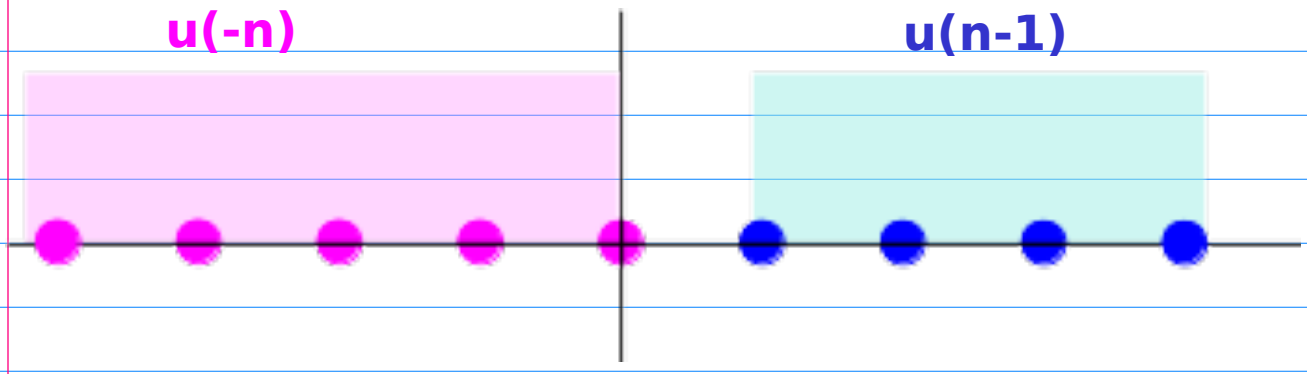
Complement Type II

Complement Type I			Complement Type II		
u(n)	(1)	(2)	u(-n)	(3)	(4)
u(-n-1)	(5)	(6)	u(n-1)	(7)	(8)

Complement Type I



Complement Type II



Causal	u(n)	(1)	(2)
	u(n-1)	(7)	(8)
Anti-Causal	u(-n-1)	(5)	(6)
	u(-n)	(3)	(4)

butterfly pair ordering

Complement Type I	u(n)	(1)	(2)
	u(-n-1)	(5)	(6)
Complement Type II	u(-n)	(3)	(4)
	u(n-1)	(7)	(8)

complementary pair ordering

		Positive Exponent	Negative Exponent
Complement Type I	u(n)	(1)	(2)
	u(-n-1)	(5)	(6)
Complement Type II	u(-n)	(3)	(4)
	u(n-1)	(7)	(8)

complementary pair ordering

Shifted Geometric Sequences

Exponent Shifting

$*a$	$*a^{-1}$
$a^{n+1} \leftarrow a^{n}$	$a^{n-1} \leftarrow a^{n}$
Left Shift	Right Shift
$a^{-n+1} \leftarrow a^{-n}$	$a^{-n-1} \leftarrow a^{-n}$
Right Shift	Left Shift

Exponent & Range Shifting

$*z$	$*z^{-1}$
$n \leftarrow n-1$	$n \leftarrow n+1$
Right Shift	Left Shift

Positive Exponent

Left Shifted

(1) *a, (7) /z

$$\frac{a}{1-az} \quad |z|<a^{-1} \qquad a^{n+1}\,u(n)$$

Right Shifted

(1) *z, (7) /a

$$\frac{z}{1-az} \quad |z|<a^{-1} \qquad a^{n-1}\,u(n-1)$$

Left Shifted

(5) *a, (3) /z

$$-\frac{z^{-1}}{1-a^{-1}z^{-1}} \quad |z|>a^{-1} \qquad a^{n+1}\,u(-n-1)$$

Right Shifted

(5) *z, (3) /a

$$-\frac{a^{-1}}{1-a^{-1}z^{-1}} \quad |z|>a^{-1} \qquad a^{n-1}\,u(-n)$$

Negative Exponent

(2) /a, (8) /z

$$\frac{a^{-1}}{1-a^{-1}z} \quad |z|<a \qquad a^{-n-1}\,u(n)$$

(2) *z, (8) *a

$$\frac{z}{1-a^{-1}z} \quad |z|<a \qquad a^{-n+1}\,u(n-1)$$

(6) /a, (4) /z

$$-\frac{z^{-1}}{1-az^{-1}} \quad |z|>a \qquad a^{-n-1}\,u(-n-1)$$

(6) *z, (4) *a

$$-\frac{a}{1-az^{-1}} \quad |z|>a \qquad a^{-n+1}\,u(-n)$$

multiplying a or a^{-1}

multiplying z or z^{-1}

Exponent Shifting

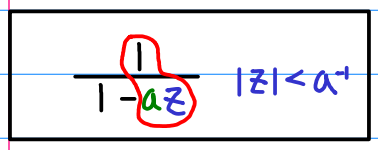
$*a$	$*a^{-1}$
$a^{n+1} \leftarrow a^{n}$ **Left Shift**	$a^{n-1} \leftarrow a^{n}$ **Right Shift**
$a^{-n+1} \leftarrow a^{-n}$ **Right Shift**	$a^{-n-1} \leftarrow a^{-n}$ **Left Shift**

Exponent & Range Shifting

$*z$	$*z^{-1}$
$n \leftarrow n-1$ **Right Shift**	$n \leftarrow n+1$ **Left Shift**

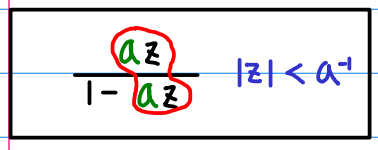
Positive Exponent

unshifted (1)



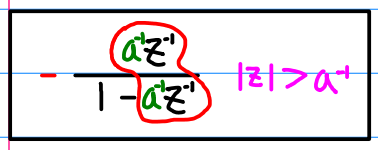
$a^n\ u(n)$

complementary (7)



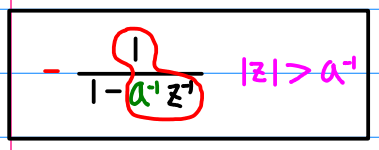
$a^n\ u(n-1)$

unshifted (5)



$a^n\ u(-n-1)$

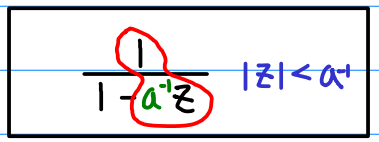
complementary (3)



$a^n\ u(-n)$

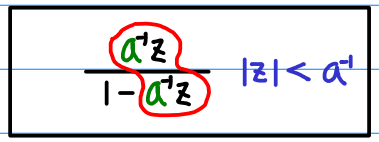
Negative Exponent

unshifted (2)



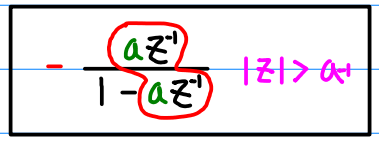
$a^{-n}\ u(n)$

complementary (8)



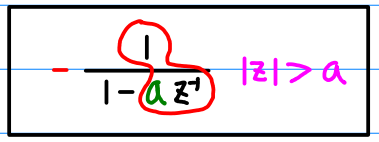
$a^{-n}\ u(n-1)$

unshifted (6)



$a^{-n}\ u(-n-1)$

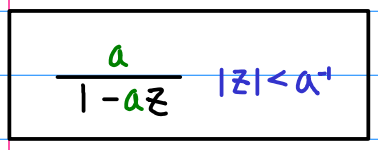
complementary (4)



$a^{-n}\ u(-n)$

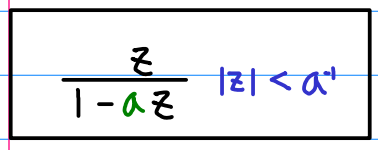
Positive Exponent

Left Shifted — (1) *a, (7) /z



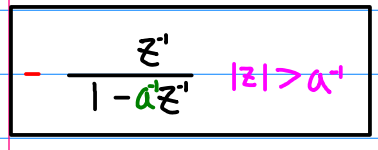
$a^{n+1}\ u(n)$

RIght Shifted — (1) *z, (7) /a



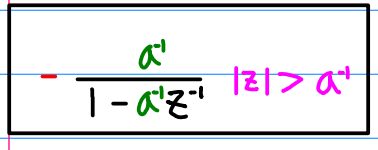
$a^{n-1}\ u(n-1)$

Left Shifted — (5) *a, (3) /z



$a^{n+1}\ u(-n-1)$

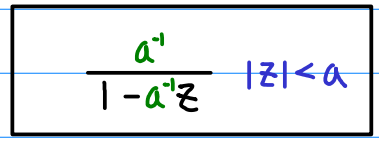
RIght Shifted — (5) *z, (3) /a



$a^{n-1}\ u(-n)$

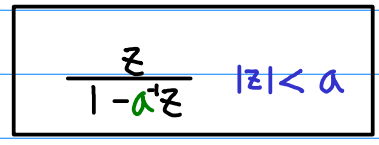
Negative Exponent

Left Shifted — (2) /a, (8) /z



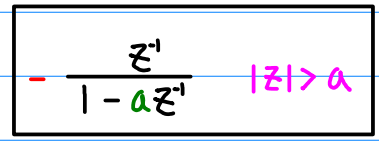
$a^{-n-1}\ u(n)$

RIght Shifted — (2) *z, (8) *a



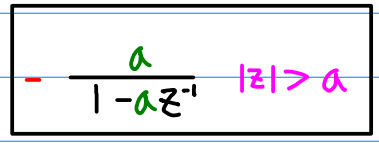
$a^{-n+1}\ u(n-1)$

Left Shifted — (6) /a, (4) /z



$a^{-n-1}\ u(-n-1)$

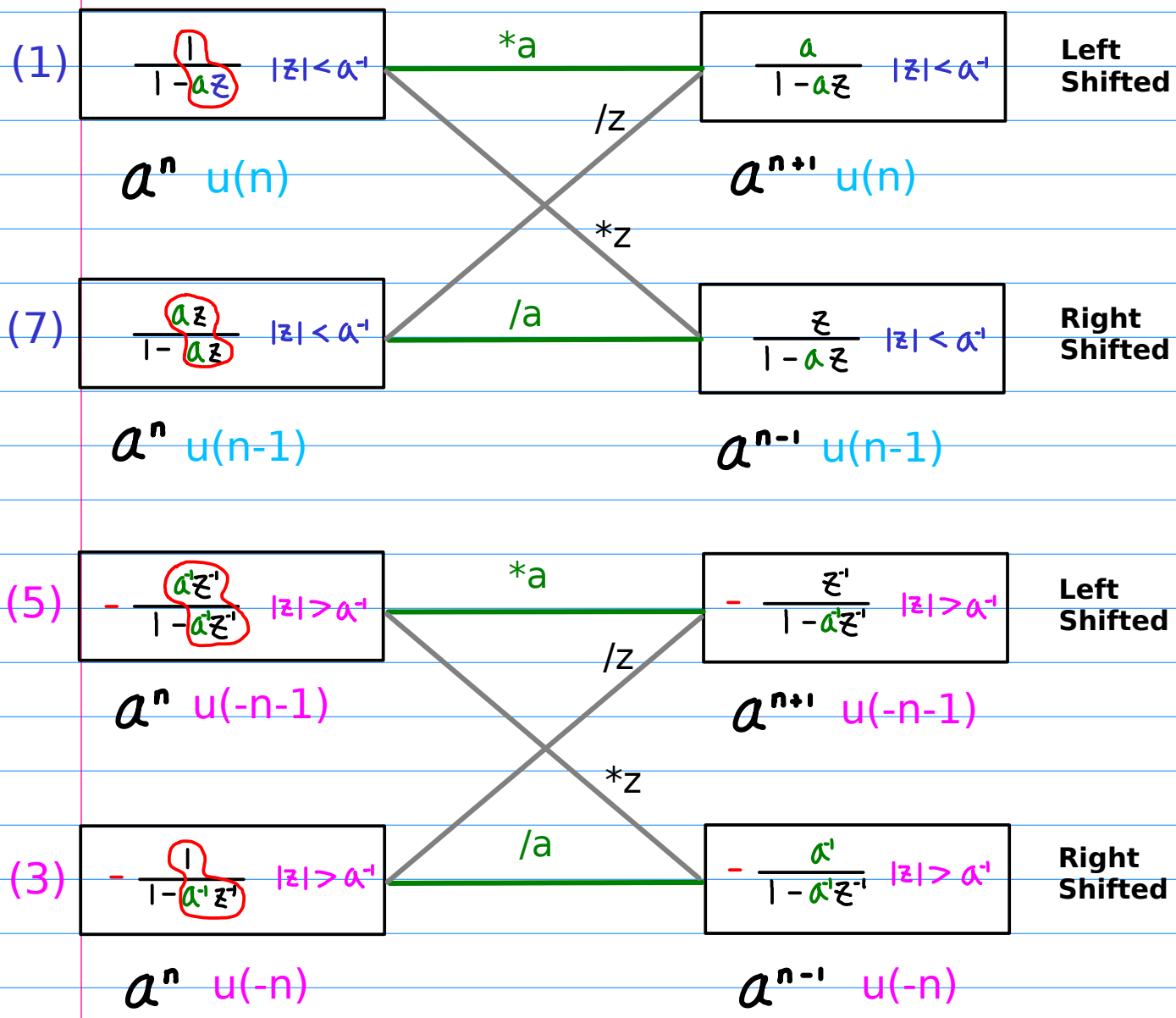
RIght Shifted — (6) *z, (4) *a



$a^{-n+1}\ u(-n)$

Combinations of Shifted Geometric Series (1)

Positive Exponent /z $n \leftarrow n+1$ *z $n \leftarrow n-1$

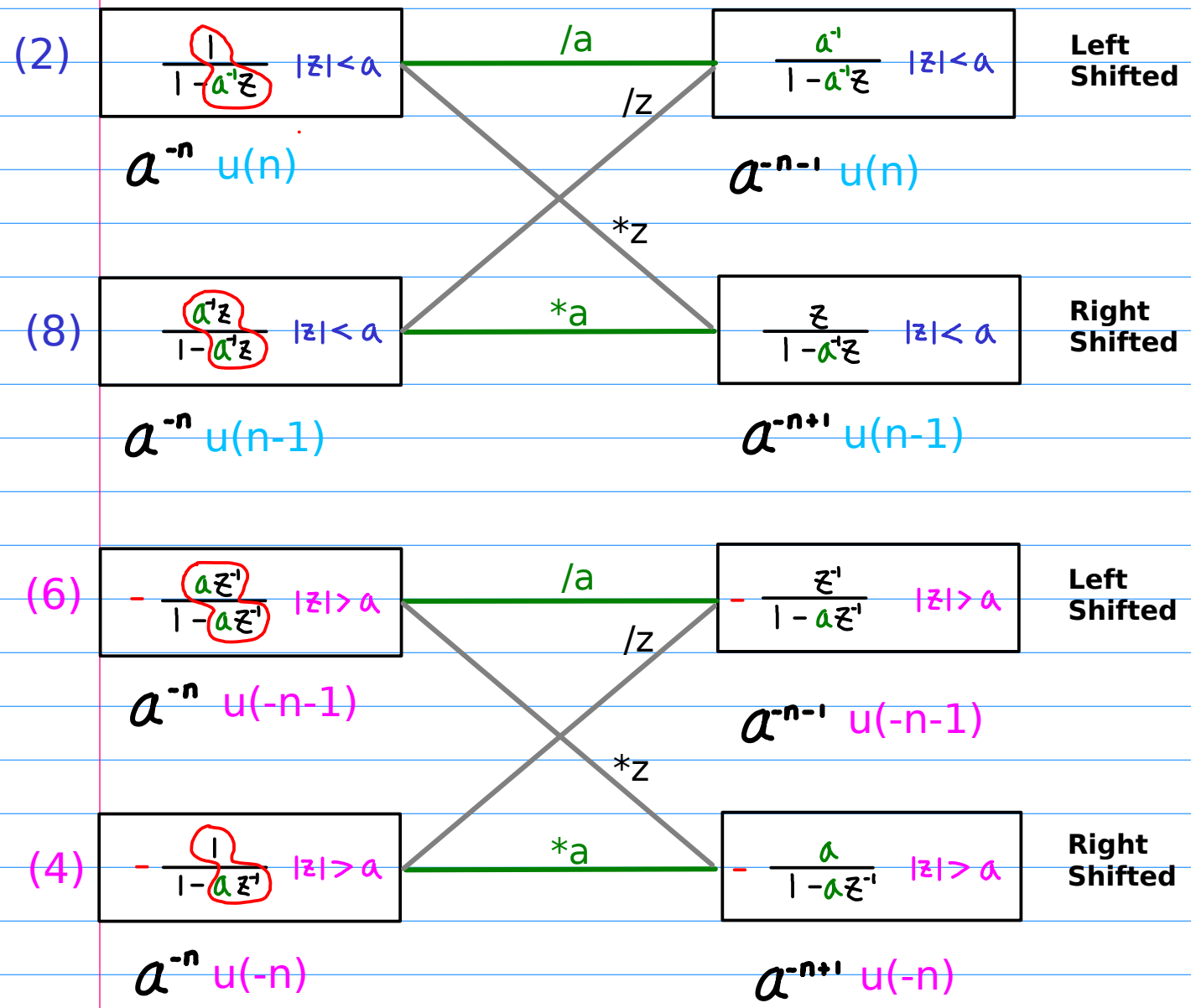


Causal	**u(n)**	(1)	(2)
	u(n-1)	(7)	(8)
Anti-Causal	**u(-n-1)**	(5)	(6)
	u(-n)	(3)	(4)

butterfly pair ordering

Combinations of Shifted Geometric Series (2)

