

First Order ODE's (1A)

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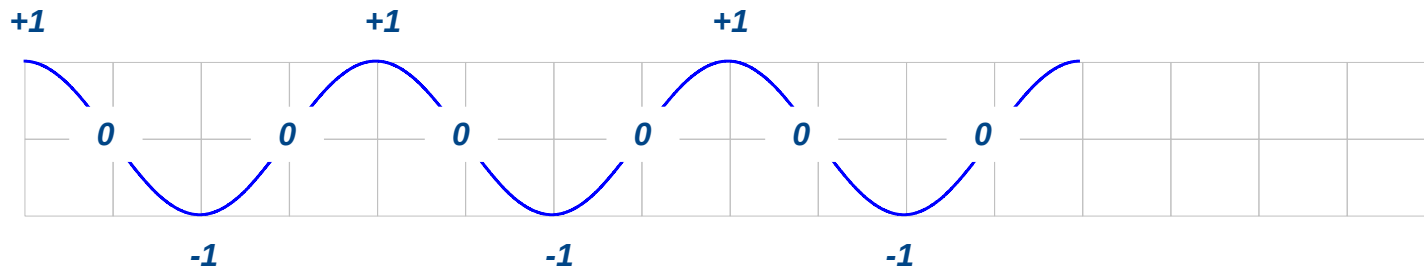
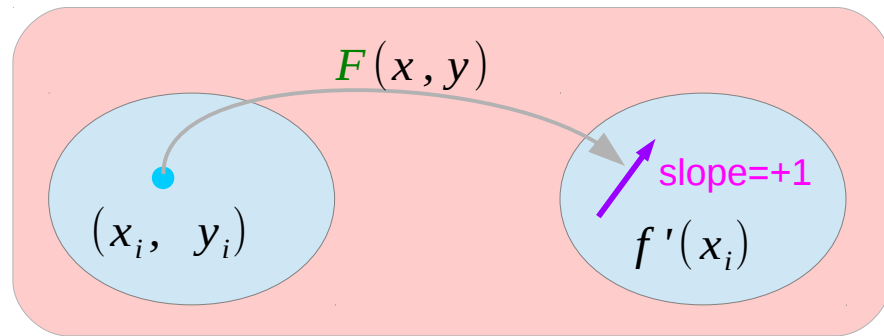
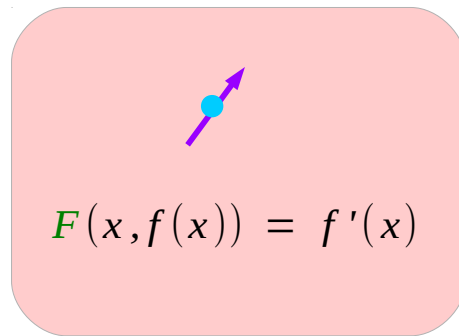
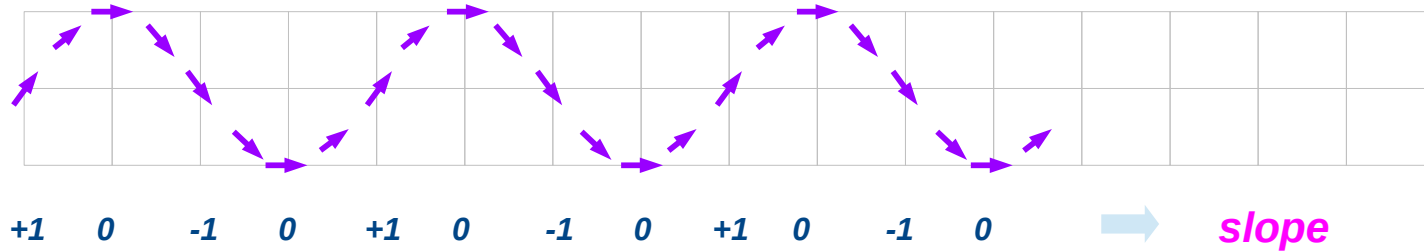
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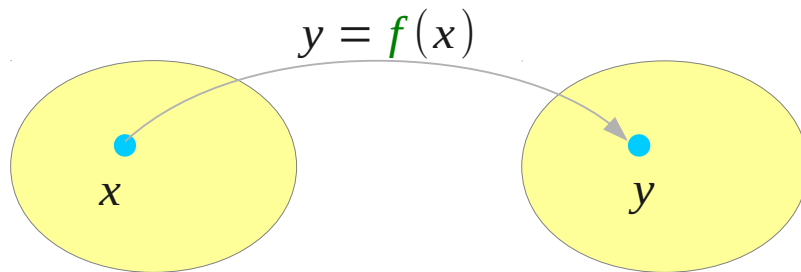
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Direction Fields

Plot of $F(x,y)=\cos(x)$

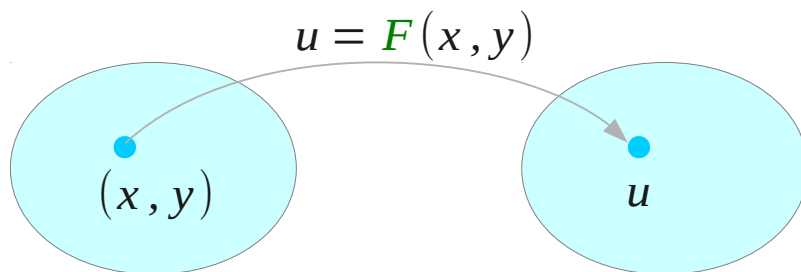


A function of two variables



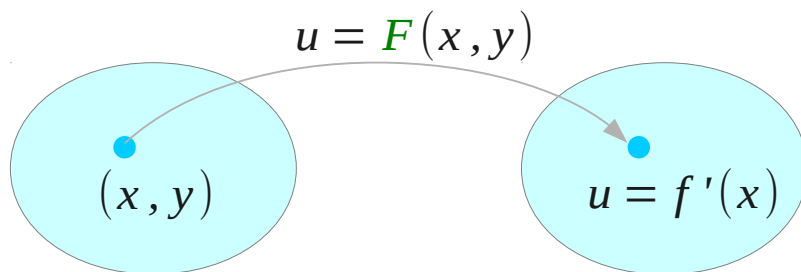
$$y = f(x)$$

f maps x to y



$$u = F(x, y)$$

F maps (x, y) to u



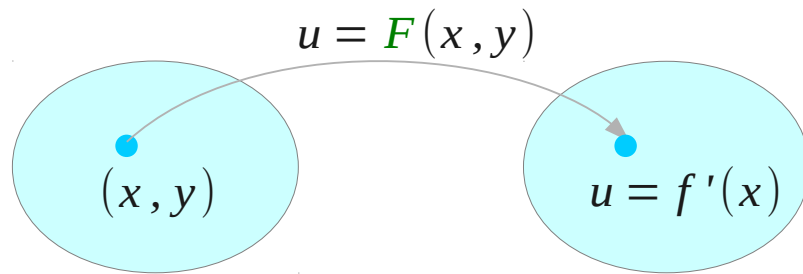
$$u = F(x, y)$$

F maps (x, y) to $f'(x)$

the derivative of $f(x)$ at x

The slope of the tangent line at $(x, f(x))$

Direction Field



$$u = F(x, y)$$

F maps (x, y) to $f'(x)$

the derivative of $f(x)$ at x

The slope of the tangent line at $(x, f(x))$

$$f'(x) = F(x, y)$$

F maps (x, y) to $f'(x)$

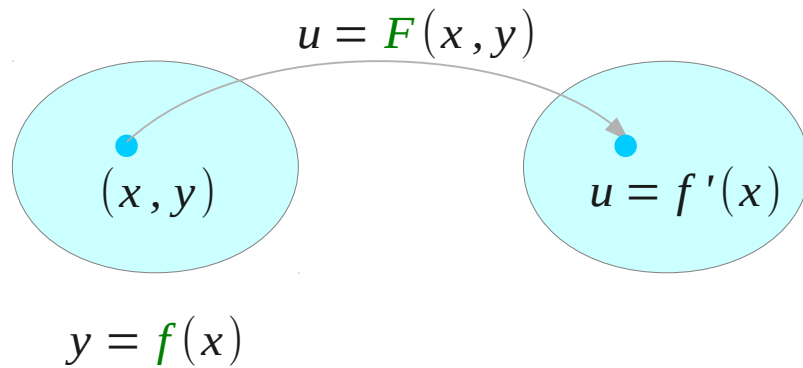
Direction Field
Slope Field

A 2-d plot representing $f'(x)$
by a **lineal element** at (x, y)



The slope of the tangent line at $(x, f(x))$

Direction Field, First Order ODE



$$u = F(x, y)$$

F maps (x, y) to **u**

$$f'(x) = F(x, y)$$

F maps (x, y) to **$f'(x)$**

the derivative of $f(x)$ at x

The slope of the tangent line at $(x, f(x))$

First Order ODE

Find solution
 $y=f(x)$

$$\frac{dy}{dx} = g(x, y)$$

where the first derivative **y'** is given by some **formula $g(x, y)$** containing variable x, y

Direction Field
Slope Field

A 2-d plot of
 $y'=f'(x)$
at (x, y)

$$g(x, y) = \frac{dy}{dx}$$

Now, it also can be viewed as a **function**

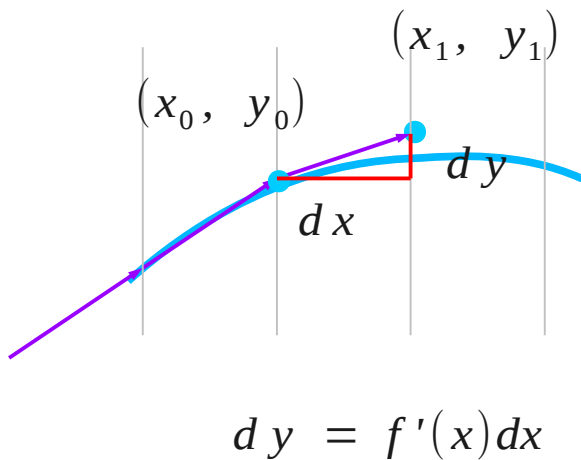
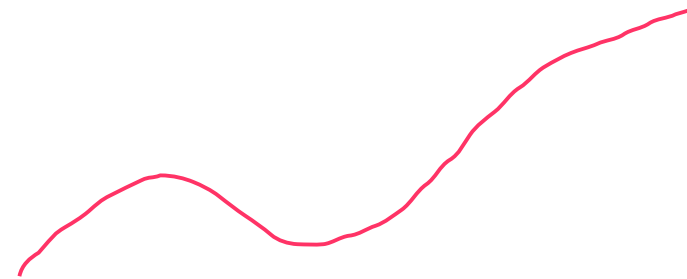
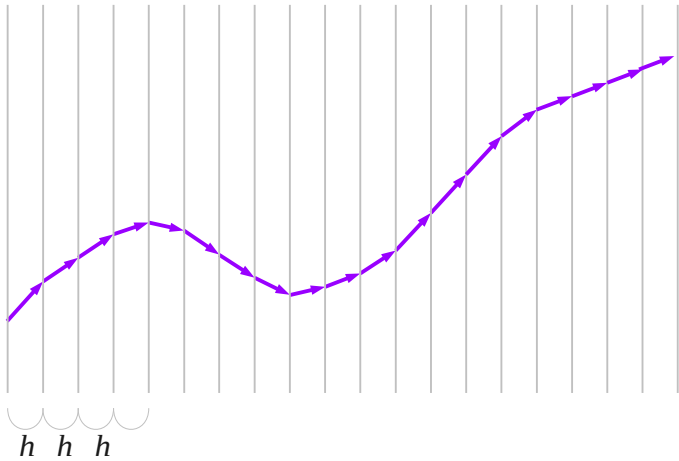
g maps (x, y) to $f'(x)$



Euler's Method

$$F(x, y) = f'(x)$$

$$y = f(x)$$



$$x_1 - x_0 = dx = h$$

$$y_1 - y_0 = dy = f'(x)dx = F(x_0, y_0)dx$$

$$y_1 = y_0 + F(x_0, y_0)h$$

$$y_{i+1} = y_i + F(x_i, y_i)h$$

Types of First Order ODEs

First & Second Order IVPs ($y=f(x)$)

First Order Initial Value Problem

$$\frac{dy}{dx} = g(x, y)$$

$$y(x_0) = y_0$$

$$y' = g(x, y)$$

$$y(x_0) = y_0$$

Second Order Initial Value Problem

$$\frac{d^2y}{dx^2} = g(x, y, y')$$

$$y(x_0) = y_0$$

$$y'(x_0) = y_1$$

$$y'' = g(x, y, y')$$

$$y(x_0) = y_0$$

$$y'(x_0) = y_1$$

Types of First Order ODEs

A General Form of First Order Differential Equations

$$\frac{dy}{dx} = g(x, y)$$

$$y' = g(x, y)$$

Separable Equations

$$\frac{dy}{dx} = g_1(x)g_2(y)$$

$$y' = g_1(x)g_2(y)$$

$$y = f(x)$$

Linear Equations

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

$$a_1(x)y' + a_0(x)y = g(x)$$

$$y = f(x)$$

Exact Equations

$$M(x, y)dx + N(x, y)dy = 0$$

$$\frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = 0$$

$$z = f(x, y)$$

Separable First Order ODEs

Separable ODEs

A General Form of First Order Differential Equations

$$\frac{dy}{dx} = g(x, y)$$

$$y' = g(x, y)$$

Separable Equations

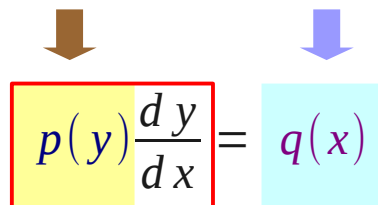
$$\frac{dy}{dx} = g_1(x)g_2(y)$$

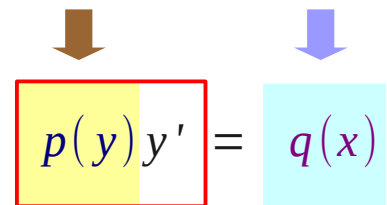
$$y' = g_1(x)g_2(y)$$

$$y = f(x)$$

$$\frac{1}{g_2(y)} \frac{dy}{dx} = g_1(x)$$

$$\frac{1}{g_2(y)} y' = g_1(x)$$


$$p(y) \frac{dy}{dx} = q(x)$$


$$p(y) y' = q(x)$$

Solving Separable ODEs (1)

$$p(y) \frac{dy}{dx} = q(x)$$

$$p(y) \frac{dy}{dx} = q(x)$$

$$p(y) \frac{dy}{dx} dx = q(x) dx$$

$$p(y) dy = q(x) dx$$

$$\int p(y) dy = \int q(x) dx$$

$$\int p(y) dy = \int q(x) dx$$

$$P(y) = Q(x) + C$$

$$p(y) y' = q(x)$$

$$p(y) y' = q(x)$$

$$p(y) y' dx = q(x) dx$$

$$p(y) dy = q(x) dx$$

$$\int p(y) dy = \int q(x) dx$$

$$\int p(y) dy = \int q(x) dx$$

$$P(y) = Q(x) + C$$

$$y = f(x)$$

not a ratio

$$dy = \frac{df}{dx} dx$$

$$\int p(y) dy$$

includes a constant

$$\int q(x) dx$$

includes another constant

implicit function y

Solving Separable ODEs (2)