Negative Exponent /z n ← n+1 *z n ← n-1

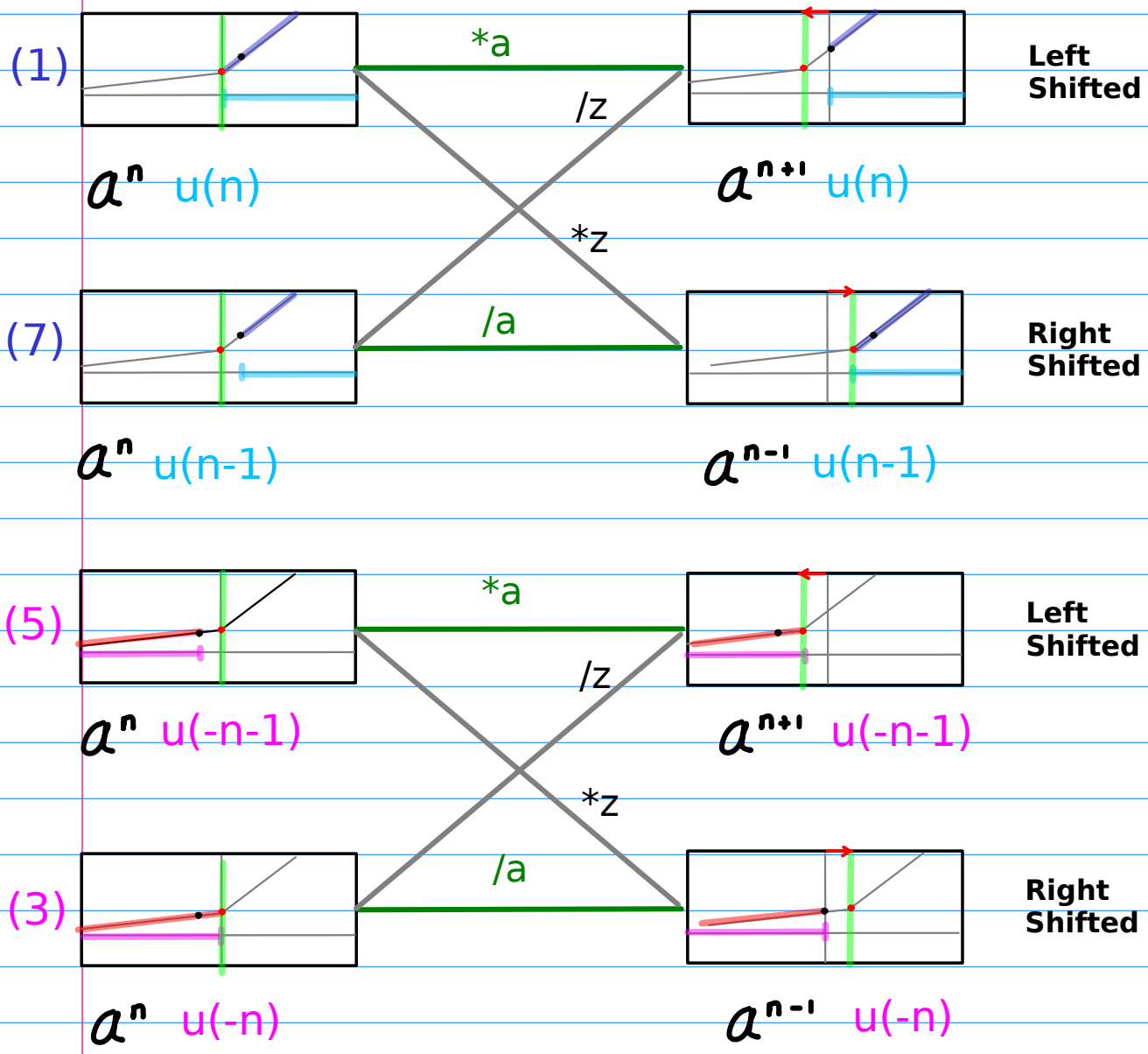


Causal	**u(n)**	(1)	(2)
	u(n-1)	(7)	(8)
Anti-Causal	**u(-n-1)**	(5)	(6)
	u(-n)	(3)	(4)

butterfly pair ordering

Graphs of Shifted Geometric Series (1)

Positive Exponent /z n ← n+1 *z n ← n-1

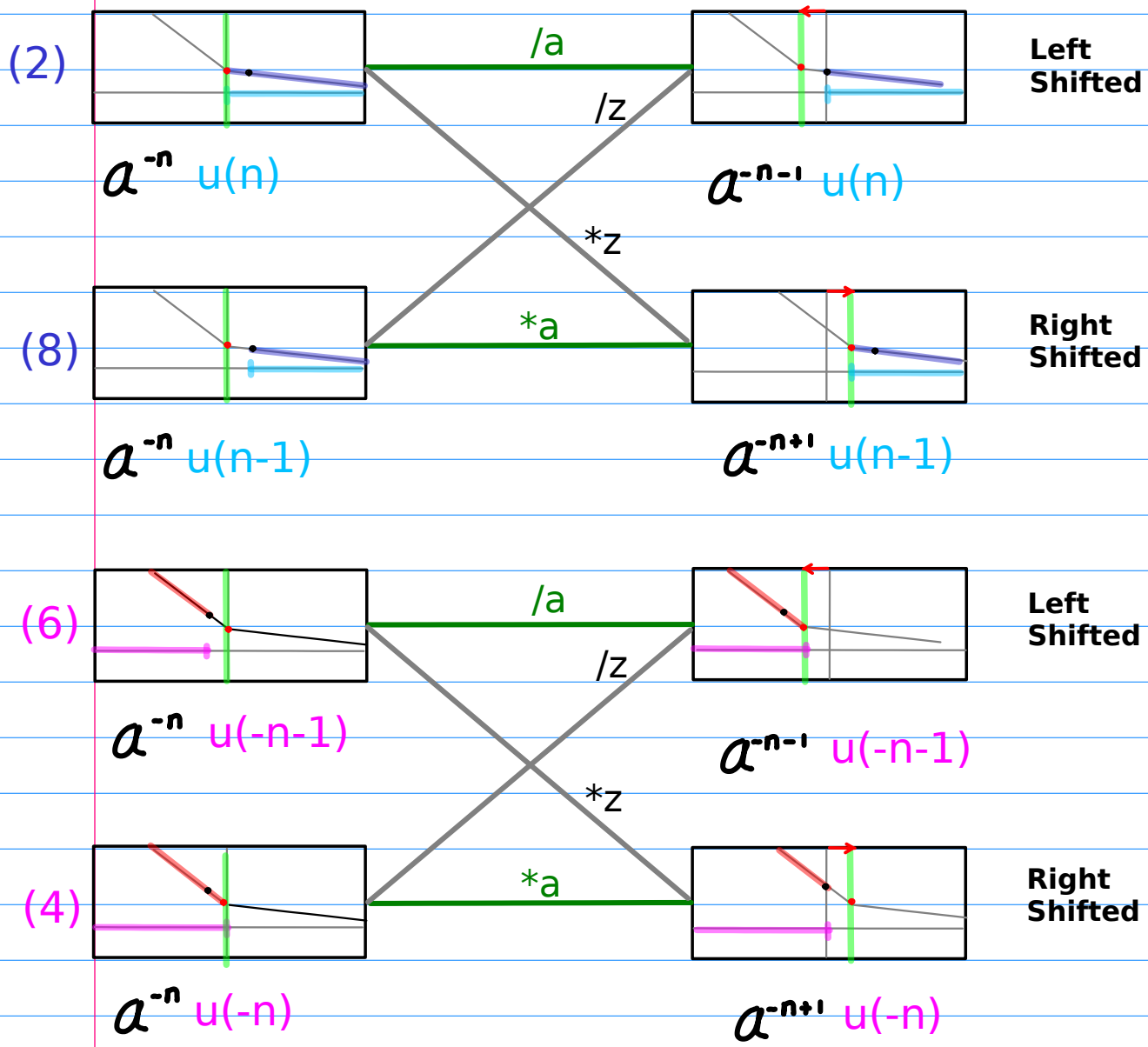


Causal	**u(n)**	(1)	(2)
	u(n-1)	(7)	(8)
Anti-Causal	**u(-n-1)**	(5)	(6)
	u(-n)	(3)	(4)

butterfly pair ordering

Graphs of Shifted Geometric Series (2)

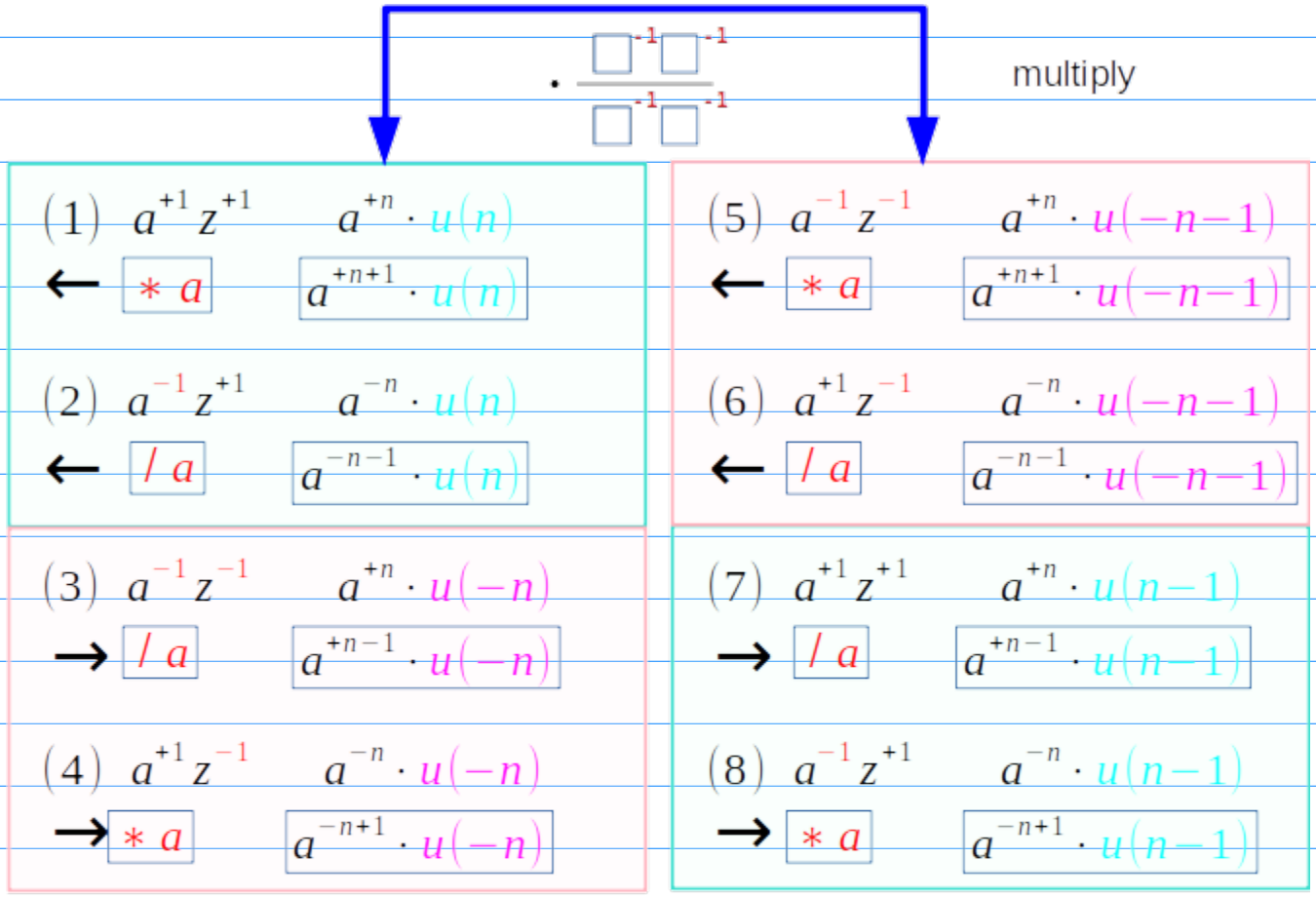
Negative Exponent /z $n \leftarrow n+1$ *z $n \leftarrow n-1$

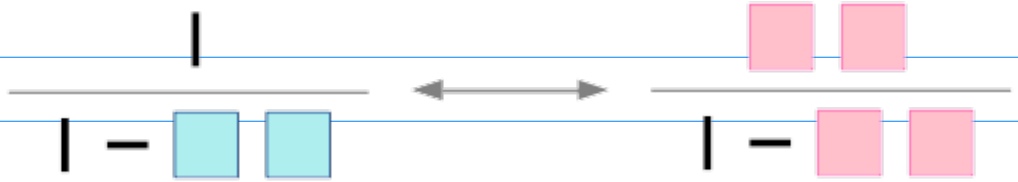


Causal	**u(n)**	(1)	(2)
	u(n-1)	(7)	(8)
Anti-Causal	**u(-n-1)**	(5)	(6)
	u(-n)	(3)	(4)

butterfly pair ordering

Shifting Geometric Series by *a or /a

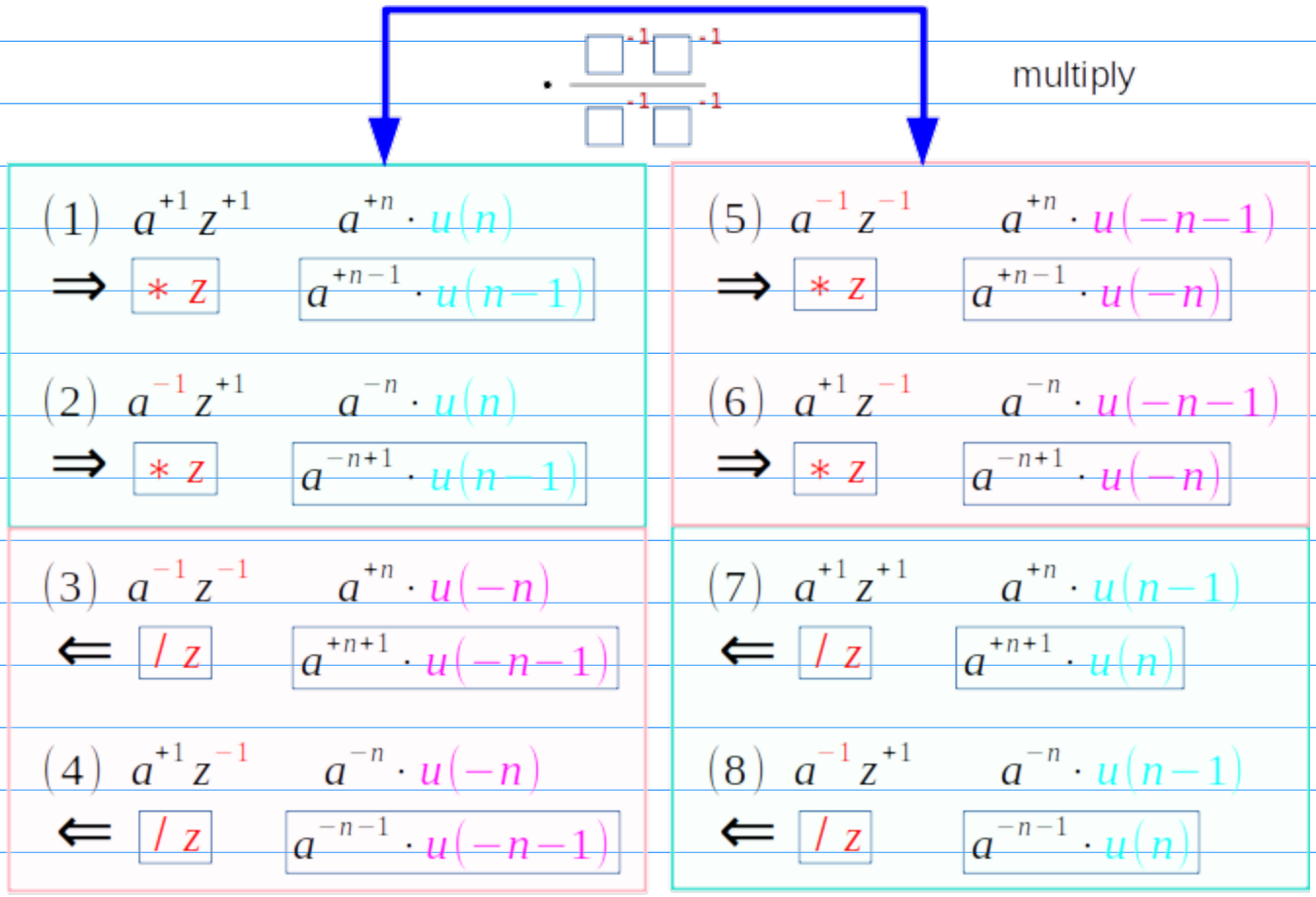


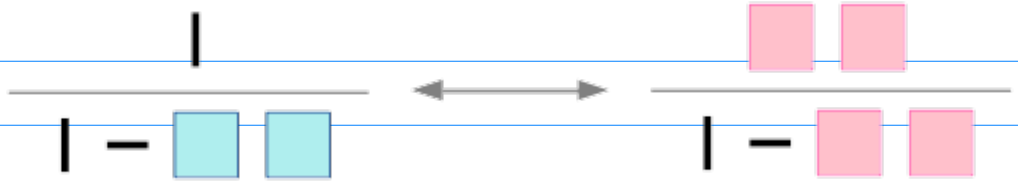


a unit start term

a non-unit start term

Shifting Geometric Series by *z or /z





a unit start term

a non-unit start term

u(n)	(1)	(2)
u(-n)	(3)	(4)
u(-n-1)	(5)	(6)
u(n-1)	(7)	(8)

row major ordering .