$$p(y) \frac{dy}{dx} = q(x)$$

$$p(y) y' = q(x)$$

given a composite function $p(y(x))$ find $y=f(x)$

$$P(y) = \int p(y) dy + c_1$$

$$\frac{d}{dy} [P(y)] = p(y)$$

$$\frac{d}{dx} \left[\int p(y) dy + c_1 \right] \cdot \frac{dy}{dx} = q(x)$$

$$\frac{d}{dy} \left[\int p(y) dy + c_1 \right] \cdot y' = q(x)$$

$$\frac{d}{dy} [P(y)] \cdot \frac{dy}{dx} = q(x)$$

$$\frac{d}{dx} \left[\int p(y) dy + c_1 \right] = q(x)$$

$$\frac{d}{dx} \left[\int p(y) dy + c_1 \right] = q(x)$$

$$\frac{d}{dx} [P(y)] = q(x)$$

$$\int p(y) dy = \int q(x) dx$$

$$\int p(y) dy = \int q(x) dx$$

$$p(y) = \frac{d}{dy} \left[\int p(y) dy + c_1 \right]$$

$$\int p(y) dy = \int q(x) dx$$

$$\int p(y) dy = \int q(x) dx$$

$$P(y) = Q(x) + C$$

$$P(y) = Q(x) + C$$

Special Cases of Separable Equations

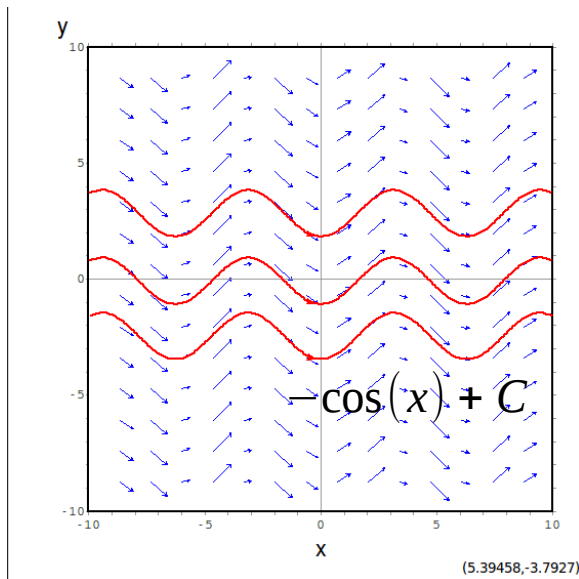
$$\frac{dy}{dx} = g_1(x)$$

$$\frac{dy}{dx} = \sin(x)$$

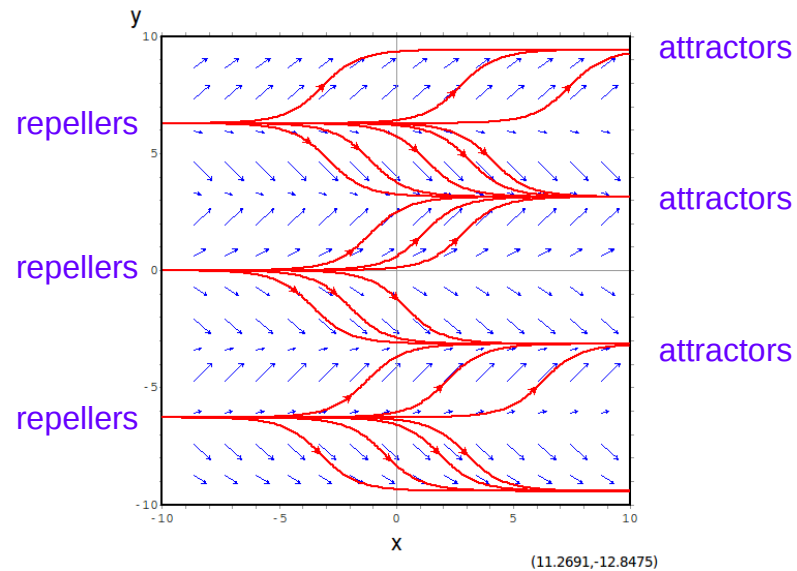
$$\frac{dy}{dx} = g_2(y)$$

$$\frac{dy}{dx} = \sin(y)$$

the only literal y
Autonomous ODEs



solution curves $y(x)$:
translated in the y direction



solution curves $y(x)$:
translated in the x direction

Autonomous First Order ODEs

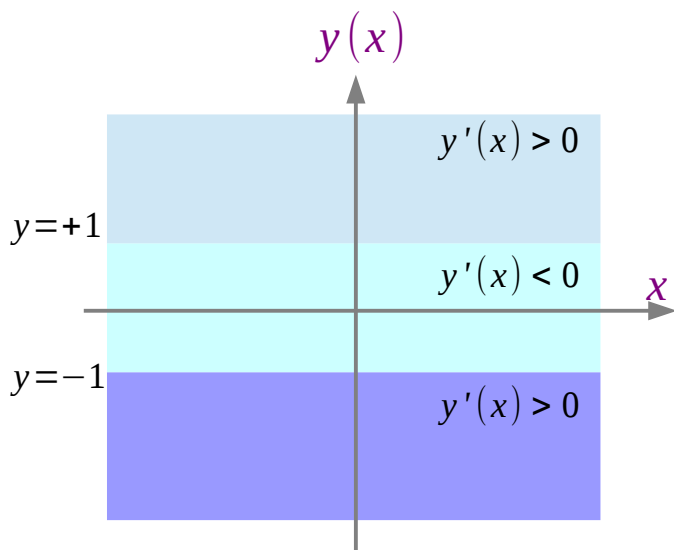
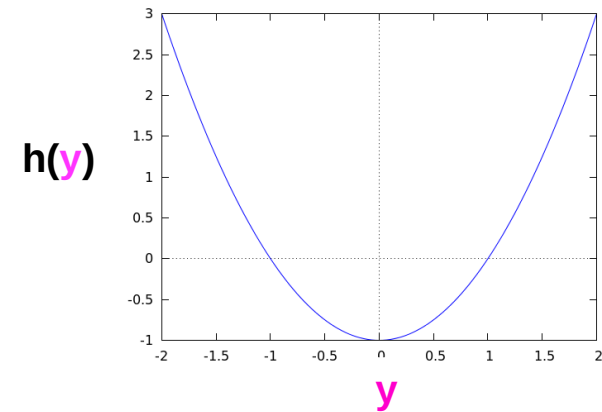
$$\frac{dy}{dx} = (y-1)(y+1)$$

$$\Rightarrow h(y)$$

$$y'(x) = h(y) > 0 \quad y > +1$$

$$y'(x) = h(y) < 0 \quad -1 < y < +1$$

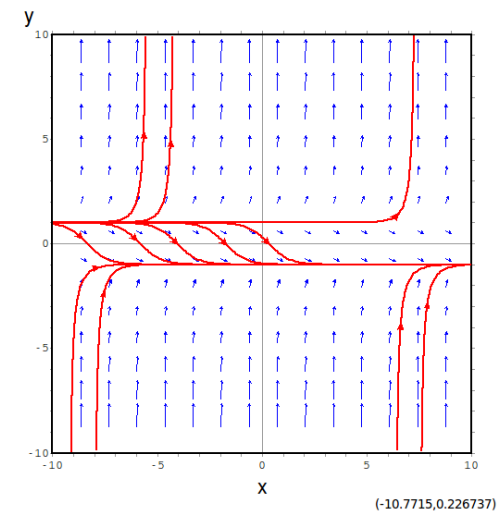
$$y'(x) = h(y) > 0 \quad y < -1$$



increasing $y(x)$ 

decreasing $y(x)$ 

increasing $y(x)$ 



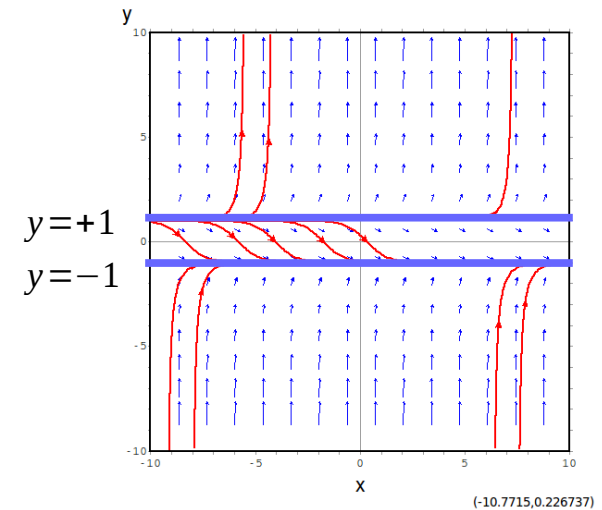
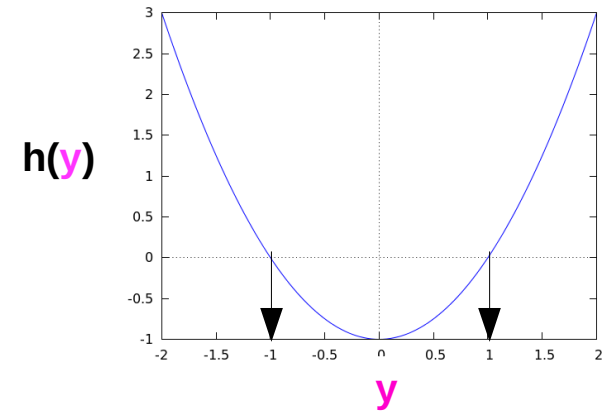
Critical Points of Autonomous ODEs

$$\frac{dy}{dx} = (y-1)(y+1) \Rightarrow h(y)$$

$$\begin{aligned} h(y) &= 0 \\ (y-1)(y+1) &= 0 \end{aligned} \quad \rightarrow \quad \begin{cases} y = +1 \\ y = -1 \end{cases}$$

critical points
(equilibrium,
stationary points) +1, -1

constant solutions $\begin{cases} y = +1 \\ y = -1 \end{cases}$



Examples of Autonomous ODEs (1)

$$\frac{dy}{dx} = (y-1)(y+1)$$

$$\frac{1}{(y^2-1)} \frac{dy}{dx} = 1$$

assumed
($y \neq -1$) \wedge ($y \neq +1$)

$$\frac{1}{(y^2-1)} dy = dx$$

$$\frac{1}{2} \left(\frac{1}{(y-1)} - \frac{1}{(y+1)} \right) dy = dx$$

$$\frac{1}{2} \left(\int \frac{1}{(y-1)} dy - \int \frac{1}{(y+1)} dy \right) = \int dx$$

$$\frac{1}{2} (\ln|y-1| - \ln|y+1|) = x + c_1$$

$$\ln \left| \frac{y-1}{y+1} \right| = 2x + c_2$$

$$\ln \left| \frac{y-1}{y+1} \right| = 2x + c_2 \quad \Rightarrow \quad \left| \frac{y-1}{y+1} \right| = e^{2x+c_2} = c_3 e^{2x}$$

$$\frac{y-1}{y+1} = +c_3 e^{2x}$$

$$\frac{y-1}{y+1} = -c_3 e^{2x}$$

$$y = \frac{1+c_3 e^{2x}}{1-c_3 e^{2x}}$$

$$y = \frac{1-c_3 e^{2x}}{1+c_3 e^{2x}}$$

$+c_3 \Rightarrow c$

$-c_3 \Rightarrow c$

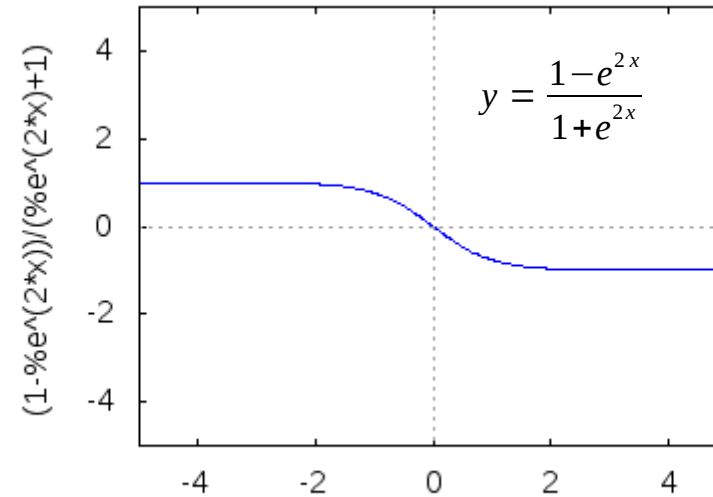
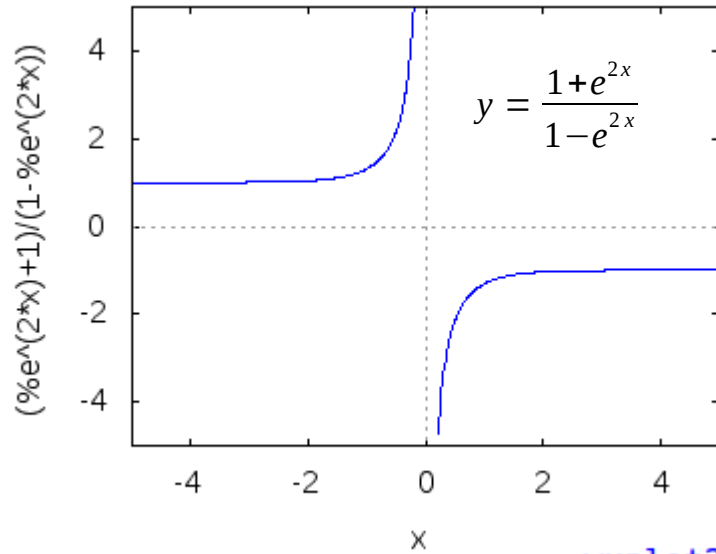
$$y = \frac{1+c e^{2x}}{1-c e^{2x}}$$

constant solutions $\left\{ \begin{array}{l} y = +1 \\ y = -1 \end{array} \right. \quad c \Rightarrow 0$

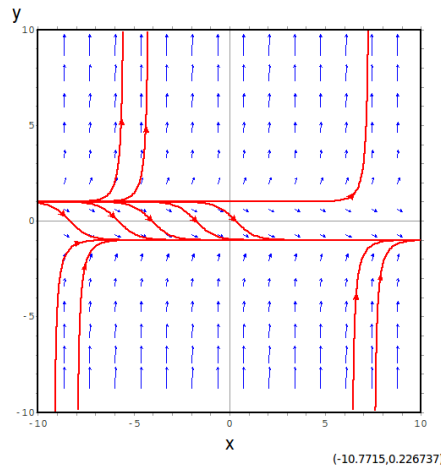
: excluded in this method

Examples of Autonomous ODEs (2)

```
wxplot2d([(1+%e^(2*x))/(1-%e^(2*x))], [x, -5, 5], [y, -5, 5]);
```



```
wxplot2d([(1-%e^(2*x))/(1+%e^(2*x))], [x, -5, 5], [y, -5, 5]);|
```



Linear First Order ODEs

Linear ODEs

A General Form of First Order Differential Equations

$$\frac{dy}{dx} = g(x, y)$$

$$y' = g(x, y)$$

Linear Equations


$$a_1(x) \frac{dy}{dx} + a_0(x)y = g(x)$$


$$a_1(x)y' + a_0(x)y = g(x)$$

$$y = f(x)$$

$$\frac{dy}{dx} + \frac{a_0(x)}{a_1(x)}y = \frac{g(x)}{a_1(x)}$$

$$y' + \frac{a_0(x)}{a_1(x)}y = \frac{g(x)}{a_1(x)}$$


$$\frac{dy}{dx} + P(x)y = Q(x)$$


$$y' + P(x)y = Q(x)$$

Standard Form of First Order ODEs

Homogeneous and Particular Solutions

Standard Form of First Order ODEs

$$\frac{dy}{dx} + P(x)y = Q(x)$$

$$y' + P(x)y = Q(x)$$

total solution

$$y = y_h + y_p$$

The Homogeneous Differential Equation

$$\frac{dy}{dx} + P(x)y = 0$$

$$y' + P(x)y = 0$$

homogeneous solution

$$y_h = f_h(x)$$

the common part of the solutions of many different differential equations whose homogeneous DE's are the same

The Nonhomogeneous Differential Equation

$$\frac{dy}{dx} + P(x)y = Q(x)$$

$$y' + P(x)y = Q(x)$$

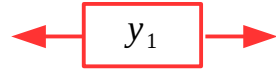
particular solution

$$y_p = f_p(x)$$

the particular solution of a specific differential equation, excluding common part of the solution

Three Different Linear ODEs (2)

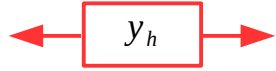
$$\frac{dy}{dx} + P(x)y = Q(x)$$



$$y' + P(x)y = Q(x)$$

EQ 1

$$\frac{dy}{dx} + P(x)y = 0$$



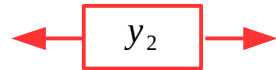
$$y' + P(x)y = 0$$

$$\frac{d[y_1+y_h]}{dx} + P(x)[y_1+y_h] = Q(x)$$

$$[y_1+y_h]' + P(x)[y_1+y_h] = Q(x)$$

$y_1 + y_h$ *solution of EQ 1*

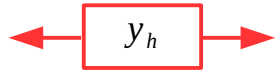
$$\frac{dy}{dx} + P(x)y = R(x)$$



$$y' + P(x)y = R(x)$$

EQ 2

$$\frac{dy}{dx} + P(x)y = 0$$



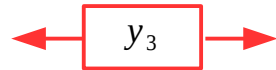
$$y' + P(x)y = 0$$

$$\frac{d[y_2+y_h]}{dx} + P(x)[y_2+y_h] = R(x)$$

$$[y_2+y_h]' + P(x)[y_2+y_h] = R(x)$$

$y_2 + y_h$ *solution of EQ 2*

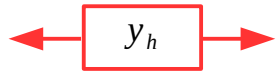
$$\frac{dy}{dx} + P(x)y = S(x)$$



$$y' + P(x)y = S(x)$$

EQ 3

$$\frac{dy}{dx} + P(x)y = 0$$



$$y' + P(x)y = 0$$

$$\frac{d[y_3+y_h]}{dx} + P(x)[y_3+y_h] = S(x)$$

$$[y_3+y_h]' + P(x)[y_3+y_h] = S(x)$$

$y_3 + y_h$ *solution of EQ 3*

Three Different Linear ODEs (2)

EQ 1

$$\frac{dy}{dx} + P(x)y = Q(x) \quad \leftarrow \quad y_1$$

$$\frac{dy}{dx} + P(x)y = 0 \quad \leftarrow \quad y_h$$

$$\frac{dy}{dx} + P(x)y = Q(x) \quad \leftarrow \quad y_1 + y_h$$

EQ 2

$$\frac{dy}{dx} + P(x)y = R(x) \quad \leftarrow \quad y_2$$

$$\frac{dy}{dx} + P(x)y = 0 \quad \leftarrow \quad y_h$$

$$\frac{dy}{dx} + P(x)y = R(x) \quad \leftarrow \quad y_2 + y_h$$

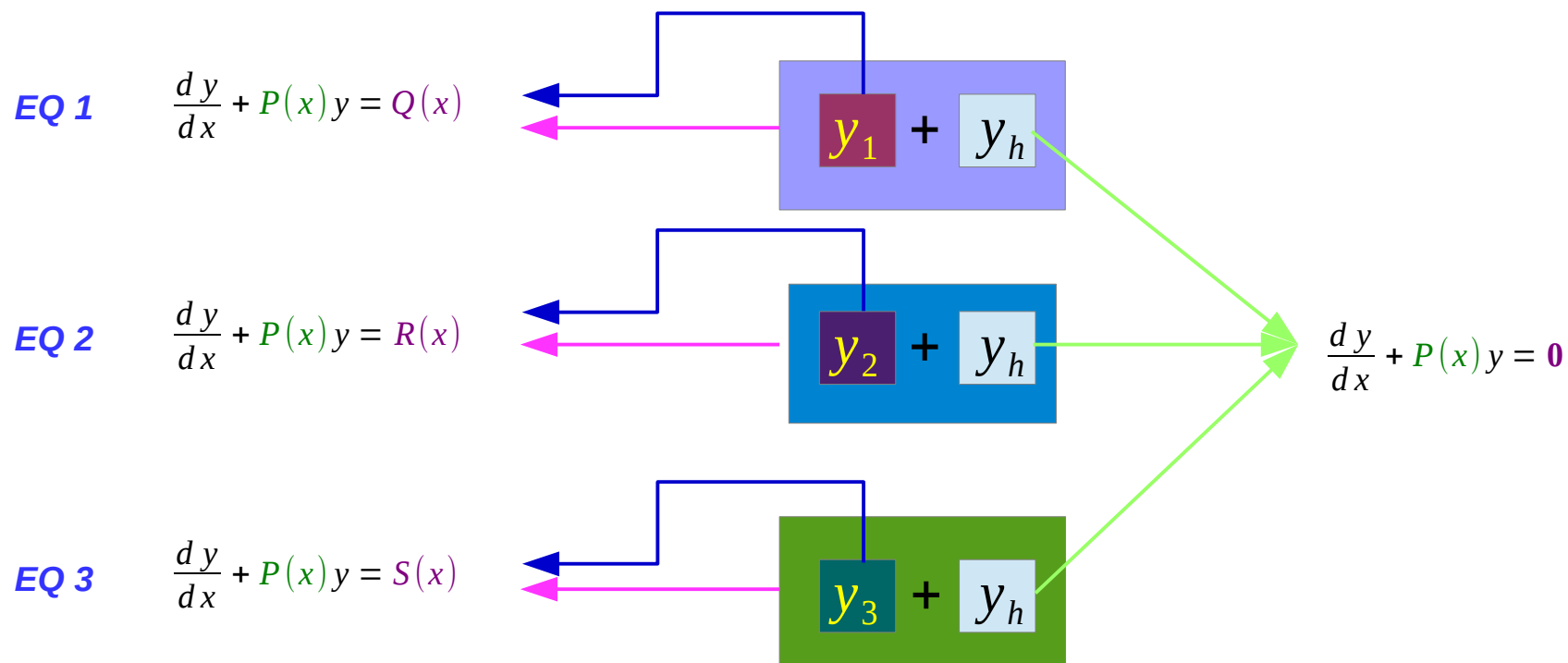
EQ 3

$$\frac{dy}{dx} + P(x)y = S(x) \quad \leftarrow \quad y_3$$

$$\frac{dy}{dx} + P(x)y = 0 \quad \leftarrow \quad y_h$$

$$\frac{dy}{dx} + P(x)y = S(x) \quad \leftarrow \quad y_3 + y_h$$

Three Different Linear ODEs (3)