(1) $a^{+1}z^{+1}$ $a^{+n}\cdot u(n)$ ← $*\,a$ $a^{+n+1}\cdot u(n)$

(2) $a^{-1}z^{+1}$ $a^{-n}\cdot u(n)$ ← $/\,a$ $a^{-n-1}\cdot u(n)$

(3) $a^{-1}z^{-1}$ $a^{+n}\cdot u(-n)$ → $/\,a$ $a^{+n-1}\cdot u(-n)$

(4) $a^{+1}z^{-1}$ $a^{-n}\cdot u(-n)$ → $*\,a$ $a^{-n+1}\cdot u(-n)$

(5) $a^{-1}z^{-1}$ $a^{+n}\cdot u(-n-1)$ ← $*\,a$ $a^{+n+1}\cdot u(-n-1)$

(6) $a^{+1}z^{-1}$ $a^{-n}\cdot u(-n-1)$ ← $/\,a$ $a^{-n-1}\cdot u(-n-1)$

(7) $a^{+1}z^{+1}$ $a^{+n}\cdot u(n-1)$ → $/\,a$ $a^{+n-1}\cdot u(n-1)$

(8) $a^{-1}z^{+1}$ $a^{-n}\cdot u(n-1)$ → $*\,a$ $a^{-n+1}\cdot u(n-1)$

(1) $a^{+1}z^{+1}$ $a^{+n}\cdot u(n)$ ⇒ $*\,z$ $a^{+n-1}\cdot u(n-1)$

(2) $a^{-1}z^{+1}$ $a^{-n}\cdot u(n)$ ⇒ $*\,z$ $a^{-n+1}\cdot u(n-1)$

(3) $a^{-1}z^{-1}$ $a^{+n}\cdot u(-n)$ ⇐ $/\,z$ $a^{+n+1}\cdot u(-n-1)$

(4) $a^{+1}z^{-1}$ $a^{-n}\cdot u(-n)$ ⇐ $/\,z$ $a^{-n-1}\cdot u(-n-1)$

(5) $a^{-1}z^{-1}$ $a^{+n}\cdot u(-n-1)$ ⇒ $*\,z$ $a^{+n-1}\cdot u(-n)$

(6) $a^{+1}z^{-1}$ $a^{-n}\cdot u(-n-1)$ ⇒ $*\,z$ $a^{-n+1}\cdot u(-n)$

(7) $a^{+1}z^{+1}$ $a^{+n}\cdot u(n-1)$ ⇐ $/\,z$ $a^{+n+1}\cdot u(n)$

(8) $a^{-1}z^{+1}$ $a^{-n}\cdot u(n-1)$ ⇐ $/\,z$ $a^{-n-1}\cdot u(n)$

Causal	**u(n)**	(1)	(2)	
	u(n-1)	(7)	(8)	
Anti-Causal	**u(-n-1)**	(5)	(6)	
	u(-n)	(3)	(4)	

butterfly pair ordering

(1) $a^{+1} z^{+1}$ $a^{+n} \cdot u(n)$ ← $*\, a$ $a^{+n+1} \cdot u(n)$

(2) $a^{-1} z^{+1}$ $a^{-n} \cdot u(n)$ ← $/\, a$ $a^{-n-1} \cdot u(n)$

(7) $a^{+1} z^{+1}$ $a^{+n} \cdot u(n-1)$ → $/\, a$ $a^{+n-1} \cdot u(n-1)$

(8) $a^{-1} z^{+1}$ $a^{-n} \cdot u(n-1)$ → $*\, a$ $a^{-n+1} \cdot u(n-1)$

(5) $a^{-1} z^{-1}$ $a^{+n} \cdot u(-n-1)$ ← $*\, a$ $a^{+n+1} \cdot u(-n-1)$

(6) $a^{+1} z^{-1}$ $a^{-n} \cdot u(-n-1)$ ← $/\, a$ $a^{-n-1} \cdot u(-n-1)$

(3) $a^{-1} z^{-1}$ $a^{+n} \cdot u(-n)$ → $/\, a$ $a^{+n-1} \cdot u(-n)$

(4) $a^{+1} z^{-1}$ $a^{-n} \cdot u(-n)$ → $*\, a$ $a^{-n+1} \cdot u(-n)$

(1) $a^{+1} z^{+1}$ $a^{+n} \cdot u(n)$ ⇒ $*\, z$ $a^{+n-1} \cdot u(n-1)$

(2) $a^{-1} z^{+1}$ $a^{-n} \cdot u(n)$ ⇒ $*\, z$ $a^{-n+1} \cdot u(n-1)$

(7) $a^{+1} z^{+1}$ $a^{+n} \cdot u(n-1)$ ⇐ $/\, z$ $a^{+n+1} \cdot u(n)$

(8) $a^{-1} z^{+1}$ $a^{-n} \cdot u(n-1)$ ⇐ $/\, z$ $a^{-n-1} \cdot u(n)$

(5) $a^{-1} z^{-1}$ $a^{+n} \cdot u(-n-1)$ ⇒ $*\, z$ $a^{+n-1} \cdot u(-n)$

(6) $a^{+1} z^{-1}$ $a^{-n} \cdot u(-n-1)$ ⇒ $*\, z$ $a^{-n+1} \cdot u(-n)$

(3) $a^{-1} z^{-1}$ $a^{+n} \cdot u(-n)$ ⇐ $/\, z$ $a^{+n+1} \cdot u(-n-1)$

(4) $a^{+1} z^{-1}$ $a^{-n} \cdot u(-n)$ ⇐ $/\, z$ $a^{-n-1} \cdot u(-n-1)$

Complement Type I	**u(n)**	(1)	(2)
	u(-n-1)	(5)	(6)
Complement Type II	**u(-n)**	(3)	(4)
	u(n-1)	(7)	(8)

complementary pair ordering

(1) $a^{+1}z^{+1}$ $a^{+n}\cdot u(n)$ (2) $a^{-1}z^{+1}$ $a^{-n}\cdot u(n)$

$\leftarrow$ $*\,a$ $a^{+n+1}\cdot u(n)$ $\leftarrow$ $/\,a$ $a^{-n-1}\cdot u(n)$

(5) $a^{-1}z^{-1}$ $a^{+n}\cdot u(-n-1)$ (6) $a^{+1}z^{-1}$ $a^{-n}\cdot u(-n-1)$

$\leftarrow$ $*\,a$ $a^{+n+1}\cdot u(-n-1)$ $\leftarrow$ $/\,a$ $a^{-n-1}\cdot u(-n-1)$

(3) $a^{-1}z^{-1}$ $a^{+n}\cdot u(-n)$ (4) $a^{+1}z^{-1}$ $a^{-n}\cdot u(-n)$

$\rightarrow$ $/\,a$ $a^{+n-1}\cdot u(-n)$ $\rightarrow$ $*\,a$ $a^{-n+1}\cdot u(-n)$

(7) $a^{+1}z^{+1}$ $a^{+n}\cdot u(n-1)$ (8) $a^{-1}z^{+1}$ $a^{-n}\cdot u(n-1)$

$\rightarrow$ $/\,a$ $a^{+n-1}\cdot u(n-1)$ $\rightarrow$ $*\,a$ $a^{-n+1}\cdot u(n-1)$

(1) $a^{+1}z^{+1}$ $a^{+n}\cdot u(n)$ (2) $a^{-1}z^{+1}$ $a^{-n}\cdot u(n)$

$\Rightarrow$ $*\,z$ $a^{+n-1}\cdot u(n-1)$ $\Rightarrow$ $*\,z$ $a^{-n+1}\cdot u(n-1)$

(5) $a^{-1}z^{-1}$ $a^{+n}\cdot u(-n-1)$ (6) $a^{+1}z^{-1}$ $a^{-n}\cdot u(-n-1)$

$\Rightarrow$ $*\,z$ $a^{+n-1}\cdot u(-n)$ $\Rightarrow$ $*\,z$ $a^{-n+1}\cdot u(-n)$

(3) $a^{-1}z^{-1}$ $a^{+n}\cdot u(-n)$ (4) $a^{+1}z^{-1}$ $a^{-n}\cdot u(-n)$

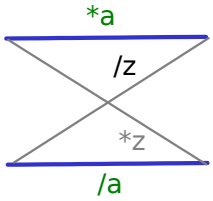
$\Leftarrow$ $/\,z$ $a^{+n+1}\cdot u(-n-1)$ $\Leftarrow$ $/\,z$ $a^{-n-1}\cdot u(-n-1)$

(7) $a^{+1}z^{+1}$ $a^{+n}\cdot u(n-1)$ (8) $a^{-1}z^{+1}$ $a^{-n}\cdot u(n-1)$

$\Leftarrow$ $/\,z$ $a^{+n+1}\cdot u(n)$ $\Leftarrow$ $/\,z$ $a^{-n-1}\cdot u(n)$

Shifted Geometric Series (1)

by multiplying a or a^{-1}



Positive Exponent

(1) ←

$$\frac{1}{1-az^{-1}} \quad |z|<a^{-1} \qquad a^n u(n) \times a \qquad \frac{a}{1-az^{-1}} \quad |z|<a^{-1} \qquad a^{n+1} u(n)$$

(7) →

$$\frac{az^{-1}}{1-az^{-1}} \quad |z|<a^{-1} \qquad a^n u(n-1) \times a^{-1} \qquad \frac{z^{-1}}{1-az^{-1}} \quad |z|<a^{-1} \qquad a^{n-1} u(n-1)$$

(5) ←

$$-\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}} \quad |z|>a^{-1} \qquad a^n u(-n-1) \times a \qquad -\frac{z^{-1}}{1-a^{-1}z^{-1}} \quad |z|>a^{-1} \qquad a^{n+1} u(-n-1)$$

(3) →

$$-\frac{1}{1-a^{-1}z^{-1}} \quad |z|>a^{-1} \qquad a^n u(-n) \times a^{-1} \qquad -\frac{a^{-1}}{1-a^{-1}z^{-1}} \quad |z|>a^{-1} \qquad a^{n-1} u(-n)$$

Negative Exponent

(2) ←

$$\frac{1}{1-a^{-1}z^{-1}} \quad |z|<a \qquad a^{-n} u(n) \times a^{-1} \qquad \frac{a^{-1}}{1-a^{-1}z^{-1}} \quad |z|<a \qquad a^{-n-1} u(n)$$

(8) →

$$\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}} \quad |z|<a \qquad a^{-n} u(n-1) \times a \qquad \frac{z^{-1}}{1-a^{-1}z^{-1}} \quad |z|<a \qquad a^{-n+1} u(n-1)$$

(6) ←

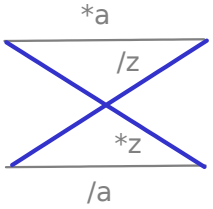
$$-\frac{az^{-1}}{1-az^{-1}} \quad |z|>a \qquad a^{-n} u(-n-1) \times a^{-1} \qquad -\frac{z^{-1}}{1-az^{-1}} \quad |z|>a \qquad a^{-n-1} u(-n-1)$$

(4) →

$$-\frac{1}{1-az^{-1}} \quad |z|>a \qquad a^{-n} u(-n) \times a \qquad -\frac{a}{1-az^{-1}} \quad |z|>a \qquad a^{-n+1} u(-n)$$

Shifted Geometric Series (2)

by multiplying z or z^{-1}



Positive Exponent

(1) ⇐ $\frac{az}{1-az}$ $|z|<a^{-1}$ | $n \leftarrow n+1$ | $a^n u(n-1)$ z^{-1} | $\frac{a}{1-az}$ $|z|<a^{-1}$ | $a^{n+1} u(n)$

(7) ⇒ $\frac{1}{1-az}$ $|z|<a^{-1}$ | $n \leftarrow n-1$ | $a^n u(n)$ $\times z$ | $\frac{z}{1-az}$ $|z|<a^{-1}$ | $a^{n-1} u(n-1)$

(5) ⇐ $-\frac{1}{1-a^{-1}z^{-1}}$ $|z|>a^{-1}$ | $n \leftarrow n+1$ | $a^n u(-n)$ $\times z^{-1}$ | $-\frac{z^{-1}}{1-a^{-1}z^{-1}}$ $|z|>a^{-1}$ | $a^{n+1} u(-n-1)$

(3) ⇒ $-\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}}$ $|z|>a^{-1}$ | $n \leftarrow n-1$ | $a^n u(-n-1) \times z$ | $-\frac{a^{-1}}{1-a^{-1}z^{-1}}$ $|z|>a^{-1}$ | $a^{n-1} u(-n)$

Negative Exponent

(2) ⇐ $\frac{a^{-1}z}{1-a^{-1}z}$ $|z|<a$ | $n \leftarrow n+1$ | $a^{-n} u(n-1) \times z^{-1}$ | $\frac{a^{-1}}{1-a^{-1}z}$ $|z|<a$ | $a^{-n-1} u(n)$

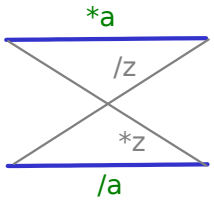
(8) ⇒ $\frac{1}{1-a^{-1}z}$ $|z|<a$ | $n \leftarrow n-1$ | $a^{-n} u(n)$ $\times z$ | $\frac{z}{1-a^{-1}z}$ $|z|<a$ | $a^{-n+1} u(n-1)$

(6) ⇐ $-\frac{1}{1-az^{-1}}$ $|z|>a$ | $n \leftarrow n+1$ | $a^{-n} u(-n)$ $\times z^{-1}$ | $-\frac{z^{-1}}{1-az^{-1}}$ $|z|>a$ | $a^{-n-1} u(-n-1)$

(4) ⇒ $-\frac{az^{-1}}{1-az^{-1}}$ $|z|>a$ | $n \leftarrow n-1$ | $a^{-n} u(-n-1) \times z$ | $-\frac{a}{1-az^{-1}}$ $|z|>a$ | $a^{-n+1} u(-n)$