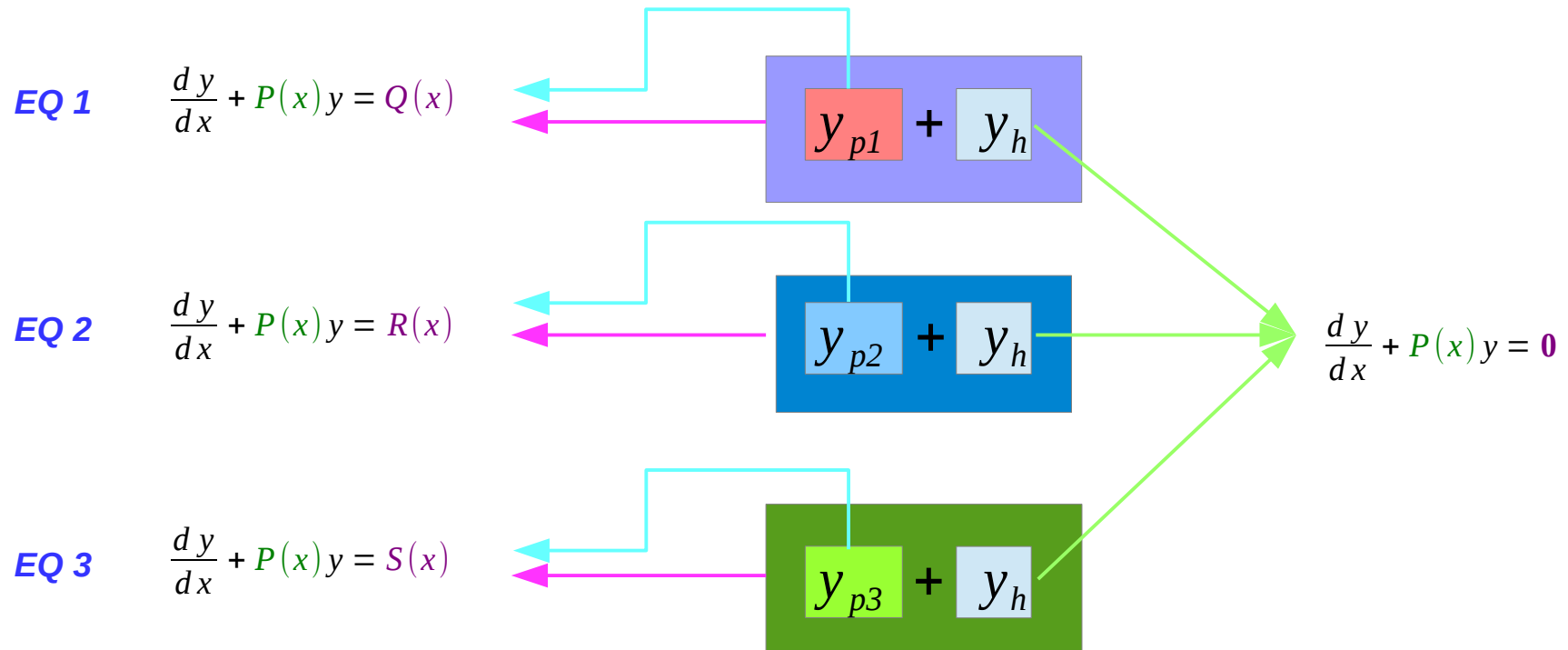


$$\begin{cases} (y_1 + y_h) + y_h \\ (y_2 + y_h) + y_h \\ (y_3 + y_h) + y_h \end{cases}$$

similarly

$$\begin{cases} y_1 + k \cdot y_h \\ y_2 + k \cdot y_h \\ y_3 + k \cdot y_h \end{cases}$$

Three Different Linear ODEs (4)



y_p *particular solution*
excluding any homogeneous solution

y_h *homogeneous solution*

Total Solution

Standard Form of First Order ODEs

$$\frac{dy}{dx} + P(x)y = Q(x)$$

$$y' + P(x)y = Q(x)$$

$$\begin{cases} y = y_p \\ y = y_h + y_p \end{cases}$$

$$\begin{cases} y = y_p \\ y = y_h + y_p \end{cases}$$

total solution

$$y = y_h + y_p$$

$$\frac{dy_h}{dx} + P(x)y_h = 0$$

$$y_h' + P(x)y_h = 0$$

homogeneous solution

+

+

$$\frac{dy_p}{dx} + P(x)y_p = Q(x)$$

$$y_p' + P(x)y_p = Q(x)$$

particular solution

||

||

$$\frac{d[y_h + y_p]}{dx} + P(x)[y_h + y_p] = Q(x)$$

$$[y_h + y_p]' + P(x)[y_h + y_p] = Q(x)$$

Solving the Homogeneous DE

$$\frac{dy}{dx} + P(x)y = 0$$

$$\frac{dy}{dx} = -P(x)y$$

$$\frac{1}{y} \frac{dy}{dx} = -P(x)$$

$$\int \frac{1}{y} \frac{dy}{dx} dx = -\int P(x) dx$$

$$\int \frac{1}{y} dy = -\int P(x) dx$$

$$\ln|y| = -\int P(x) dx + C$$

$$|y| = e^{-\int P(x) dx + C}$$

$$y = ce^{-\int P(x) dx}$$

$$y' + P(x)y = 0$$

$$y' = -P(x)y$$

$$\frac{1}{y} y' = -P(x)$$

$$\int \frac{1}{y} y' dx = -\int P(x) dx$$

$$\int \frac{1}{y} dy = -\int P(x) dx$$

$$\ln|y| = -\int P(x) dx + C$$

$$|y| = e^{-\int P(x) dx + C}$$

$$y = ce^{-\int P(x) dx}$$

homogeneous solution

$$y_h = f_h(x)$$

not a ratio

$$dy = \frac{df}{dx} dx$$

$$y = \pm e^{-\int P(x) dx} \cdot e^C$$
$$= \pm e^C \cdot e^{-\int P(x) dx}$$

Integrating Factor

$$\frac{dy}{dx} + P(x)y = 0$$

$$y = ce^{-\int P(x)dx}$$

$$y_h = cy_1$$

$$y_1 = e^{-\int P(x)dx}$$

$$y_p = u(x)y_1$$

$$\frac{dy_p}{dx} + P(x)y_p = Q(x)$$

$$y' + P(x)y = 0$$

$$y = ce^{-\int P(x)dx}$$

$$y_h = cy_1$$

$$y_1 = e^{-\int P(x)dx}$$

$$y_p = u(x)y_1$$

$$y_p' + P(x)y_p = Q(x)$$

homogeneous solution

$$y_h = f_h(x)$$

Integrating factor

$$\frac{1}{y_1} = e^{+\int P(x)dx}$$

particular solution

$$y_p = f_p(x)$$

Solving the Non-homogeneous DE (1)

$$\frac{d y_p}{d x} + P(x) y_p = Q(x)$$

$$y_p = u(x) y_1$$

$$\left\{ \frac{d u}{d x} \cdot y_1 + u \cdot \frac{d y_1}{d x} \right\} + P(x) u y_1 = Q(x)$$

$$u \cdot \left\{ \frac{d y_1}{d x} + P(x) y_1 \right\} + \frac{d u}{d x} \cdot y_1 = Q(x)$$

$$\frac{d u}{d x} d x = \frac{Q(x)}{y_1(x)} d x$$

$$u(x) = \int \frac{Q(x)}{y_1(x)} d x + c$$

$$y_p(x) = \left[\int \frac{Q(x)}{y_1(x)} d x + c \right] \cdot y_1$$

$$y_p' + P(x) y_p = Q(x)$$

$$y_p = u(x) y_1$$

$$\{u' \cdot y_1 + u \cdot y_1'\} + P(x) u y_1 = Q(x)$$

$$u \cdot \{y_1' + P(x) y_1\} + u' \cdot y_1 = Q(x)$$

$$u' d x = \frac{Q(x)}{y_1(x)} d x$$

$$u(x) = \int \frac{Q(x)}{y_1(x)} d x + c$$

$$y_p(x) = \left[\int \frac{Q(x)}{y_1(x)} d x + c \right] \cdot y_1$$

$$y_p = f_p(x)$$

$$y_h(x) = c \cdot y_1$$

$$\left\{ \frac{d y_h}{d x} + P(x) y_h \right\} = 0$$

$$c \left\{ \frac{d y_1}{d x} + P(x) y_1 \right\} = 0$$

$$y_p(x) = u \cdot y_1$$

$$y_h(x) = c \cdot y_1$$

excluding homogeneous parts of the solution

Solving the Non-homogeneous DE (2)

$$\frac{d y_p}{d x} + P(x) y_p = Q(x)$$

$$y_p' + P(x) y_p = Q(x)$$

$$y_p = u(x) y_1$$

$$\frac{d}{d x} \{u \cdot y_1\} + P(x) u \cdot y_1 = Q(x)$$

$$(u \cdot y_1)' + P(x) u \cdot y_1 = Q(x)$$

$$\frac{d u}{d x} \cdot y_1 = Q(x)$$

$$u' \cdot y_1 = Q(x)$$

$$y_1 = e^{-\int P(x) dx}$$

$$u(x) = \int \frac{Q(x)}{y_1(x)} dx$$

$$u(x) = \int \frac{Q(x)}{y_1(x)} dx$$

Integrating factor

$$y_p(x) = \left[\int \frac{Q(x)}{y_1(x)} dx \right] \cdot y_1$$

$$y_p(x) = \left[\int \frac{Q(x)}{y_1(x)} dx \right] \cdot y_1$$

$$\frac{1}{y_1} = e^{+\int P(x) dx}$$

$$y_p(x) = \left[\int \left\{ Q(x) \cdot e^{+\int P(x) dx} \right\} dx \right] \cdot y_1 = \left[\int \left\{ Q(x) \cdot e^{+\int P(x) dx} \right\} dx \right] \cdot e^{-\int P(x) dx}$$

$$y(x) = y_h(x) + y_p(x) = c e^{-\int P(x) dx} + e^{-\int P(x) dx} \cdot \left[\int \left\{ Q(x) \cdot e^{+\int P(x) dx} \right\} dx \right]$$

Example of $y_p(x) = u(x) y_h(x) - (1)$

$$\frac{dy_p}{dx} + 2y_p = x$$

$$Q(x) = x$$

$$y_1 = e^{-\int 2dx} = e^{-2x}$$

$$y_h = ce^{-2x}$$

$$\frac{dy_h}{dx} + 2y_h = -2ce^{-2x} + 2ce^{-2x} = 0$$

$$u(x) = \int \frac{Q(x)}{y_1(x)} dx = \int \frac{x}{e^{-2x}} dx = \int x e^{+2x} dx$$

$$= \frac{1}{2} \left(x e^{+2x} - \frac{1}{2} e^{+2x} \right) = \frac{1}{2} \left(x - \frac{1}{2} \right) e^{+2x}$$

(a general function of $Q(x)$) / y_1

$$y_p(x) = u(x) e^{-2x} = \frac{1}{2} \left(x - \frac{1}{2} \right) e^{+2x} e^{-2x} = \frac{1}{2} \left(x - \frac{1}{2} \right)$$

a general function of x

$$\frac{dy_p}{dx} + 2y_p = \frac{1}{2} + 2 \frac{1}{2} \left(x - \frac{1}{2} \right) = x$$

Example of $y_p(x) = u(x) y_h(x) - (2)$

$$\frac{dy_p}{dx} + 2y_p = e^x$$

$$Q(x) = e^x$$

$$y_1 = e^{-\int 2dx} = e^{-2x}$$

$$y_h = ce^{-2x}$$

$$\frac{dy_h}{dx} + 2y_h = -2ce^{-2x} + 2ce^{-2x} = 0$$

$$u(x) = \int \frac{Q(x)}{y_1(x)} dx = \int \frac{e^x}{e^{-2x}} dx = \int e^x e^{+2x} dx = \frac{1}{3} e^x e^{+2x}$$

(a general function of $Q(x)$) / y_1

$$y_p(x) = u(x) e^{-2x} = \frac{1}{3} e^x e^{+2x} e^{-2x} = \frac{1}{3} e^x$$

a general function of e^x

$$\frac{dy_p}{dx} + 2y_p = \frac{1}{3} e^x + 2 \frac{1}{3} e^x = e^x$$

Example of $y_p(x) = u(x) y_h(x) - (3)$

$$\frac{d y_p}{d x} + 2 y_p = \cos(x)$$

$$Q(x) = \cos(x)$$

$$= \frac{1}{5}(\cos(x) - 2 \sin(x)) + \frac{1}{5}(2 \sin(x) + 4 \cos(x))$$

$$y_1 = e^{-\int 2 dx} = e^{-2x}$$

$$y_h = c e^{-2x}$$

$$\frac{d y_h}{d x} + 2 y_h = -2 c e^{-2x} + 2 c e^{-2x} = 0$$

$$u(x) = \int \frac{Q(x)}{y_1(x)} dx = \int \frac{\cos(x)}{e^{-2x}} dx = \int \cos(x) e^{+2x} dx$$

$$= \frac{1}{5}(\sin(x) + 2 \cos(x)) e^{+2x} \quad \text{(a general function of } Q(x) \text{) / } y_1$$

$$y_p(x) = u(x) e^{-2x}$$

$$= \frac{1}{5}(\sin(x) + 2 \cos(x)) e^{+2x} e^{-2x}$$

$$\frac{d y_p}{d x} + 2 y_p = \cos(x)$$

$$= \frac{1}{5}(\sin(x) + 2 \cos(x))$$

a general function of $\cos(x)$

Homogeneous Equation

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 \frac{dy}{dx} + a_0 y = g(x)$$

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 \frac{dy}{dx} + a_0 y = 0$$

*Associated
Homogeneous Equation
with constant coefficients*

$$a_n m^n + a_{n-1} m^{n-1} + \cdots + a_1 m + a_0 = 0$$

Auxiliary Equation

↓ $m = m_1, m_2, \dots, m_n$

*n solutions of the
Auxiliary Equation*

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x}$$

*General Solutions of the
Homogeneous Equation*

Particular Solution

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 \frac{dy}{dx} + a_0 y = g(x)$$

$$y_1 = \cdots + e^{-m_i x} + \cdots$$

$$y_h = \cdots + c_i e^{-m_i x} + \cdots$$

$$u(x) = \int \frac{Q(x)}{y_1(x)} dx = \cdots + \int Q(x) c_i e^{+m_i x} dx \cdots +$$

(a general function of $Q(x)$) / y_1

$$y_p(x) = u(x) e^{-2x} = \cdots + \int Q(x) c_i e^{+m_i x} dx e^{-m_i x} + \cdots$$

a general function of $\cos(x)$

Total Solution

$$\frac{dy}{dx} + P(x)y = Q(x)$$

$$y_1 = e^{-\int P(x)dx}$$

total solution

$$c y_1 + u(x) y_1$$

$$u(x) y_1$$

homogeneous solution

particular solution

$$c e^{-\int P(x)dx} + e^{-\int P(x)dx} \cdot \left[\int \left\{ Q(x) \cdot e^{+\int P(x)dx} \right\} dx \right]$$

Integrating factor

$$c y_1$$

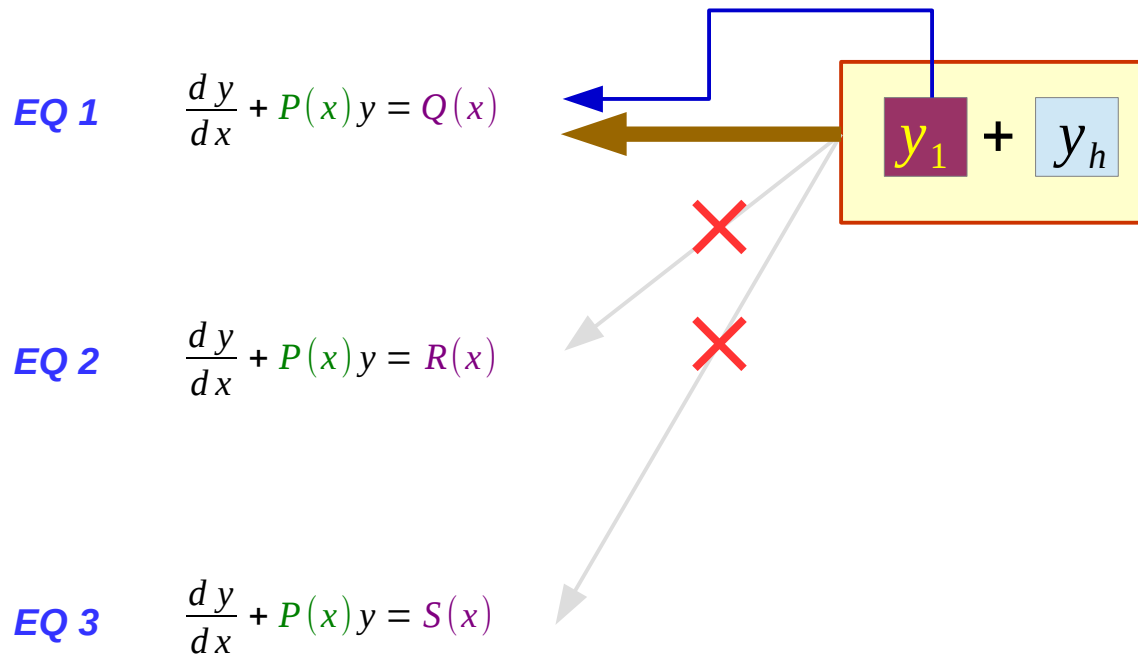
$$\frac{dy}{dx} + P(x)y = 0$$

The solution of an ODE (1)

if there is any solution, it is in the form of S

A $\frac{dy}{dx} + P(x)y = Q(x)$ \iff $y(x) = ce^{-\int P(x)dx} + e^{-\int P(x)dx} \cdot \left[\int \left[Q(x) \cdot e^{+\int P(x)dx} \right] dx \right]$ S

B is the solution of A ODE only



The solution of an ODE (2)

if there is any solution, it is in the form of S

$$A \quad \frac{dy}{dx} + P(x)y = Q(x) \quad \longleftrightarrow \quad y(x) = ce^{-\int P(x)dx} + e^{-\int P(x)dx} \cdot \left[\int \left[Q(x) \cdot e^{+\int P(x)dx} \right] dx \right] \quad S$$

B is the solution of A ODE only

$$e^{+\int P(x)dx} \times y(x) = \left(ce^{-\int P(x)dx} + e^{-\int P(x)dx} \cdot \left[\int \left[Q(x) \cdot e^{+\int P(x)dx} \right] dx \right] \right) \times e^{+\int P(x)dx}$$

$$\left\{ e^{+\int P(x)dx} \right\} \cdot y(x) = c + \left[\int \left[Q(x) \cdot e^{+\int P(x)dx} \right] dx \right]$$

$$\frac{d}{dx} \left[\left\{ e^{+\int P(x)dx} \right\} \cdot y(x) \right] = Q(x) \cdot \left\{ e^{+\int P(x)dx} \right\}$$

$$\left[\left\{ e^{+\int P(x)dx} \right\} \cdot \frac{dy}{dx} + \left\{ e^{+\int P(x)dx} \right\} P(x) \cdot y(x) \right] = Q(x) \cdot \left\{ e^{+\int P(x)dx} \right\}$$

$$\frac{dy}{dx} + P(x)y = Q(x)$$

Method of Solving First Order ODEs

$$\frac{dy}{dx} + P(x)y = Q(x)$$

$$y' + P(x)y = Q(x)$$

$$y = f(x)$$

$$\left[e^{+\int P(x)dx} \right] \cdot \left[\frac{dy}{dx} + P(x)y \right] = \left[e^{+\int P(x)dx} \right] Q(x)$$

$$y_1 = e^{-\int P(x)dx} \quad \frac{1}{y_1} = e^{+\int P(x)dx}$$

$$\left[e^{+\int P(x)dx} \right] \frac{dy}{dx} + \left[\left[e^{+\int P(x)dx} \right] P(x) \right] y = \left[e^{+\int P(x)dx} \right] Q(x)$$

$$\left[e^{+\int P(x)dx} \right] \cdot P(x) = \frac{d}{dx} \left[e^{+\int P(x)dx} \right] \quad f'(g(x))g'(x)$$

$$\left[e^{+\int P(x)dx} \right] \frac{dy}{dx} + \frac{d}{dx} \left[e^{+\int P(x)dx} \right] y = \left[e^{+\int P(x)dx} \right] Q(x)$$

$$\left[e^{+\int P(x)dx} \right] \frac{dy}{dx} + \frac{d}{dx} \left[e^{+\int P(x)dx} \right] y = \frac{d}{dx} \left[e^{+\int P(x)dx} \cdot y \right]$$

$$\frac{d}{dx} \left[\left[e^{+\int P(x)dx} \right] \cdot y \right] = \left[e^{+\int P(x)dx} \right] Q(x)$$

$$f'(x)g(x) + f(x)g'(x)$$

$$\int \frac{d}{dx} \left[\left[e^{+\int P(x)dx} \right] \cdot y \right] dx = \int \left[e^{+\int P(x)dx} \right] Q(x) dx + c$$

$$\left[\left[e^{+\int P(x)dx} \right] \cdot y \right] = \int \left[e^{+\int P(x)dx} \right] Q(x) dx + c \quad \longrightarrow \quad y(x) = c e^{-\int P(x)dx} + e^{-\int P(x)dx} \cdot \left[\int \left[Q(x) \cdot \left[e^{+\int P(x)dx} \right] dx \right] \right]$$