Shifted Geometric Series (3)

by multiplying a or a^{-1} Assume a > 1



(1)	a^{n} u(n)	*a	←	a^{n+1}	u(n)
(2)	a^{-n} u(n)	/a	←	a^{-n-1}	u(n)
(3)	a^{n} u(-n)	/a	→	a^{n-1}	u(-n)
(4)	a^{-n} u(-n)	*a	→	a^{-n+1}	u(-n)
(5)	a^{n} u(-n-1)	*a	←	a^{n+1}	u(-n-1)
(6)	a^{-n} u(-n-1)	/a	←	a^{-n-1}	u(-n-1)
(7)	a^{n} u(n-1)	/a	→	a^{n-1}	u(n-1)
(8)	a^{-n} u(n-1)	*a	→	a^{-n+1}	u(n-1)

row major ordering

(1)	(2)	*a	/a	←	←
(3)	(4)	/a	*a	→	→
(5)	(6)	*a	/a	←	←
(7)	(8)	/a	*a	→	→

complementary pair ordering

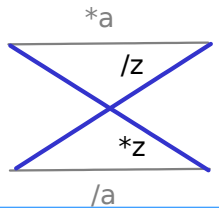
(1)	(2)	*a	/a	←	←
(5)	(6)	*a	/a	←	←
(3)	(4)	/a	*a	→	→
(7)	(8)	/a	*a	→	→

butterfly pair ordering

(1)	(2)	*a	/a	←	←
(7)	(8)	/a	*a	→	→
(5)	(6)	*a	/a	←	←
(3)	(4)	/a	*a	→	→

Shifted Geometric Series (4)

by multiplying z or z^{-1} Assume $a > 1$



(1)	a^n u(n)	*z	$\Rightarrow$	a^{n-1}	u(n-1)
(2)	a^{-n} u(n)	*z	$\Rightarrow$	a^{-n+1}	u(n-1)
(3)	a^n u(-n)	/z	$\Leftarrow$	a^{n+1}	u(-n-1)
(4)	a^{-n} u(-n)	/z	$\Leftarrow$	a^{-n-1}	u(-n-1)
(5)	a^n u(-n-1)	*z	$\Rightarrow$	a^{n-1}	u(-n)
(6)	a^{-n} u(-n-1)	*z	$\Rightarrow$	a^{-n+1}	u(-n)
(7)	a^n u(n-1)	/z	$\Leftarrow$	a^{n+1}	u(n)
(8)	a^{-n} u(n-1)	/z	$\Leftarrow$	a^{-n-1}	u(n)

row major ordering

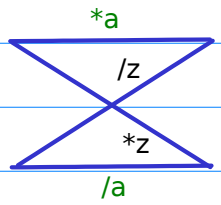
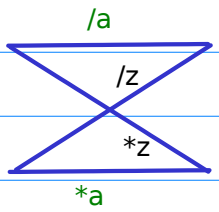
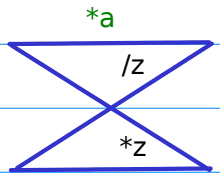
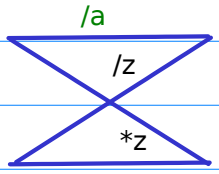
(1)	(2)	*z	*z	$\Rightarrow$	$\Rightarrow$		
(3)	(4)	/z	/z	$\Leftarrow$	$\Leftarrow$		
(5)	(6)	*z	*z	$\Rightarrow$	$\Rightarrow$		
(7)	(8)	/z	/z	$\Leftarrow$	$\Leftarrow$		

complementary pair ordering

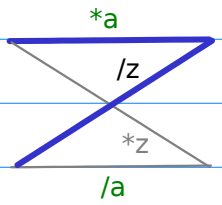
(1)	(2)	*z	*z	$\Rightarrow$	$\Rightarrow$		
(5)	(6)	*z	*z	$\Rightarrow$	$\Rightarrow$		
(3)	(4)	/z	/z	$\Leftarrow$	$\Leftarrow$		
(7)	(8)	/z	/z	$\Leftarrow$	$\Leftarrow$		

butterfly pair ordering

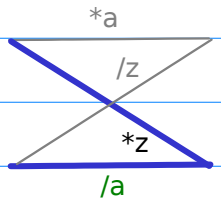
(1)	(2)	*z	*z	$\Rightarrow$	$\Rightarrow$		
(7)	(8)	/z	/z	$\Leftarrow$	$\Leftarrow$		
(5)	(6)	*z	*z	$\Rightarrow$	$\Rightarrow$		
(3)	(4)	/z	/z	$\Leftarrow$	$\Leftarrow$		

(1) $a^n u(n)$  $a^{n+1} u(n)$	(2) $a^{-n} u(n)$  $a^{-n-1} u(n)$
(7) $a^n u(n-1)$ $a^{n-1} u(n-1)$	(8) $a^{-n} u(n-1)$ $a^{-n+1} u(n-1)$
(5) $a^n u(-n-1)$  $a^{n+1} u(-n-1)$	(6) $a^{-n} u(-n-1)$  $a^{-n-1} u(-n-1)$
(3) $a^n u(-n)$ $a^{n-1} u(-n)$	(4) $a^{-n} u(-n)$ $a^{-n+1} u(-n)$

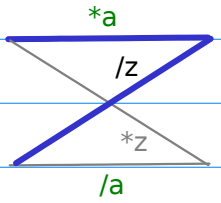
← (1) *a = (7) /z ⇐



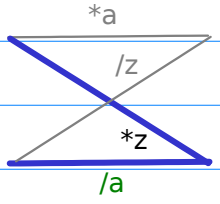
⇒ (1) *z = (7) /a →



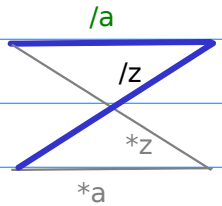
← (5) *a = (3) /z ⇐



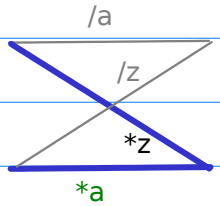
⇒ (5) *z = (3) /a →



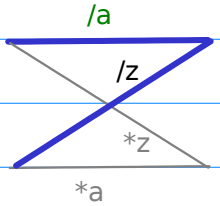
← (2) /a = (8) /z ⇐



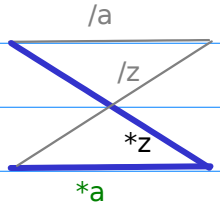
⇒ (2) *z = (8) *a →

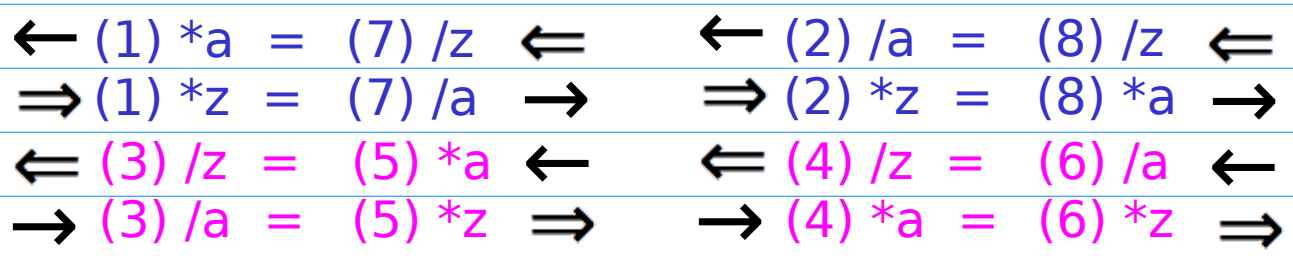


← (6) /a = (4) /z ⇐

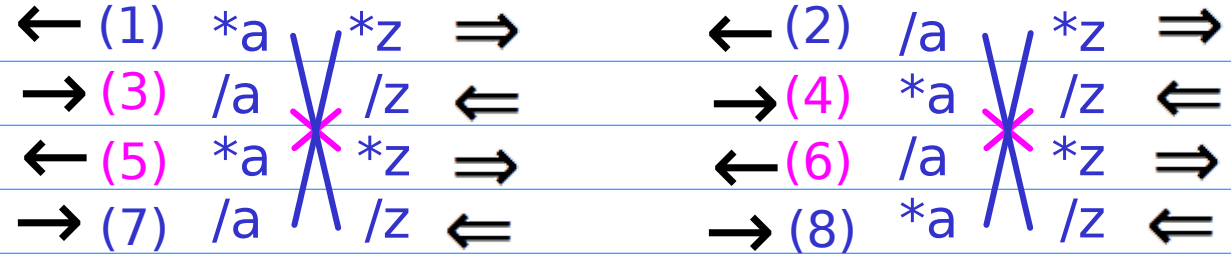


⇒ (6) *z = (4) *a →

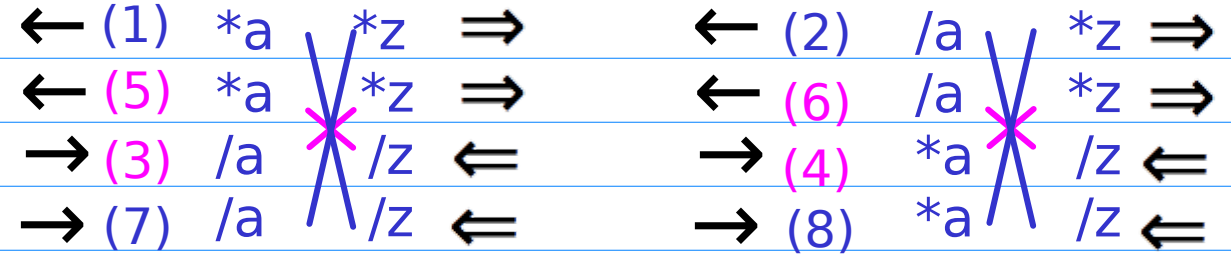




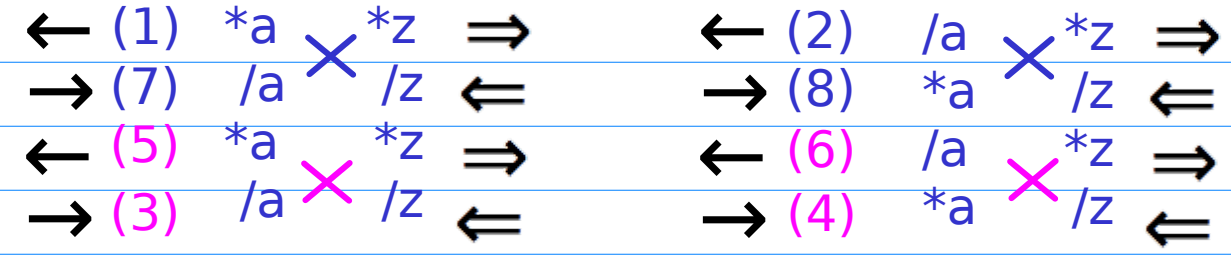
row major ordering



complementary pair ordering



butterfly pair ordering



Assume $a > 1$

(1)	a^n	u(n)	$*a \leftarrow$ a^{n+1}	u(n)
(7)	a^n	u(n-1)	$/a \rightarrow$ a^{n-1}	u(n-1)
(5)	a^n	u(-n-1)	$*a \leftarrow$ a^{n+1}	u(-n-1)
(3)	a^n	u(-n)	$/a \rightarrow$ a^{n-1}	u(-n)
(2)	a^{-n}	u(n)	$/a \leftarrow$ a^{-n-1}	u(n)
(8)	a^{-n}	u(n-1)	$*a \rightarrow$ a^{-n+1}	u(n-1)
(6)	a^{-n}	u(-n-1)	$/a \leftarrow$ a^{-n-1}	u(-n-1)
(4)	a^{-n}	u(-n)	$*a \rightarrow$ a^{-n+1}	u(-n)

2^n	u(n)	2^{n+1}	u(n)
2^n	u(n-1)	2^{n-1}	u(n-1)
2^n	u(-n-1)	2^{n+1}	u(-n-1)
2^n	u(-n)	2^{n-1}	u(-n)

$(\frac{1}{2})^{-n}$	u(n)	$(\frac{1}{2})^{-n-1}$	u(n)
$(\frac{1}{2})^{-n}$	u(n-1)	$(\frac{1}{2})^{-n+1}$	u(n-1)
$(\frac{1}{2})^{-n}$	u(-n-1)	$(\frac{1}{2})^{-n-1}$	u(-n-1)
$(\frac{1}{2})^{-n}$	u(-n)	$(\frac{1}{2})^{-n+1}$	u(-n)

$(\frac{1}{2})^n$	u(n)	$(\frac{1}{2})^{n+1}$	u(n)
$(\frac{1}{2})^n$	u(n-1)	$(\frac{1}{2})^{n-1}$	u(n-1)
$(\frac{1}{2})^n$	u(-n-1)	$(\frac{1}{2})^{n+1}$	u(-n-1)
$(\frac{1}{2})^n$	u(-n)	$(\frac{1}{2})^{n-1}$	u(-n)

2^{-n}	u(n)	2^{-n-1}	u(n)
2^{-n}	u(n-1)	2^{-n+1}	u(n-1)
2^{-n}	u(-n-1)	2^{-n-1}	u(-n-1)
2^{-n}	u(-n)	2^{-n+1}	u(-n)

Complement Type I	**u(n)**	(1)	(2)
	u(-n-1)	(5)	(6)
Complement Type II	**u(-n)**	(3)	(4)
	u(n-1)	(7)	(8)

complementary pair ordering

(1) a z	a^{n}	(2) $a^{-1}z$	a^{-n}
(5) $a^{-1}z^{-1}$	a^{n}	(6) $a\ z^{-1}$	a^{-n}
(3) $a^{-1}z^{-1}$	a^{n}	(4) $a\ z^{-1}$	a^{-n}
(7) a z	a^{n}	(8) $a^{-1}z$	a^{-n}

(1) a z	$*a$	(2) $a^{-1}z$	$/a$
(5) $a^{-1}z^{-1}$	$*a$	(6) $a\ z^{-1}$	$/a$
(3) $a^{-1}z^{-1}$	$/a$	(4) $a\ z^{-1}$	$*a$
(7) a z	$/a$	(8) $a^{-1}z$	$*a$

(1) a z	$*z$	(2) $a^{-1}z$	$*z$
(5) $a^{-1}z^{-1}$	$*z$	(6) $a\ z^{-1}$	$*z$
(3) $a^{-1}z^{-1}$	$/z$	(4) $a\ z^{-1}$	$/z$
(7) a z	$/z$	(8) $a^{-1}z$	$/z$

Geometric Series Combinations

complementary pair ordering

(1)

unit	$\frac{1}{1-az}$ $\quad \lvert z\rvert < a^{-1}$	$a^n \quad u(n)$
non-unit	$-\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}}$ $\quad \lvert z\rvert > a^{-1}$	$-a^n \quad u(-n-1)$

(2)

$\frac{1}{1-a^{-1}z}$ $\quad \lvert z\rvert < a$	$\left(\frac{1}{a}\right)^n \quad u(n)$
$-\frac{az^{-1}}{1-az^{-1}}$ $\quad \lvert z\rvert > a$	$-\left(\frac{1}{a}\right)^n \quad u(-n-1)$

(5)

(6)

(3)

unit	$-\frac{1}{1-a^{-1}z^{-1}}$ $\quad \lvert z\rvert > a^{-1}$	$-a^n \quad u(-n)$
non-unit	$\frac{az}{1-az}$ $\quad \lvert z\rvert < a^{-1}$	$a^n \quad u(n-1)$

(4)

$-\frac{1}{1-az^{-1}}$ $\quad \lvert z\rvert > a$	$-\left(\frac{1}{a}\right)^n \quad u(-n)$
$\frac{a^{-1}z}{1-a^{-1}z}$ $\quad \lvert z\rvert < a$	$\left(\frac{1}{a}\right)^n \quad u(n-1)$

(7)

(8)

		Positive Exponent	Negative Exponent
Complement Type I	**u(n)**	(1)	(2)
	u(-n-1)	(5)	(6)
Complement Type II	**u(-n)**	(3)	(4)
	u(n-1)	(7)	(8)

Shifted Combinations
(I) by scaling *a /a

complementary pair ordering

(1) *a

$\frac{a}{1-az}$	$\lvert z\rvert < a^{-1}$	$a^{n+1}\ u(n)$
$-\frac{z^{-1}}{1-a^{-1}z^{-1}}$	$\lvert z\rvert > a^{-1}$	$-a^{n+1}\ u(-n-1)$

(2) /a

$\frac{a^{-1}}{1-a^{-1}z}$	$\lvert z\rvert < a$	$\left(\frac{1}{a}\right)^{n+1}\ u(n)$
$-\frac{z^{-1}}{1-az^{-1}}$	$\lvert z\rvert > a$	$-\left(\frac{1}{a}\right)^{n+1}\ u(-n-1)$

(5) *a
(3) /a

$-\frac{a^{-1}}{1-a^{-1}z^{-1}}$	$\lvert z\rvert > a^{-1}$	$-a^{n-1}\ u(-n)$
$\frac{z}{1-az}$	$\lvert z\rvert < a^{-1}$	$a^{n-1}\ u(n-1)$

(7) /a

(6) /a
(4) *a

$-\frac{a}{1-az^{-1}}$	$\lvert z\rvert > a$	$-\left(\frac{1}{a}\right)^{n-1}\ u(-n)$
$\frac{z}{1-a^{-1}z}$	$\lvert z\rvert < a$	$\left(\frac{1}{a}\right)^{n-1}\ u(n-1)$