Linear First Order ODEs – Summary

Linear First-Order ODE

$$y' + P y = Q \quad P(x), Q(x)$$

$$y' + P y = 0 \quad P(x), Q(x) = 0$$

$$\frac{dy}{dx} = -P \cdot y \Rightarrow \frac{dy}{y} = -P \cdot dx \Rightarrow \ln y = -\int P dx + C \Rightarrow y = e^{-\int P dx + C}$$

$$y = A e^{-I} \quad (I = \int P dx, A = e^C) \Rightarrow y e^I = A$$

$$y' + P y = Q \quad P(x), Q(x) \neq 0$$

$$\frac{d}{dx}(y e^I) = y' e^I + y e^I \frac{dI}{dx} = y' e^I + y e^I P = e^I (y' + P y) = e^I Q \quad (\neq 0)$$

$$\frac{d}{dx}(y e^I) = e^I Q \Rightarrow y e^I = \int e^I Q dx + c \Rightarrow y = e^{-I} \left(\int e^I Q dx + c \right)$$

$$y = e^{-I} \left(\int e^I Q dx + c \right) \quad (I = \int P dx)$$

Standard Form

$$\frac{dy}{dx} + P(x)y = f(x) \quad P(x), f(x)$$

$$y_c' + P y_c = 0 \quad P(x), f(x) = 0$$

$$y_g = y_c + y_p \quad \text{general solution} = \text{homogeneous solution} + \text{particular solution}$$

$$\frac{d}{dx} [y_c + y_p] + P(x)[y_c + y_p] = f(x)$$

$$[y_c' + P(x)y_c] + [y_p' + P(x)y_p] = f(x)$$

$$y_c' + P y_c = 0 \quad y_p' + P y_p = f(x)$$

The Homogeneous DE

$$\frac{dy}{dx} + P(x)y = f(x) \quad P(x), f(x)$$

$$y_c' + P y_c = 0 \quad P(x), f(x) = 0$$

$$\frac{dy_c}{dx} = -P \cdot y_c \Rightarrow \frac{dy_c}{y_c} = -P \cdot dx \Rightarrow \ln y_c = -\int P dx + C \Rightarrow y_c = e^{-\int P dx + C}$$

$$y_c = c y_1 \quad (y_1 = e^{-\int P dx}, c = e^{+C}) \quad \text{homogeneous solution}$$

$$\frac{dy_c}{dx} + P \cdot y_c = c \left(\frac{d}{dx} (e^{-\int P dx}) + P \cdot e^{-\int P dx} \right) = c (-P \cdot e^{-\int P dx} + P \cdot e^{-\int P dx}) = 0$$

The Non-homogeneous DE

$$\frac{dy}{dx} + P(x)y = f(x) \quad P(x), f(x)$$

$$y_p' + P y_p = f(x) \quad P(x), f(x) \neq 0$$

$$y_c = c y_1 \quad (y_1 = e^{-\int P dx}, c = e^{+C})$$

homogeneous solution

$$y_p = u y_1 \quad (y_1 = e^{-\int P dx}, u(x) \neq \text{const})$$

particular solution

$$\frac{d}{dx}(y_1 u) + P \cdot y_1 u = f(x) \quad \left[\frac{dy_1}{dx} + P \cdot y_1 \right] u + y_1 \frac{du}{dx} = f(x)$$

$$y_1 \frac{du}{dx} = f(x) \quad du = \frac{f(x)}{y_1(x)} dx$$

$$u = \int \frac{f(x)}{y_1(x)} dx$$

$$y_p = u y_1 \quad y_p = y_1 \int \frac{f(x)}{y_1(x)} dx = e^{-\int P dx} \int f(x) e^{+\int P dx} dx$$

$$y_p = u y_1 \quad (y_1 = e^{-\int P dx}, u = \int f(x) e^{+\int P dx} dx)$$

particular solution

The Non-homogeneous DE

$$\frac{dy}{dx} + P(x)y = f(x) \quad P(x), f(x)$$

$$y_p' + P y_p = f(x) \quad P(x), f(x) \neq 0$$

$$y_p = u y_1 \quad (y_1 = e^{-\int P dx}, u = \int f(x) e^{+\int P dx} dx) \quad \text{particular solution}$$

$$y_g = y_c + y_p = c e^{-\int P dx} + e^{-\int P dx} \int f(x) e^{+\int P dx} dx \quad \text{general solution}$$

$$y_g e^{+\int P dx} = c + \int f(x) e^{+\int P dx} dx \quad \frac{d}{dx} (y_g e^{+\int P dx}) = f(x) e^{+\int P dx}$$

$$\left(\frac{dy_g}{dx} \right) e^{+\int P dx} + (P y_g) e^{+\int P dx} = f(x) e^{+\int P dx}$$

$$\frac{dy_g}{dx} + P y_g = f(x)$$

The Non-homogeneous DE

$$\frac{dy}{dx} + P(x)y = f(x) \quad P(x), f(x)$$

$$\left(\frac{dy_g}{dx}\right)e^{+\int P dx} + (Py_g)e^{+\int P dx} = f(x)e^{+\int P dx} \quad \begin{array}{l} \text{integration} \\ \text{factor} \end{array} \quad e^{+\int P dx}$$

$$\frac{d}{dx} (y_g e^{+\int P dx}) = f(x) e^{+\int P dx}$$

$$y_g e^{+\int P dx} = c + \int f(x) e^{+\int P dx} dx$$

$$y_g = y_c + y_p = c e^{-\int P dx} + e^{-\int P dx} \int f(x) e^{+\int P dx} dx$$

Exact First Order ODEs

Differential Form & Equation

A differential form

$$P(x, y)dx + Q(x, y)dy$$

A first order differential equation

$$P(x, y)dx + Q(x, y)dy = 0$$

this differential form is **exact**
in a region **R** if there is a function
 $f(x, y)$ such that

$$\begin{aligned} P(x, y)dx + Q(x, y)dy \\ = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = df \end{aligned}$$

exact equation
in a region **R** if there is a function
 $f(x, y)$ such that

$$\begin{aligned} P(x, y)dx + Q(x, y)dy \\ = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = df = 0 \end{aligned}$$

$$P(x, y)dx + Q(x, y)dy$$

is an **exact differential** in a region **R**
if it corresponds to
the **total differential** of
some function $f(x, y)$

$$df(x, y) = 0$$

$$f(x, y) = c$$

To be exact

$$P(x, y)dx + Q(x, y)dy = df \quad \xrightarrow{\text{to be exact}} \quad df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

$$P(x, y) = \frac{\partial f}{\partial x}$$

$$Q(x, y) = \frac{\partial f}{\partial y}$$

$$\frac{\partial P}{\partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

$$\frac{\partial Q}{\partial x} = \frac{\partial^2 f}{\partial x \partial y}$$

$$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y \partial x} \text{ all defined and continuous} \quad \iff \quad \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

$$P(x, y)dx + Q(x, y)dy \text{ is an exact (total) differential} \quad \iff \quad \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

Path Independence

A differential df of the following form

$$P(x, y)dx + Q(x, y)dy = df$$

is **exact**, if $\int df$ is path independent

$\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\frac{\partial^2 f}{\partial x \partial y}$, $\frac{\partial^2 f}{\partial y \partial x}$ all defined and continuous



$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

$P(x, y)dx + Q(x, y)dy$
is an **exact (total) differential**



$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

path independent $\int df$



$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

Exact and Inexact Differential Examples

$$df = 2xy^3 dx + 3x^2y^2 dy$$

Is there a function $f=f(x,y)$ such that

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

$$\frac{\partial f}{\partial x} = 2xy^3$$

$$\frac{\partial f}{\partial y} = 3x^2y^2$$

$$\frac{\partial^2 f}{\partial y \partial x} = 6xy^2$$

$$\frac{\partial^2 f}{\partial x \partial y} = 6xy^2$$

$$f(x,y) = x^2y^3 \quad \text{exact}$$

$$\int_{(x_1, y_1)}^{(x_2, y_2)} df = f(x_2, y_2) - f(x_1, y_1)$$

Only initial & final points
Path independent

$$df = 2x^2y^3 dx + 3x^3y^2 dy$$

Is there a function $f=f(x,y)$ such that

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

$$\frac{\partial f}{\partial x} = 2x^2y^3$$

$$\frac{\partial f}{\partial y} = 3x^3y^2$$

$$\frac{\partial^2 f}{\partial y \partial x} = 6x^2y^2$$

$$\frac{\partial^2 f}{\partial x \partial y} = 9x^2y^2$$

$$\text{no } f(x,y) \quad \text{inexact}$$

Integration result depends
on the path also,
in addition to initial & final points