(8) *a

	Positive Exponent	Negative Exponent		
Complement Type I	(1) u(n) (5) u(-n-1)	(2) u(n) (6) u(-n-1)	*a u(n) *a u(-n-1)	/a u(n) /a u(-n-1)
Complement Type II	(3) u(-n) (7) u(n-1)	(4) u(-n) (8) u(n-1)	/a u(-n) /a u(n-1)	*a u(-n) *a u(n-1)

Shifted Combinations
(II) by scaling *z /z

complementary pair ordering

(1) ***z**

$\frac{z}{1-az}$	$\lvert z\rvert < a^{-1}$	$a^{n-1}\,u(n-1)$
$-\frac{a^{-1}}{1-a^{-1}z^{-1}}$	$\lvert z\rvert > a^{-1}$	$-a^{n-1}u(-n)$

(2) ***z**

$\frac{z}{1-a^{-1}z}$	$\lvert z\rvert < a$	$\left(\frac{1}{a}\right)^{n-1}\,u(n-1)$
$-\frac{a}{1-az^{-1}}$	$\lvert z\rvert > a$	$-\left(\frac{1}{a}\right)^{n-1}\,u(-n)$

(5) ***z**

(3) **/z**

$-\frac{z^{-1}}{1-a^{-1}z^{-1}}$	$\lvert z\rvert > a^{-1}$	$-a^{n+1}\,u(-n-1)$
$\frac{a}{1-az}$	$\lvert z\rvert < a^{-1}$	$a^{n+1}\,u(n)$

(7) **/z**

(6) ***z**

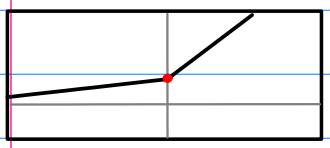
(4) **/z**

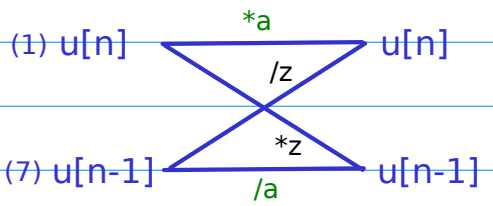
$-\frac{z^{-1}}{1-az^{-1}}$	$\lvert z\rvert > a$	$-\left(\frac{1}{a}\right)^{n+1}u(-n-1)$
$\frac{a^{-1}}{1-a^{-1}z}$	$\lvert z\rvert < a$	$\left(\frac{1}{a}\right)^{n+1}\,u(n)$

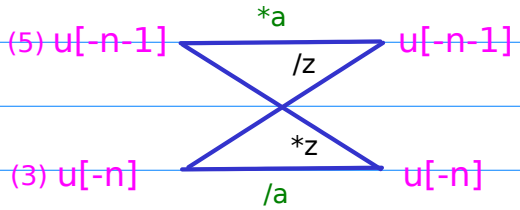
(8) **/z**

	Positive Exponent	Negative Exponent		
Complement Type I	(1) u(n) (5) u(-n-1)	(2) u(n) (6) u(-n-1)	*z u(n-1) *z u(-n)	*z u(n-1) *z u(-n)
Complement Type II	(3) u(-n) (7) u(n-1)	(4) u(-n) (8) u(n-1)	/z u(-n-1) /z u(n)	/z u(-n-1) /z u(n)

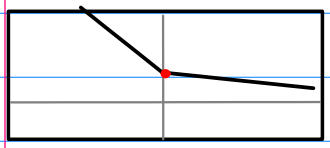
a^n

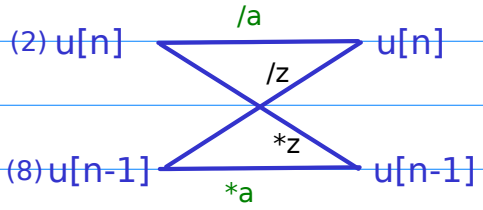


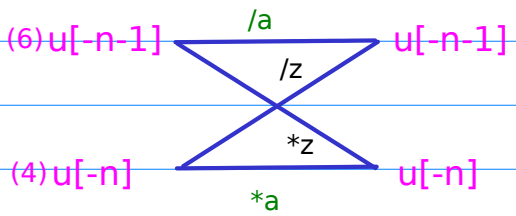




a^{-n}





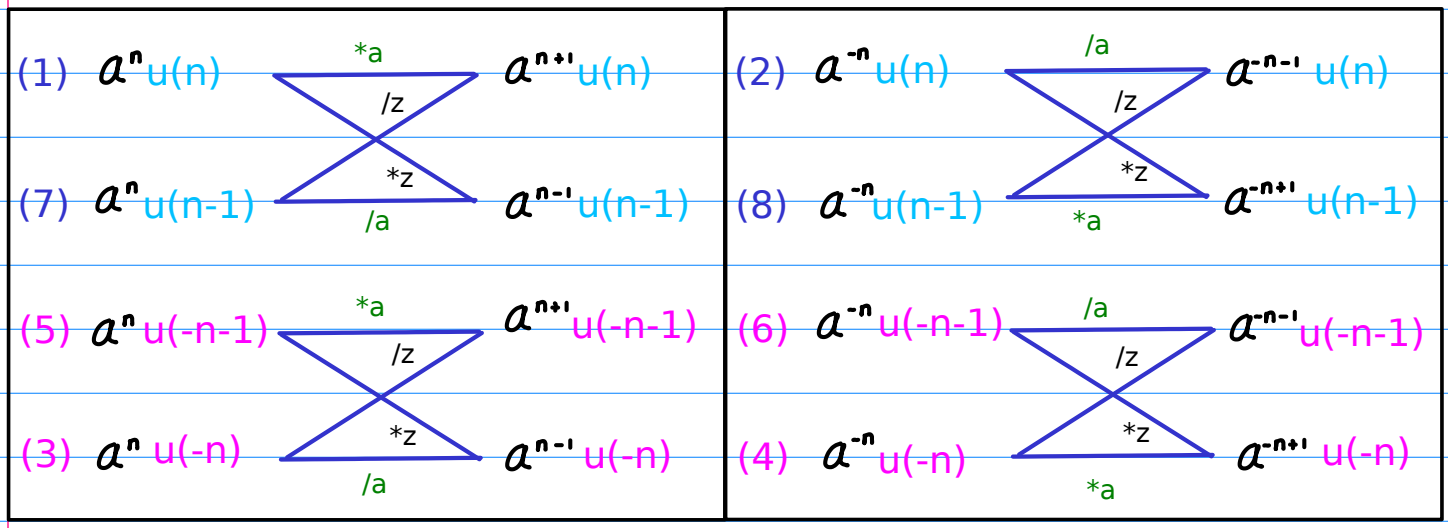


Causal	**u(n)**	(1)	(2)
	u(n-1)	(7)	(8)
Anti-Causal	**u(-n-1)**	(5)	(6)
	u(-n)	(3)	(4)

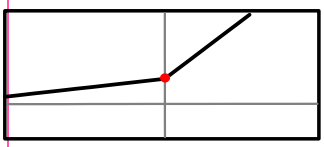
butterfly pair ordering

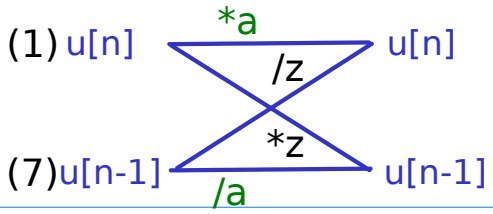
Positive Exponent	**Negative Exponent**
(1)	(2)
(3)	(4)
(5)	(6)
(7)	(8)

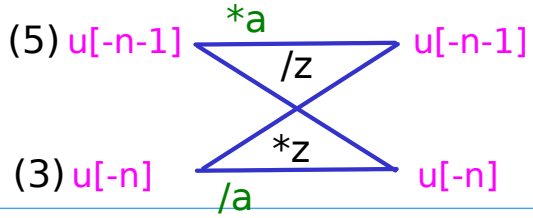
row major ordering .


(1) $a^n u(n)$
*a
/z
$a^{n+1} u(n)$
(7) $a^n u(n-1)$
*z
/a
$a^{n-1} u(n-1)$
(2) $a^{-n} u(n)$
/a
/z
$a^{-n-1} u(n)$
(8) $a^{-n} u(n-1)$
*z
*a
$a^{-n+1} u(n-1)$
(5) $a^n u(-n-1)$
*a
/z
$a^{n+1} u(-n-1)$
(3) $a^n u(-n)$
*z
/a
$a^{n-1} u(-n)$
(6) $a^{-n} u(-n-1)$
/a
/z
$a^{-n-1} u(-n-1)$
(4) $a^{-n} u(-n)$
*z
*a
$a^{-n+1} u(-n)$

a^n







(1) ***a**

$\frac{1}{1-az}$ $\lvert z\rvert < a^{-1}$	$\frac{a}{1-az}$ $\lvert z\rvert < a^{-1}$
$\frac{az}{1-az}$ $\lvert z\rvert < a^{-1}$	$\frac{z}{1-az}$ $\lvert z\rvert < a^{-1}$

(7) **/a**

(1) ***a**

a^n u(n) $(a^0, a^1, a^2, \cdots)$	a^{n+1} u(n) $(a^1, a^2, a^3, \cdots)$
a^n u(n-1) $(a^1, a^2, a^3, \cdots)$	a^{n-1} u(n-1) $(a^0, a^1, a^2, \cdots)$

(7) **/a**

(1) ***z**

$\frac{1}{1-az}$ $\lvert z\rvert < a^{-1}$	$\frac{z}{1-az}$ $\lvert z\rvert < a^{-1}$
$\frac{az}{1-az}$ $\lvert z\rvert < a^{-1}$	$\frac{a}{1-az}$ $\lvert z\rvert < a^{-1}$

(7) **/z**

(1) ***z**

a^n u(n) $(a^0, a^1, a^2, \cdots)$	a^{n-1} u(n-1) $(a^0, a^1, a^2, \cdots)$
a^n u(n-1) $(a^1, a^2, a^3, \cdots)$	a^{n+1} u(n) $(a^1, a^2, a^3, \cdots)$

(7) **/z**

(5) ***a**

$-\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}}$ $\lvert z\rvert > a^{-1}$	$-\frac{z^{-1}}{1-a^{-1}z^{-1}}$ $\lvert z\rvert > a^{-1}$
$-\frac{1}{1-a^{-1}z^{-1}}$ $\lvert z\rvert > a^{-1}$	$-\frac{a^{-1}}{1-a^{-1}z^{-1}}$ $\lvert z\rvert > a^{-1}$

(3) **/a**

(5) ***a**

$-a^n$ u(-n-1) $-(\cdots, \frac{1}{a^2}, \frac{1}{a^1}, \frac{1}{a^0})$	$-a^{n+1}$ u(-n-1) $-(\cdots, \frac{1}{a^2}, \frac{1}{a^1}, \frac{1}{a^0})$
$-a^n$ u(-n) $-(\cdots, \frac{1}{a^2}, \frac{1}{a^1}, \frac{1}{a^0})$	$-a^{n-1}$ u(-n) $-(\cdots, \frac{1}{a^3}, \frac{1}{a^2}, \frac{1}{a^1})$

(3) **/a**

(5) ***z**

$-\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}}$ $\lvert z\rvert > a^{-1}$	$-\frac{a^{-1}}{1-a^{-1}z^{-1}}$ $\lvert z\rvert > a^{-1}$
$-\frac{1}{1-a^{-1}z^{-1}}$ $\lvert z\rvert > a^{-1}$	$-\frac{z^{-1}}{1-a^{-1}z^{-1}}$ $\lvert z\rvert > a^{-1}$

(3) **/z**

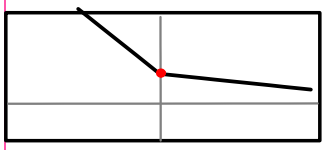
(5) ***z**

$-a^n$ u(-n-1) $-(\cdots, \frac{1}{a^2}, \frac{1}{a^1}, \frac{1}{a^0})$	$-a^{n-1}$ u(-n) $-(\cdots, \frac{1}{a^3}, \frac{1}{a^2}, \frac{1}{a^1})$
$-a^n$ u(-n) $-(\cdots, \frac{1}{a^2}, \frac{1}{a^1}, \frac{1}{a^0})$	$-a^{n+1}$ u(-n-1) $-(\cdots, \frac{1}{a^2}, \frac{1}{a^1}, \frac{1}{a^0})$

(3) **/z**

butterfly pair ordering

a^{-n}



(2) u[n] — /a → u[n]; /z → u[n]

(8) u[n-1] — *a → u[n-1]; *z → u[n-1]

(6) u[-n-1] — /a → u[-n-1]; /z → u[-n-1]

(4) u[-n] — *a → u[-n]; *z → u[-n]

(2) **/a**

$\frac{1}{1-a^{-1}z}$ $\lvert z\rvert < a$	$\frac{a^{-1}}{1-a^{-1}z}$ $\lvert z\rvert < a$
$\frac{a^{-1}z}{1-a^{-1}z}$ $\lvert z\rvert < a$	$\frac{z}{1-a^{-1}z}$ $\lvert z\rvert < a$

(8) ***a**

(2) **/a**

$\left(\frac{1}{a}\right)^n$ u(n) $\left(\frac{1}{a^0}, \frac{1}{a^1}, \frac{1}{a^2}, \cdots\right)$	$\left(\frac{1}{a}\right)^{n+1}$ u(n) $\left(\frac{1}{a^1}, \frac{1}{a^2}, \frac{1}{a^3}, \cdots\right)$
$\left(\frac{1}{a}\right)^n$ u(n-1) $\left(\frac{1}{a^1}, \frac{1}{a^2}, \frac{1}{a^3}, \cdots\right)$	$\left(\frac{1}{a}\right)^{n-1}$ u(n-1) $\left(\frac{1}{a^0}, \frac{1}{a^1}, \frac{1}{a^2}, \cdots\right)$

(8) ***a**

(2) ***z**

$\frac{1}{1-a^{-1}z}$ $\lvert z\rvert < a$	$\frac{z}{1-a^{-1}z}$ $\lvert z\rvert < a$
$\frac{a^{-1}z}{1-a^{-1}z}$ $\lvert z\rvert < a$	$\frac{a^{-1}}{1-a^{-1}z}$ $\lvert z\rvert < a$

(8) **/z**

(2) ***z**

$\left(\frac{1}{a}\right)^n$ u(n) $\left(\frac{1}{a^0}, \frac{1}{a^1}, \frac{1}{a^2}, \cdots\right)$	$\left(\frac{1}{a}\right)^{n-1}$ u(n-1) $\left(\frac{1}{a^0}, \frac{1}{a^1}, \frac{1}{a^2}, \cdots\right)$
$\left(\frac{1}{a}\right)^n$ u(n-1) $\left(\frac{1}{a^1}, \frac{1}{a^2}, \frac{1}{a^3}, \cdots\right)$	$\left(\frac{1}{a}\right)^{n+1}$ u(n) $\left(\frac{1}{a^1}, \frac{1}{a^2}, \frac{1}{a^3}, \cdots\right)$

(8) **/z**

(6) **/a**

$-\frac{az^{-1}}{1-az^{-1}}$ $\lvert z\rvert > a$	$-\frac{z^{-1}}{1-az^{-1}}$ $\lvert z\rvert > a$
$-\frac{1}{1-az^{-1}}$ $\lvert z\rvert > a$	$-\frac{a}{1-az^{-1}}$ $\lvert z\rvert > a$

(4) ***a**

(6) **/a**

$-\left(\frac{1}{a}\right)^n$ u(-n-1) $-\left(\cdots, a^3, a^2, a^1\right)$	$-\left(\frac{1}{a}\right)^{n+1}$ u(-n-1) $-\left(\cdots, a^2, a^1, a^0\right)$
$-\left(\frac{1}{a}\right)^n$ u(-n) $-\left(\cdots, a^3, a^1, a^0\right)$	$-\left(\frac{1}{a}\right)^{n-1}$ u(-n) $-\left(\cdots, a^3, a^2, a^1\right)$

(4) ***a**

(6) ***z**

$-\frac{az^{-1}}{1-az^{-1}}$ $\lvert z\rvert > a$	$-\frac{a}{1-az^{-1}}$ $\lvert z\rvert > a$
$-\frac{1}{1-az^{-1}}$ $\lvert z\rvert > a$	$-\frac{z^{-1}}{1-az^{-1}}$ $\lvert z\rvert > a$

(4) **/z**

(6) ***z**

$-\left(\frac{1}{a}\right)^n$ u(-n-1) $-\left(\cdots, a^3, a^2, a^1\right)$	$-\left(\frac{1}{a}\right)^{n-1}$ u(-n) $-\left(\cdots, a^3, a^2, a^1\right)$
$-\left(\frac{1}{a}\right)^n$ u(-n) $-\left(\cdots, a^2, a^1, a^0\right)$	$-\left(\frac{1}{a}\right)^{n+1}$ u(-n-1) $-\left(\cdots, a^2, a^1, a^0\right)$

(4) **/z**

butterfly pair ordering

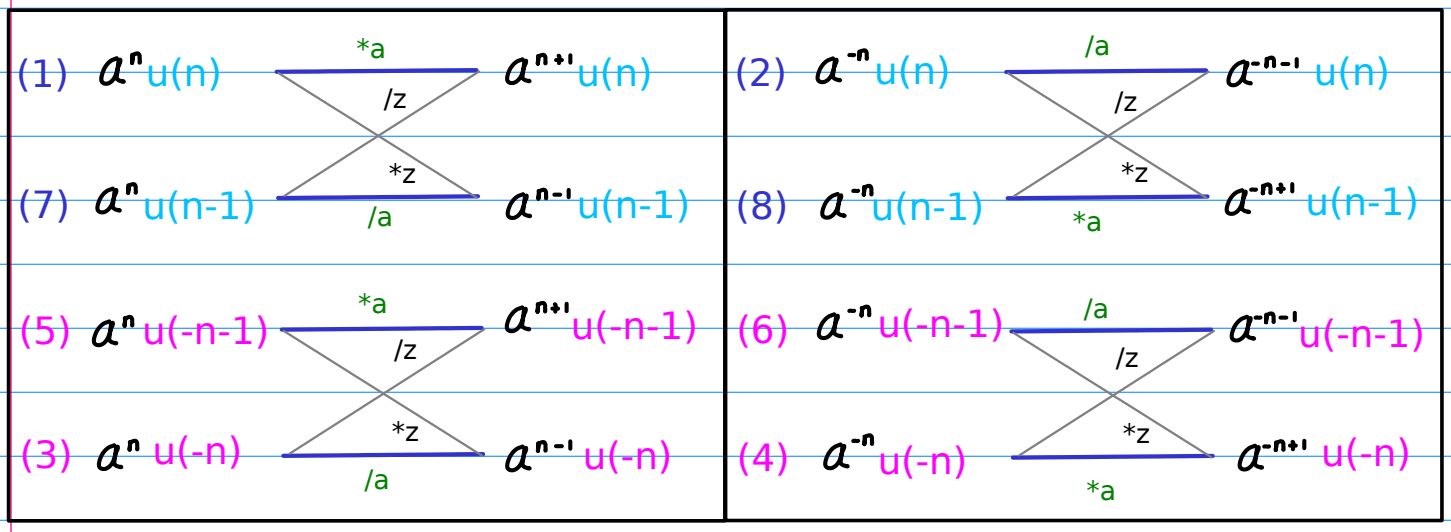
Scale by **a**
1. Geometric Series

complementary pair ordering

	(1)	*a	(2)	/a
	$\frac{1}{1-az}$ $\lvert z\rvert < a^{-1}$	$\frac{a}{1-az}$ $\lvert z\rvert < a^{-1}$	$\frac{1}{1-a^{-1}z}$ $\lvert z\rvert < a$	$\frac{a^{-1}}{1-a^{-1}z}$ $\lvert z\rvert < a$
Comp.ROC	$-\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}}$ $\lvert z\rvert > a^{-1}$	$-\frac{z^{-1}}{1-a^{-1}z^{-1}}$ $\lvert z\rvert > a^{-1}$	$-\frac{az^{-1}}{1-az^{-1}}$ $\lvert z\rvert > a$	$-\frac{z^{-1}}{1-az^{-1}}$ $\lvert z\rvert > a$

	(5)	*a	(6)	/a
	(3)	/a	(4)	*a
	$-\frac{1}{1-a^{-1}z^{-1}}$ $\lvert z\rvert > a^{-1}$	$-\frac{a^{-1}}{1-a^{-1}z^{-1}}$ $\lvert z\rvert > a^{-1}$	$-\frac{1}{1-az^{-1}}$ $\lvert z\rvert > a$	$-\frac{a}{1-az^{-1}}$ $\lvert z\rvert > a$
Comp.ROC	$\frac{az}{1-az}$ $\lvert z\rvert < a^{-1}$	$\frac{z}{1-az}$ $\lvert z\rvert < a^{-1}$	$\frac{a^{-1}z}{1-a^{-1}z}$ $\lvert z\rvert < a$	$\frac{z}{1-a^{-1}z}$ $\lvert z\rvert < a$

	(7)	/a	(8)	*a



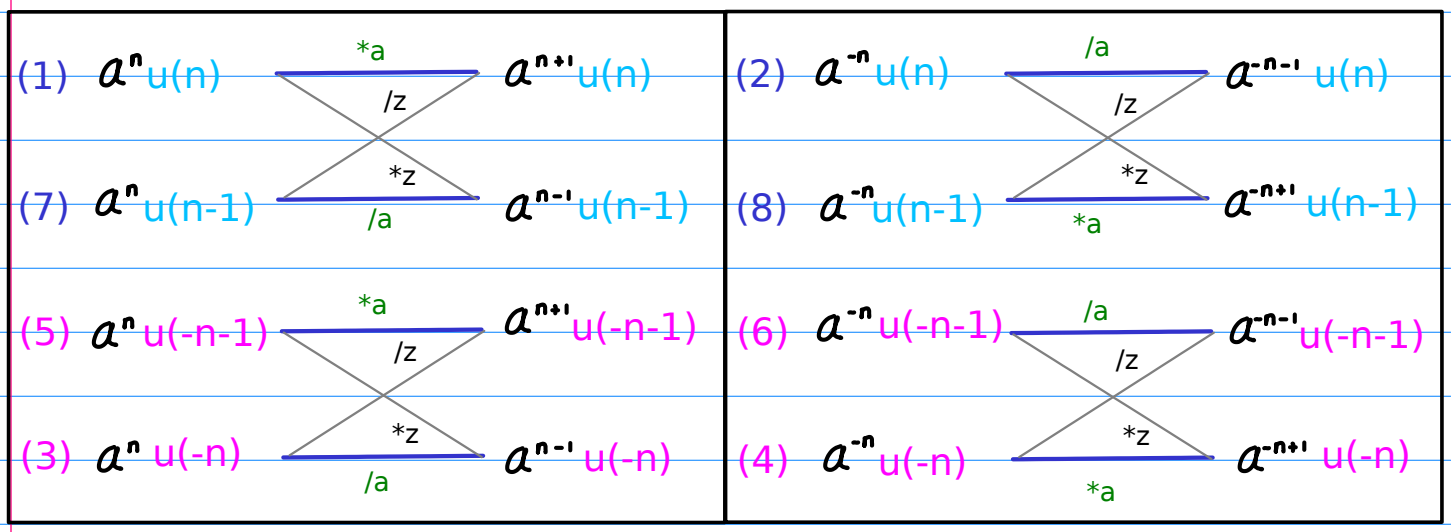
Scale by **a**
2. Sequences

complementary pair ordering

	(1)	***a**	(2)	**/a**
	$a^n\ u(n)$	$a^{n+1}\ u(n)$	$(\frac{1}{a})^n\ u(n)$	$(\frac{1}{a})^{n+1}\ u(n)$
Comp.ROC	$-a^n\ u(-n-1)$	$-a^{n+1}\ u(-n-1)$	$-(\frac{1}{a})^n\ u(-n-1)$	$-(\frac{1}{a})^{n+1}u(-n-1)$

	(5)	***a**	(6)	**/a**
	(3)	**/a**	(4)	***a**
	$-a^n\ u(-n)$	$-a^{n-1}\ u(-n)$	$-(\frac{1}{a})^n\ u(-n)$	$-(\frac{1}{a})^{n-1}\ u(-n)$
Comp.ROC	$a^n\ u(n-1)$	$a^{n-1}\ u(n-1)$	$(\frac{1}{a})^n\ u(n-1)$	$(\frac{1}{a})^{n-1}\ u(n-1)$

	(7)	**/a**	(8)	***a**



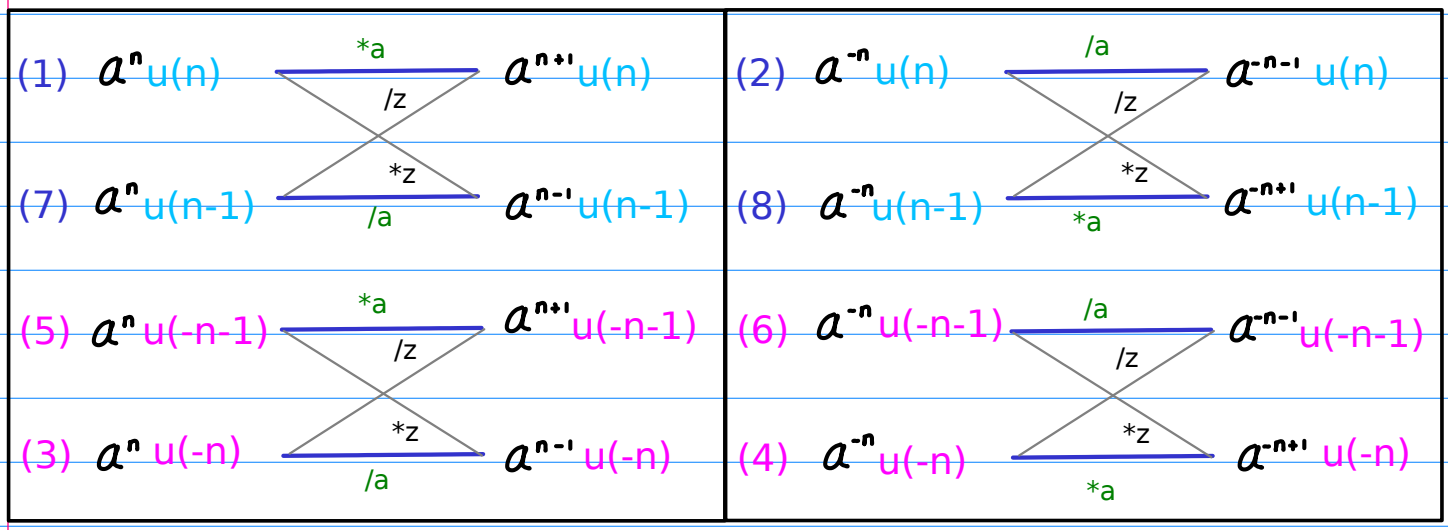
Scale by **a**
3. Sequence values

complementary pair ordering

(1)	*a	(2)	/a
$(a^0, a^1, a^2, \cdots)$	$(a^1, a^2, a^3, \cdots)$	$(\frac{1}{a^0}, \frac{1}{a^1}, \frac{1}{a^2}, \cdots)$	$(\frac{1}{a^1}, \frac{1}{a^2}, \frac{1}{a^3}, \cdots)$
Comp.ROC $-(\cdots, \frac{1}{a^3}, \frac{1}{a^2}, \frac{1}{a^1})$	$-(\cdots, \frac{1}{a^3}, \frac{1}{a^2}, \frac{1}{a^1}, \frac{1}{a^0})$	$-(\cdots, a^3, a^2, a^1)$	$-(\cdots, a^2, a^1, a^0)$

(5)	*a	(6)	/a
(3)	/a	(4)	*a
$-(\cdots, \frac{1}{a^2}, \frac{1}{a^1}, \frac{1}{a^0})$	$-(\cdots, \frac{1}{a^3}, \frac{1}{a^2}, \frac{1}{a^1})$	$-(\cdots, a^2, a^1, a^0)$	$-(\cdots, a^3, a^2, a^1)$
Comp.ROC $(a^1, a^2, a^3, \cdots)$	$(a^0, a^1, a^2, \cdots)$	$(\frac{1}{a^1}, \frac{1}{a^2}, \frac{1}{a^3}, \cdots)$	$(\frac{1}{a^0}, \frac{1}{a^1}, \frac{1}{a^2}, \cdots)$

(7)	/a	(8)	*a

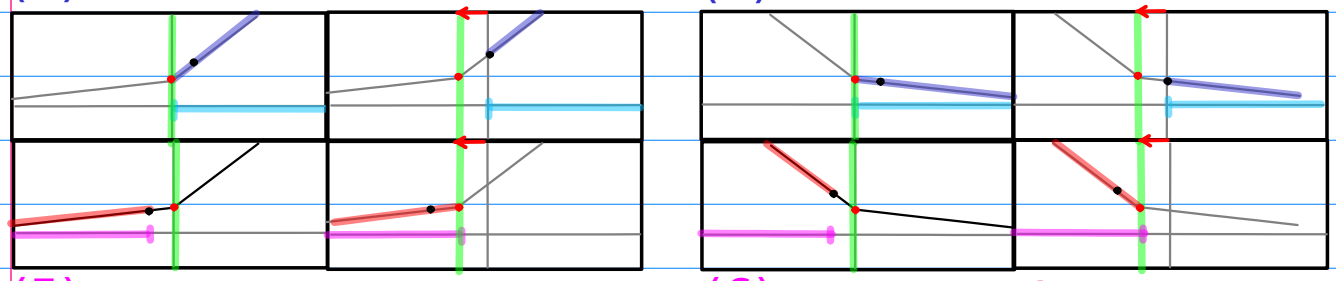


Scale by **a**
4. Graphs

complementary pair ordering

(1) ***a** (2) **/a**

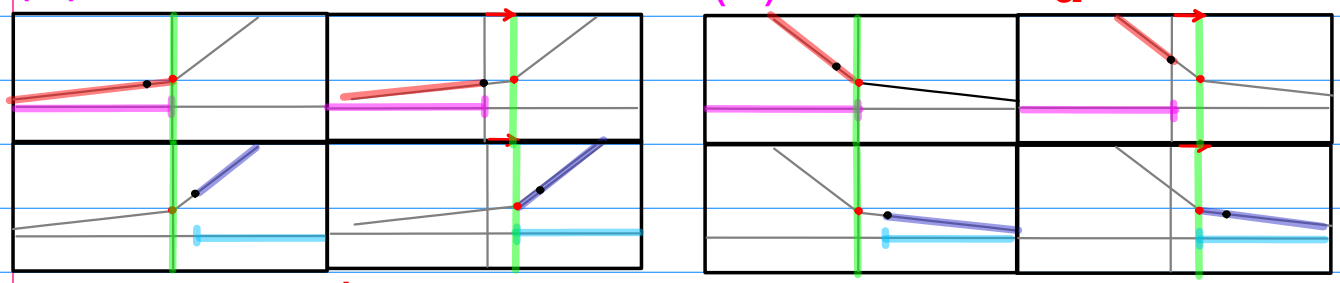
Comp.ROC



(5) ***a** (6) **/a**

(3) **/a** (4) ***a**

Comp.ROC

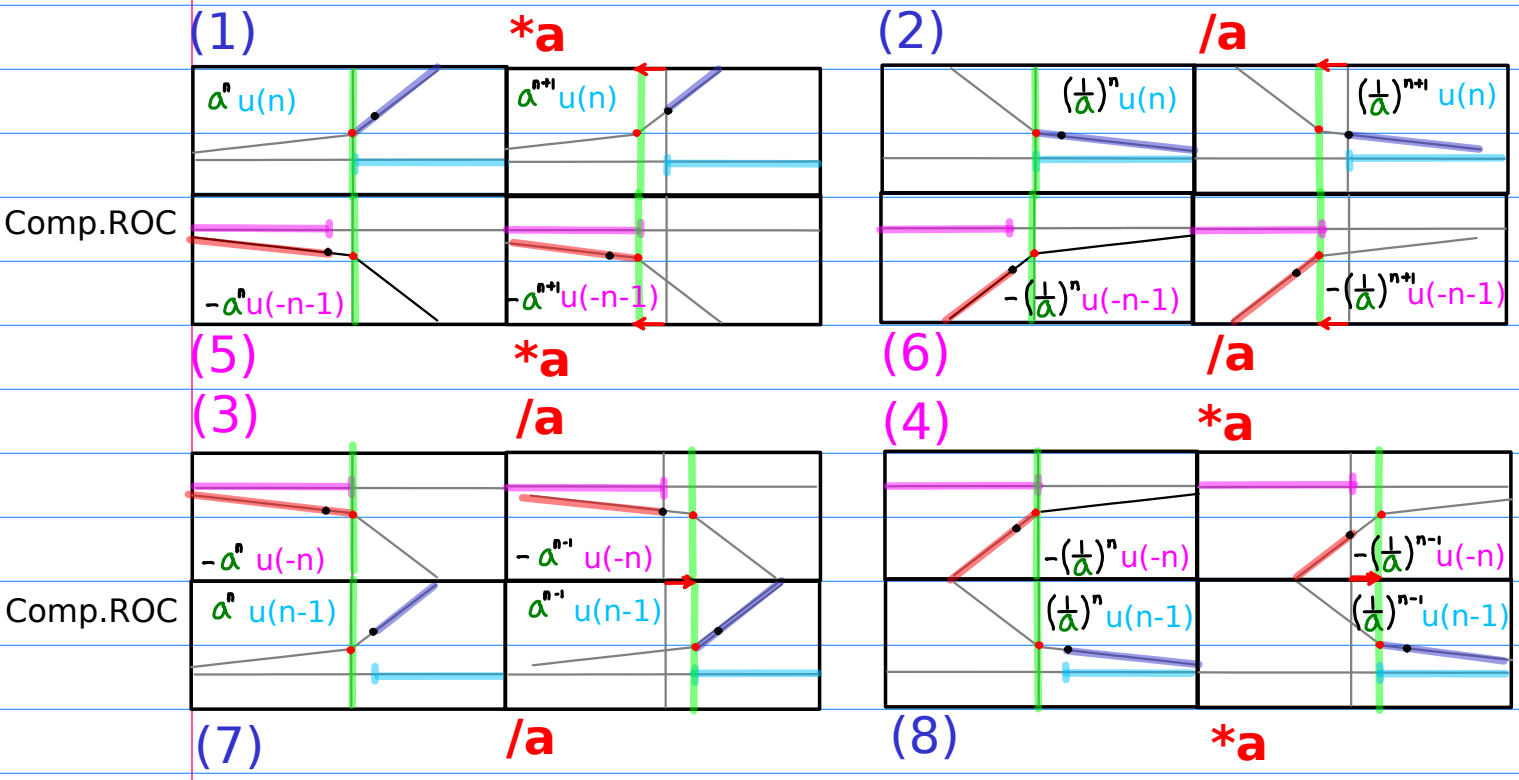


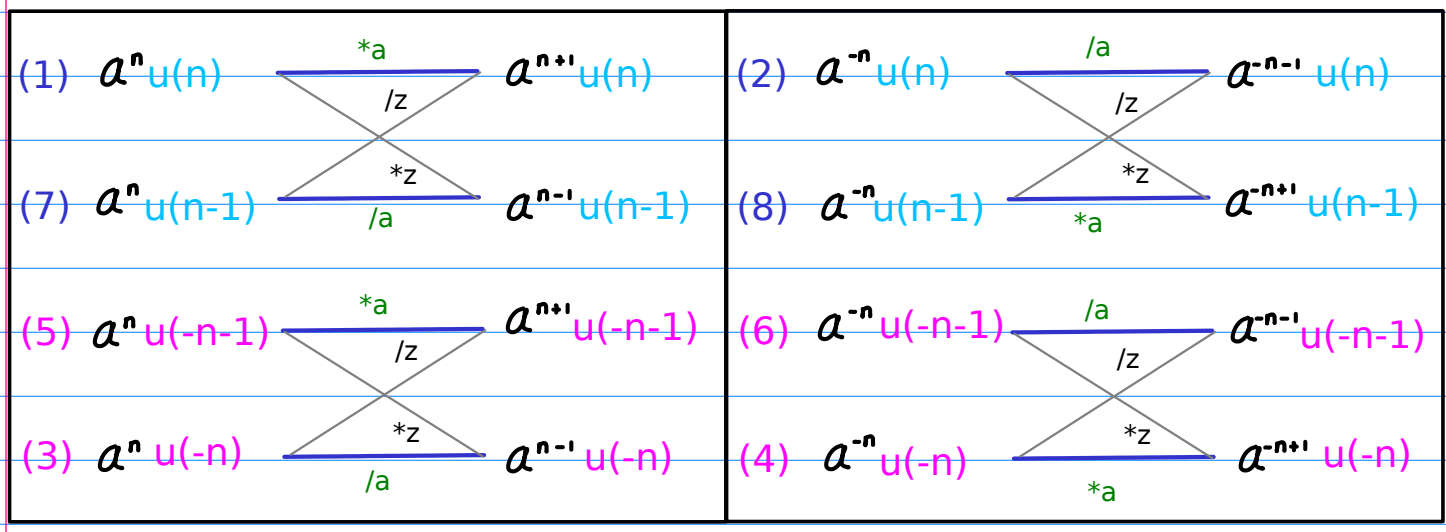
(7) **/a** (8) ***a**

(1) $a^n u(n)$	*a	$a^{n+1} u(n)$	(2) $a^{-n} u(n)$	/a	$a^{-n-1} u(n)$
(7) $a^n u(n-1)$	/a	$a^{n-1} u(n-1)$	(8) $a^{-n} u(n-1)$	*a	$a^{-n+1} u(n-1)$
(5) $a^n u(-n-1)$	*a	$a^{n+1} u(-n-1)$	(6) $a^{-n} u(-n-1)$	/a	$a^{-n-1} u(-n-1)$
(3) $a^n u(-n)$	/a	$a^{n-1} u(-n)$	(4) $a^{-n} u(-n)$	*a	$a^{-n+1} u(-n)$

Scale by **a**
5. Graphs - signs

complementary pair ordering

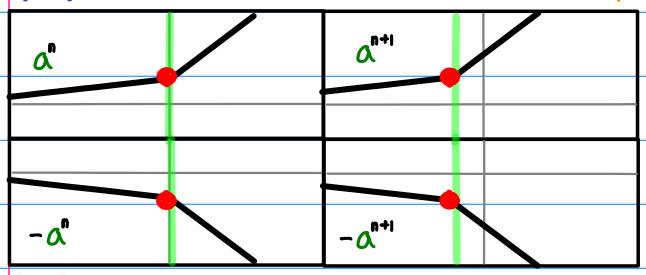




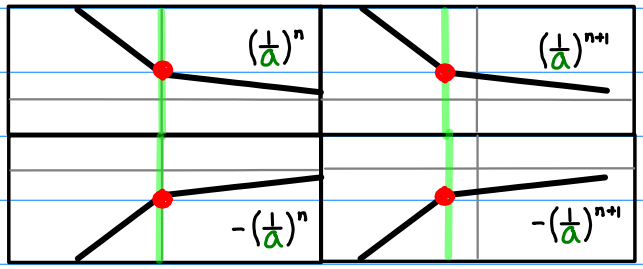
complementary pair ordering

Scale by **a**
6. Graphs - Exponents

(1) ***a** ← SHL.Exp



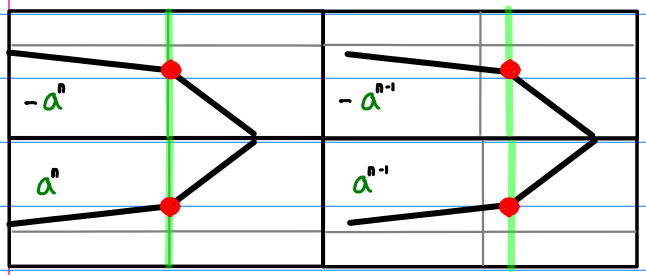
(2) **/a** ← SHL.Exp



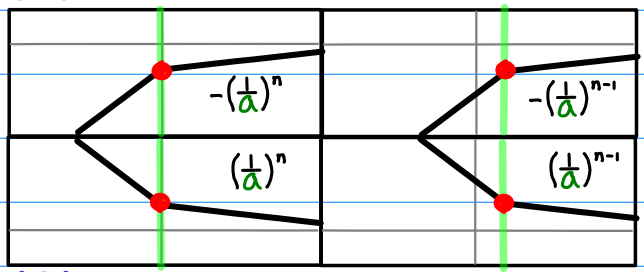
(5) ***a** ← SHL.Exp

(6) **/a** ← SHL.Exp

(3) **/a** → SHR.Exp



(4) ***a** → SHR.Exp



(7) **/a** → SHR.Exp

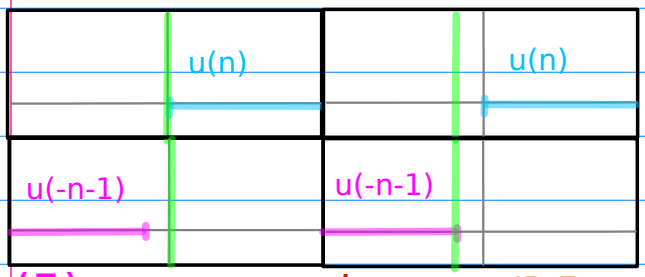
(8) ***a** → SHR.Exp

(1) $a^n u(n)$ — *a, /z → $a^{n+1} u(n)$	(2) $a^{-n} u(n)$ — /a, /z → $a^{-n-1} u(n)$
(7) $a^n u(n-1)$ — *z, /a → $a^{n-1} u(n-1)$	(8) $a^{-n} u(n-1)$ — *z, *a → $a^{-n+1} u(n-1)$
(5) $a^n u(-n-1)$ — *a, /z → $a^{n+1} u(-n-1)$	(6) $a^{-n} u(-n-1)$ — /a, /z → $a^{-n-1} u(-n-1)$
(3) $a^n u(-n)$ — *z, /a → $a^{n-1} u(-n)$	(4) $a^{-n} u(-n)$ — *z, *a → $a^{-n+1} u(-n)$

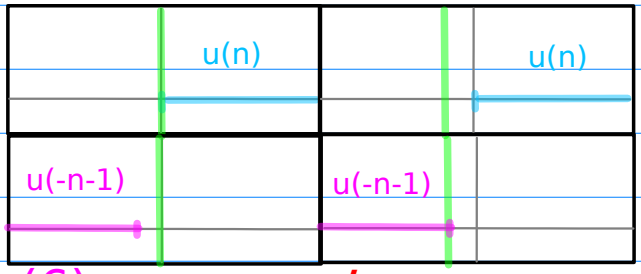
Scale by **a**
7. Graphs - Ranges

complementary pair ordering

(1) ***a** ID.Exp



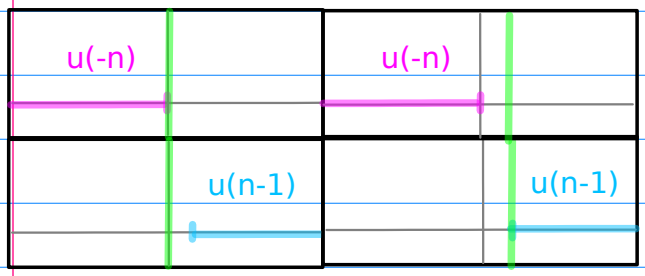
(2) **/a** ID.Exp



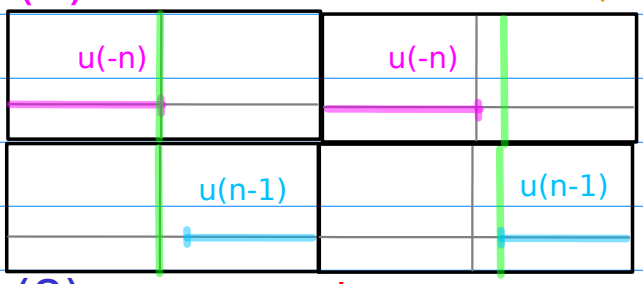
(5) ***a** ID.Exp

(6) **/a** ID.Exp

(3) **/a** ID.Exp

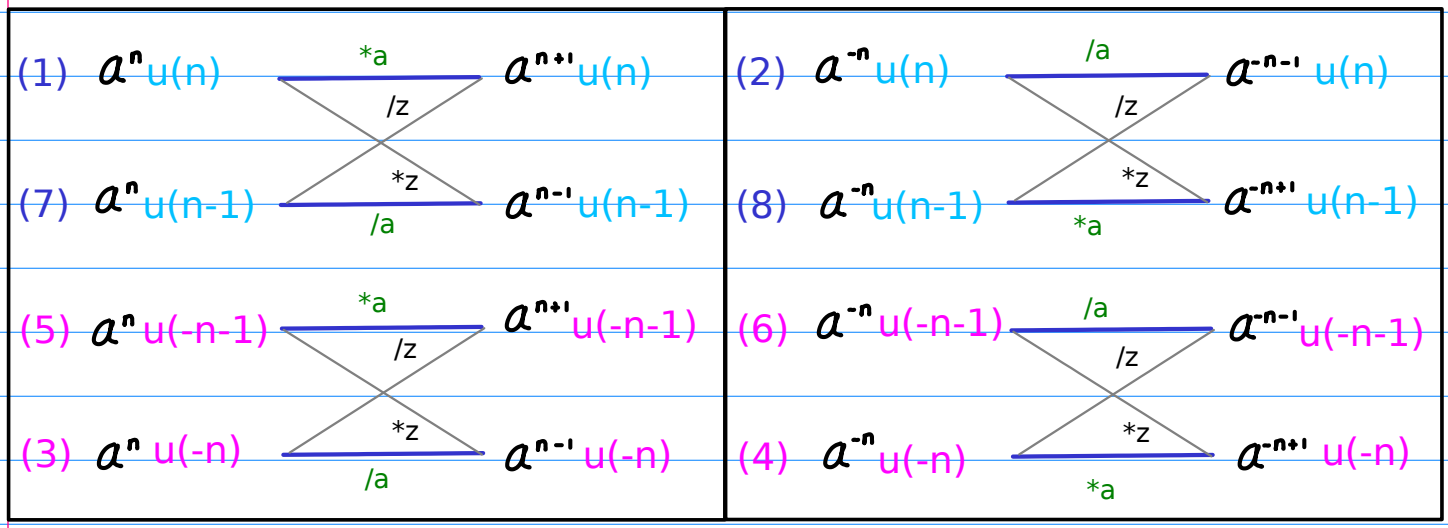


(4) ***a** ID.Exp



(7) **/a** ID.Exp

(8) ***a** ID.Exp



Scale by z
1. Geometric Series

complementary pair ordering

(1)	*z	(2)	*z
$\frac{1}{1-az}$ $\lvert z\rvert < a^{-1}$	$\frac{z}{1-az}$ $\lvert z\rvert < a^{-1}$	$\frac{1}{1-a^{-1}z}$ $\lvert z\rvert < a$	$\frac{z}{1-a^{-1}z}$ $\lvert z\rvert < a$
Comp.ROC $-\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}}$ $\lvert z\rvert > a^{-1}$	$-\frac{a^{-1}}{1-a^{-1}z^{-1}}$ $\lvert z\rvert > a^{-1}$	$-\frac{az^{-1}}{1-az^{-1}}$ $\lvert z\rvert > a$	$-\frac{a}{1-az^{-1}}$ $\lvert z\rvert > a$

(5)	*z	(6)	*z
(3)	/z	(4)	/z
$-\frac{1}{1-a^{-1}z^{-1}}$ $\lvert z\rvert > a^{-1}$	$-\frac{z^{-1}}{1-a^{-1}z^{-1}}$ $\lvert z\rvert > a^{-1}$	$-\frac{1}{1-az^{-1}}$ $\lvert z\rvert > a$	$-\frac{z^{-1}}{1-az^{-1}}$ $\lvert z\rvert > a$
Comp.ROC $\frac{az}{1-az}$ $\lvert z\rvert < a^{-1}$	$\frac{a}{1-az}$ $\lvert z\rvert < a^{-1}$	$\frac{a^{-1}z}{1-a^{-1}z}$ $\lvert z\rvert < a$	$\frac{a^{-1}}{1-a^{-1}z}$ $\lvert z\rvert < a$
(7)	/z	(8)	/z

(1) $a^n u(n)$	*a, /z → $a^{n+1}u(n)$	(2) $a^{-n}u(n)$	/a, /z → $a^{-n-1}u(n)$
(7) $a^n u(n-1)$	*z, /a → $a^{n-1}u(n-1)$	(8) $a^{-n}u(n-1)$	*z, *a → $a^{-n+1}u(n-1)$
(5) $a^n u(-n-1)$	*a, /z → $a^{n+1}u(-n-1)$	(6) $a^{-n}u(-n-1)$	/a, /z → $a^{-n-1}u(-n-1)$
(3) $a^n u(-n)$	*z, /a → $a^{n-1}u(-n)$	(4) $a^{-n}u(-n)$	*z, *a → $a^{-n+1}u(-n)$

Scale by z
2. Sequences

complementary pair ordering

(1)	*z	(2)	*z
$a^n\ u(n)$	$a^{n-1}\ u(n-1)$	$(\frac{1}{a})^n\ u(n)$	$(\frac{1}{a})^{n-1}\ u(n-1)$
Comp.ROC $-a^n\ u(-n-1)$	$-a^{n-1}\ u(-n)$	$-(\frac{1}{a})^n\ u(-n-1)$	$-(\frac{1}{a})^{n-1}\ u(-n)$

(5)	*z	(6)	*z
(3)	/z	(4)	/z
$-a^n\ u(-n)$	$-a^{n+1}\ u(-n-1)$	$-(\frac{1}{a})^n\ u(-n)$	$-(\frac{1}{a})^{n+1}u(-n-1)$
Comp.ROC $a^n\ u(n-1)$	$a^{n+1}\ u(n)$	$(\frac{1}{a})^n\ u(n-1)$	$(\frac{1}{a})^{n+1}\ u(n)$
(7)	/z	(8)	/z

(1) $a^n u(n)$	*a, /z, *z, /a	$a^{n+1}u(n)$	(2) $a^{-n}u(n)$	/a, /z, *z, *a	$a^{-n-1}u(n)$
(7) $a^n u(n-1)$		$a^{n-1}u(n-1)$	(8) $a^{-n}u(n-1)$		$a^{-n+1}u(n-1)$
(5) $a^n u(-n-1)$	*a, /z, *z, /a	$a^{n+1}u(-n-1)$	(6) $a^{-n}u(-n-1)$	/a, /z, *z, *a	$a^{-n-1}u(-n-1)$
(3) $a^n u(-n)$		$a^{n-1}u(-n)$	(4) $a^{-n}u(-n)$		$a^{-n+1}u(-n)$

Scale by z
3. Sequence values

complementary pair ordering

(1) *z (2) *z

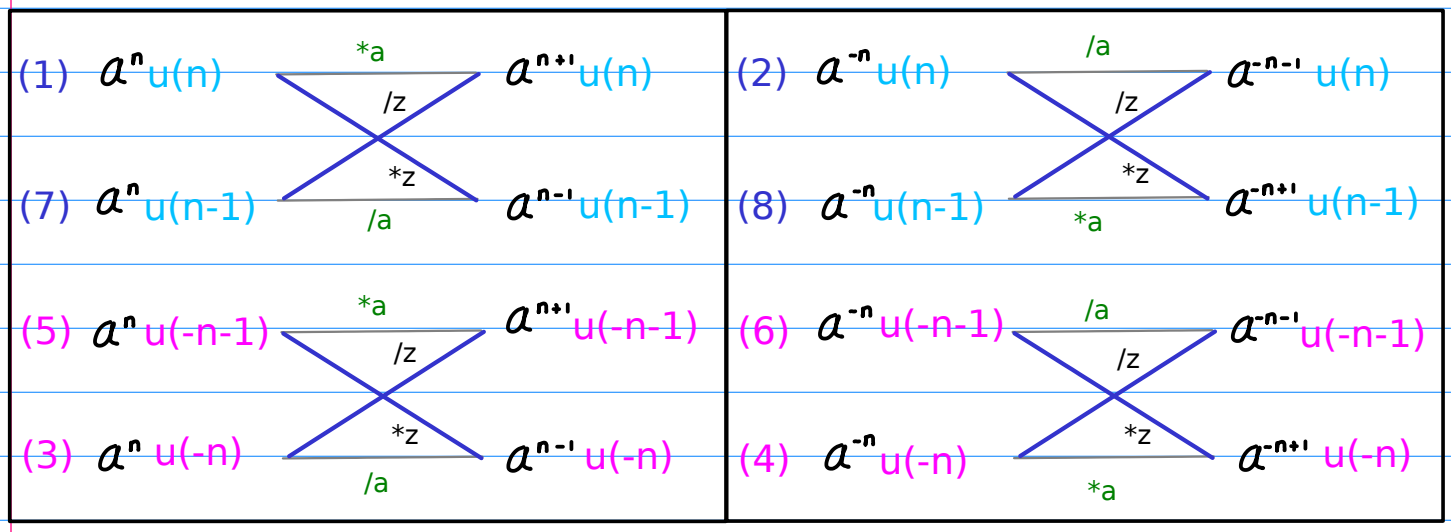
$(a^0, a^1, a^2, \cdots)$	$(a^0, a^1, a^2, \cdots)$	$(\frac{1}{a^0}, \frac{1}{a^1}, \frac{1}{a^2}, \cdots)$	$(\frac{1}{a^1}, \frac{1}{a^2}, \frac{1}{a^3}, \cdots)$
Comp.ROC	$-(\cdots, \frac{1}{a^3}, \frac{1}{a^2}, \frac{1}{a^1})$	$-(\cdots, \frac{1}{a^3}, \frac{1}{a^2}, \frac{1}{a^1})$	$-(\cdots, a^3, a^2, a^1)$	$-(\cdots, a^3, a^2, a^1)$

(5) *z (6) *z

(3) /z (4) /z

$-(\cdots, \frac{1}{a^2}, \frac{1}{a^1}, \frac{1}{a^0})$	$-(\cdots, \frac{1}{a^2}, \frac{1}{a^1}, \frac{1}{a^0})$	$-(\cdots, a^2, a^1, a^0)$	$-(\cdots, a^2, a^1, a^0)$
Comp.ROC	$(a^1, a^2, a^3, \cdots)$	$(a^1, a^2, a^3, \cdots)$	$(\frac{1}{a^1}, \frac{1}{a^2}, \frac{1}{a^3}, \cdots)$	$(\frac{1}{a^1}, \frac{1}{a^2}, \frac{1}{a^3}, \cdots)$

(7) /z (8) /z

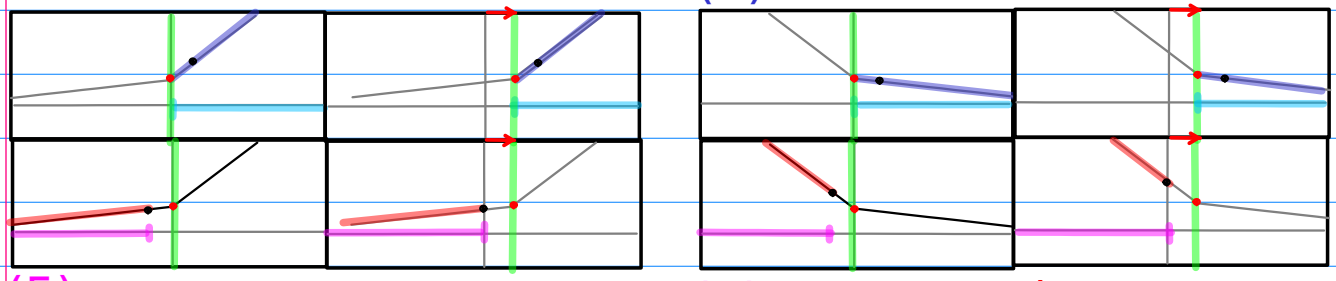


Scale by z
4. Graphs

complementary pair ordering

(1) *z (2) *z

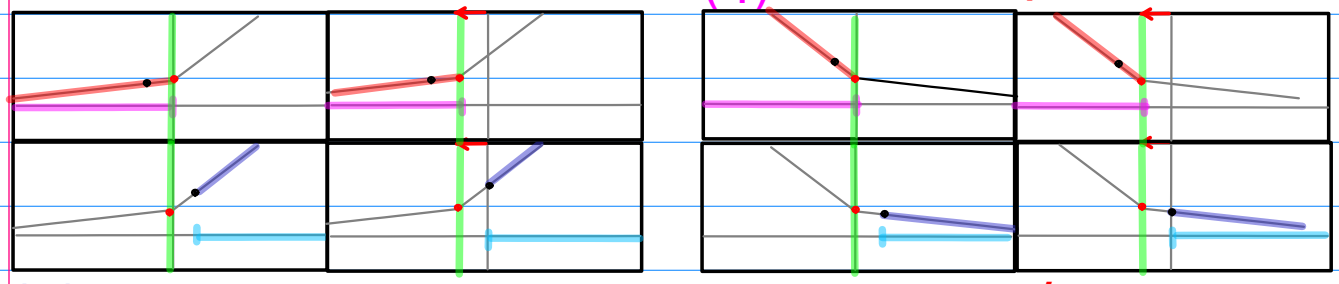
Comp.ROC



(5) *z (6) *z

(3) /a (4) /z

Comp.ROC



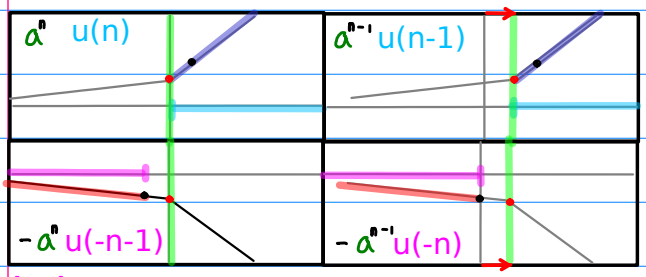
(7) /z (8) /z

(1) $a^n u(n)$	*a	$a^{n+1} u(n)$	(2) $a^{-n} u(n)$	/a	$a^{-n-1} u(n)$
(7) $a^n u(n-1)$	/z, *z	$a^{n-1} u(n-1)$	(8) $a^{-n} u(n-1)$	/z, *z	$a^{-n+1} u(n-1)$
	/a			*a	
(5) $a^n u(-n-1)$	*a	$a^{n+1} u(-n-1)$	(6) $a^{-n} u(-n-1)$	/a	$a^{-n-1} u(-n-1)$
(3) $a^n u(-n)$	/z, *z	$a^{n-1} u(-n)$	(4) $a^{-n} u(-n)$	/z, *z	$a^{-n+1} u(-n)$
	/a			*a	

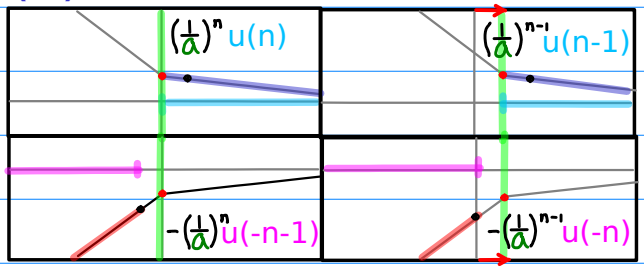
Scale by z
5. Graphs - signs

complementary pair ordering

(1) *z

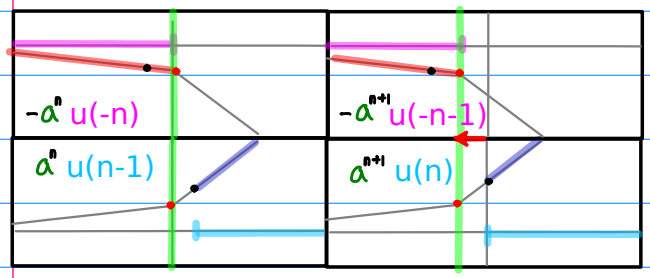


(2) *z



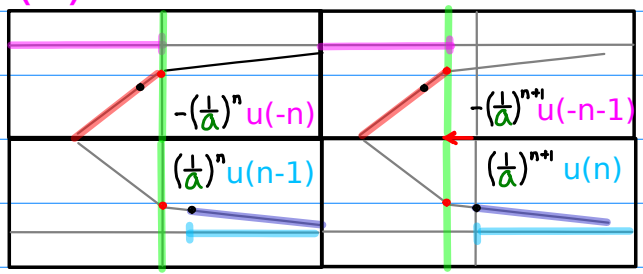
(5) *z

(3) /z



(6) *z

(4) /z



(7) /z

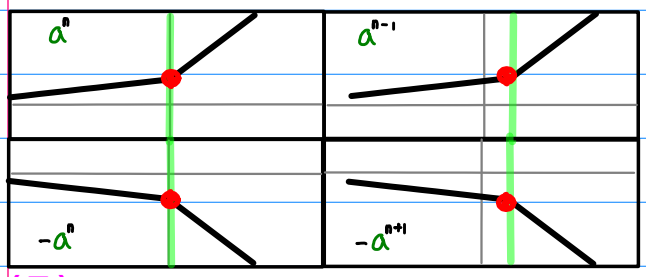
(8) /z

(1) $a^n u(n)$ —*a→ $a^{n+1}u(n)$	(2) $a^{-n}u(n)$ —/a→ $a^{-n-1}u(n)$
(7) $a^n u(n-1)$ —/a→ $a^{n-1}u(n-1)$	(8) $a^{-n}u(n-1)$ —*a→ $a^{-n+1}u(n-1)$
(5) $a^n u(-n-1)$ —*a→ $a^{n+1}u(-n-1)$	(6) $a^{-n}u(-n-1)$ —/a→ $a^{-n-1}u(-n-1)$
(3) $a^n u(-n)$ —/a→ $a^{n-1}u(-n)$	(4) $a^{-n}u(-n)$ —*a→ $a^{-n+1}u(-n)$

Scale by z
6. Graphs - Exponents

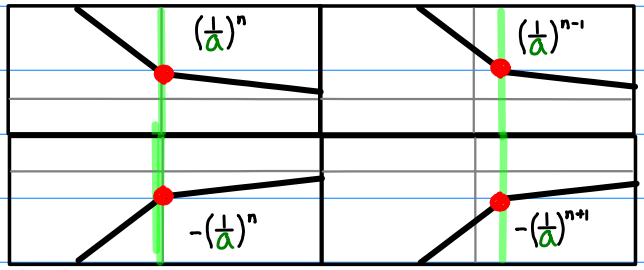
complementary pair ordering

(1) *z → SHR.Exp



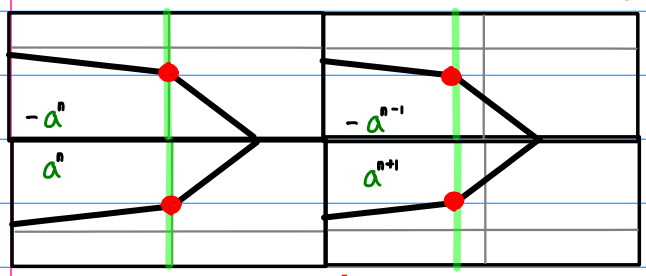
Comp.ROC

(2) *z → SHR.Exp



(5) *z → SHR.Exp

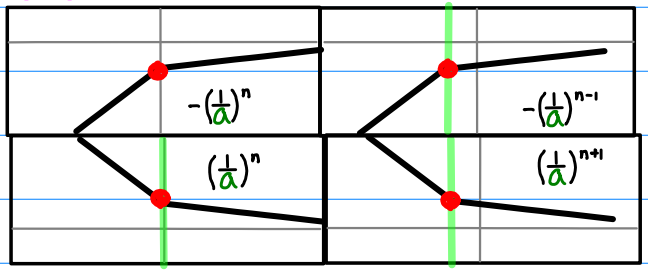
(3) /z ← SHL.Exp



Comp.ROC

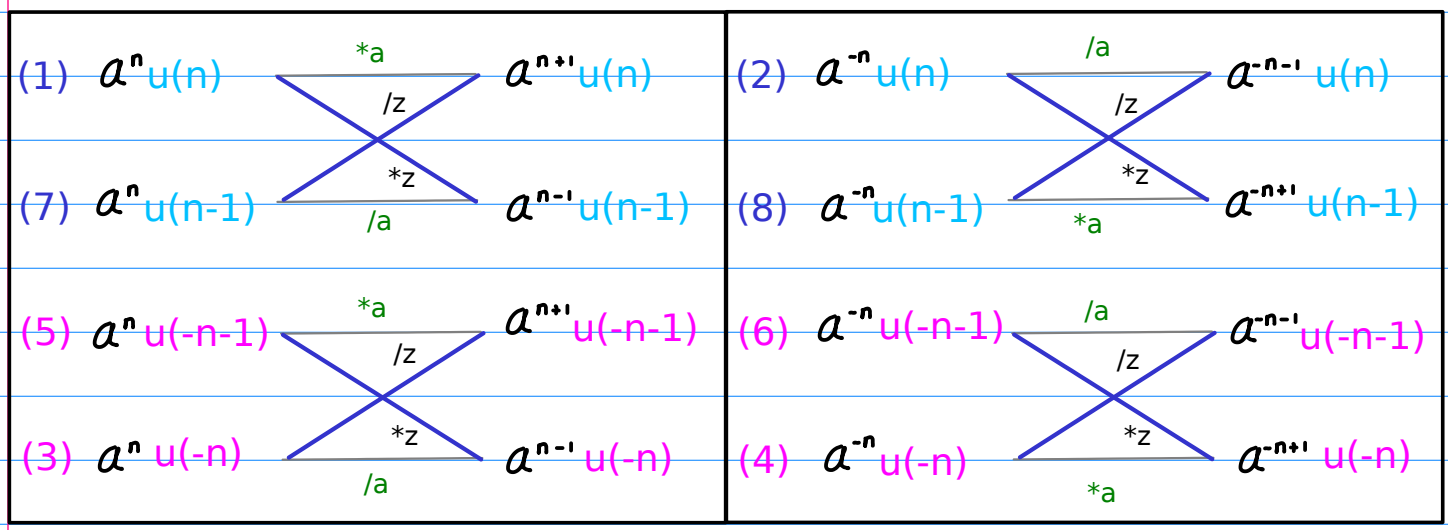
(6) *z → SHR.Exp

(4) /z ← SHL.Exp



(7) /z ← SHL.Exp

(8) /z ← SHL.Exp



Scale by z
7. Graphs - Ranges

complementary pair ordering