Exact Equations (1)

Exact Equations

$$M(x, y)dx + N(x, y)dy = 0$$

$$\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = 0$$

$$z = f(x, y)$$

$$\frac{\partial f}{\partial x} = M(x, y)$$

$$\frac{\partial f}{\partial y} = N(x, y)$$

$$\frac{df}{dx} = M(x) \Rightarrow$$

$$f(x) = \int M(x)dx + c$$

$$\int \frac{\partial f}{\partial x} dx = \int M(x, y) dx + c$$

$$\int \frac{\partial f}{\partial y} dy = \int N(x, y) dy + c$$

$$f(x, y) = \int M(x, y) dx + g(y)$$

$$f(x, y) = \int N(x, y) dy + h(x)$$

$$\frac{\partial f}{\partial y} = N(x, y)$$

$$\frac{\partial f}{\partial x} = M(x, y)$$

$$= \frac{\partial}{\partial y} \int M(x, y) dx + g'(y)$$

$$= \frac{\partial}{\partial x} \int N(x, y) dy + h'(x)$$

$$g'(y) = N(x, y) - \frac{\partial}{\partial y} \int M(x, y) dx$$

$$h'(x) = M(x, y) - \frac{\partial}{\partial x} \int N(x, y) dy$$

Exact Equations (2)

$$f(x, y) = \int M(x, y) dx + g(y)$$

$$g'(y) = N(x, y) - \frac{\partial}{\partial y} \int M(x, y) dx$$

$$g(y) = \int g'(y) dy$$

$$= \int \left\{ N(x, y) - \frac{\partial}{\partial y} \int M(x, y) dx \right\} dy$$

$$f(x, y) = \int M(x, y) dx + \int \left\{ N(x, y) - \frac{\partial}{\partial y} \int M(x, y) dx \right\} dy$$

$$f(x, y) = \int N(x, y) dy + h(x)$$

$$h'(x) = M(x, y) - \frac{\partial}{\partial x} \int N(x, y) dy$$

$$h(x) = \int h'(x) dx$$

$$= \int \left\{ M(x, y) - \frac{\partial}{\partial x} \int N(x, y) dy \right\} dx$$

$$f(x, y) = \int N(x, y) dy + \int \left\{ M(x, y) - \frac{\partial}{\partial x} \int N(x, y) dy \right\} dx$$

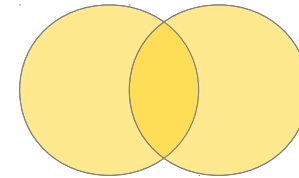
Exact Equations (3)

Exact Equations

$$M(x, y)dx + N(x, y)dy = 0 \quad \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = 0$$

$$f(x, y) = \int M(x, y) dx + \int \left\{ N(x, y) - \frac{\partial}{\partial y} \int M(x, y) dx \right\} dy$$

$$f(x, y) = \int N(x, y) dy + \int \left\{ M(x, y) - \frac{\partial}{\partial x} \int N(x, y) dy \right\} dx$$



$$f(x, y) = \int M(x, y) dx + \int N(x, y) dy - \int \frac{\partial}{\partial y} \int M(x, y) dx dy \quad \int \frac{\partial}{\partial y} \int \frac{\partial f}{\partial x} dx dy$$

$$f(x, y) = \int N(x, y) dy + \int M(x, y) dx - \int \frac{\partial}{\partial x} \int N(x, y) dy dx \quad \int \frac{\partial}{\partial x} \int \frac{\partial f}{\partial y} dy dx$$

Exact Equations (4)

Exact Equations

$$M(x, y)dx + N(x, y)dy = 0 \quad \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = 0$$

$$f(x, y) = \int M(x, y) dx + \int \left\{ N(x, y) - \frac{\partial}{\partial y} \int M(x, y) dx \right\} dy \quad \frac{\partial}{\partial y} \frac{\partial}{\partial x} \int M(x, y) dx = \frac{\partial M}{\partial y}$$

$$\frac{\partial f}{\partial x} = M(x, y) + \int \left\{ \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right\} dy = M(x, y)$$

$$\frac{\partial f}{\partial y} = \int \frac{\partial M}{\partial y} dx + N(x, y) - \int \frac{\partial M}{\partial y} dx = N(x, y)$$

$$f(x, y) = \int N(x, y) dy + \int \left\{ M(x, y) - \frac{\partial}{\partial x} \int N(x, y) dy \right\} dx \quad \frac{\partial}{\partial x} \frac{\partial}{\partial y} \int N(x, y) dy = \frac{\partial N}{\partial x}$$

$$\frac{\partial f}{\partial x} = \int \frac{\partial N}{\partial x} dy + M(x, y) - \int \frac{\partial N}{\partial x} dy = M(x, y)$$

$$\frac{\partial f}{\partial y} = N(x, y) + \int \left\{ \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right\} dx = N(x, y)$$

NonExact First Order ODEs

NonExact Equations (1)

Exact Equations

$$M(x, y)dx + N(x, y)dy = 0 \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$z = f(x, y)$$

NonExact Equations

$$M(x, y)dx + N(x, y)dy = 0 \quad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$$z = f(x, y)$$

Exact Equations

$$\mu(x, y)M(x, y)dx + \mu(x, y)N(x, y)dy = 0 \quad \frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x}$$

If we can find $\mu(x, y)$

Exact Equations

$$\mu(x)M(x, y)dx + \mu(x)N(x, y)dy = 0 \quad \frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x}$$

If we can find $\mu(x)$
or $\mu(y)$

$$\mu(y)$$

$$\mu(y)$$

Multiplying NonExact Equations by $\mu(x, y)$

NonExact Equations

$$M(x, y)dx + N(x, y)dy = 0$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$$z = f(x, y)$$

Exact Equations

$$\mu(x, y)M(x, y)dx + \mu(x, y)N(x, y)dy = 0$$

$$\frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x}$$

If we can find $\mu(x, y)$

$$\frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x} \quad \leftarrow \quad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$$\frac{\partial \mu}{\partial y} M + \mu \frac{\partial M}{\partial y} = \frac{\partial \mu}{\partial x} N + \mu \frac{\partial N}{\partial x}$$

$$\mu_y M + \mu M_y = \mu_x N + \mu N_x$$

$$\mu M_y - \mu N_x = \mu_x N - \mu_y M \quad \text{Partial Differential Equation}$$

difficult to find $\mu(x, y)$

Multiplying NonExact Equations by $\mu(x)$

NonExact Equations

$$M(x, y)dx + N(x, y)dy = 0 \quad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$$z = f(x, y)$$

Exact Equations

$$\mu(x)M(x, y)dx + \mu(x)N(x, y)dy = 0 \quad \frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x}$$

If we can find $\mu(x)$

$$\frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x}$$



$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$$\cancel{\frac{\partial \mu}{\partial y}} M + \mu \frac{\partial M}{\partial y} = \frac{\partial \mu}{\partial x} N + \mu \frac{\partial N}{\partial x}$$

$$\mu \frac{\partial M}{\partial y} = \frac{d\mu}{dx} N + \mu \frac{\partial N}{\partial x}$$

$$\frac{d\mu}{dx} N = \mu \frac{\partial M}{\partial y} - \mu \frac{\partial N}{\partial x}$$

$$\frac{d\mu}{dx} N = \mu M_y - \mu N_x$$

$$\frac{d\mu}{dx} = \left(\frac{M_y - N_x}{N} \right) \mu$$

generally $P(x, y)$
sometimes $P(x)$

$$\frac{d\mu}{dx} - P(x)\mu = 0$$

Multiplying NonExact Equations by $\mu(y)$

NonExact Equations

$$M(x, y)dx + N(x, y)dy = 0$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$$z = f(x, y)$$

Exact Equations

$$\mu(y)M(x, y)dx + \mu(y)N(x, y)dy = 0$$

$$\frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x}$$

If we can find $\mu(y)$

$$\frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x}$$



$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$$\frac{\partial \mu}{\partial y} M + \mu \frac{\partial M}{\partial y} = \frac{\partial \mu}{\partial x} N + \mu \frac{\partial N}{\partial x}$$

$$M \frac{\partial \mu}{\partial y} = \mu \frac{\partial N}{\partial x} - \mu \frac{\partial M}{\partial y}$$

$$M \frac{d\mu}{dy} = \mu \frac{\partial N}{\partial x} - \mu \frac{\partial M}{\partial y}$$

$$M \frac{d\mu}{dy} = \mu N_x - \mu M_y$$

$$\frac{d\mu}{dy} = \left(\frac{N_x - M_y}{M} \right) \mu$$

generally $P(x, y)$
sometimes $P(y)$

$$\frac{d\mu}{dy} - P(y)\mu = 0$$

Solving NonExact Equations

$$\frac{\partial}{\partial y}[\mu(x)M(x,y)] = \frac{\partial}{\partial x}[\mu(x)N(x,y)]$$

$$\frac{d\mu}{dx} = \left(\frac{M_y - N_x}{N}\right)\mu = P(x)\mu$$

$$\frac{d\mu}{dx} - P(x)\mu = 0$$

$$\mu(x) = ce^{\int P(x)dx}$$

$$\mu(x) = ce^{\int \left(\frac{M_y - N_x}{N}\right)dx}$$

$$\mu(x)M(x,y)dx + \mu(x)N(x,y)dy = 0$$

$$\frac{\partial}{\partial y}[\mu(x)M(x,y)] = \frac{\partial}{\partial x}[\mu(x)N(x,y)]$$

$$\frac{d\mu}{dy} = \left(\frac{N_x - M_y}{M}\right)\mu = P(y)\mu$$

$$\frac{d\mu}{dy} - P(y)\mu = 0$$

$$\mu(y) = ce^{\int P(y)dy}$$

$$\mu(y) = ce^{\int \left(\frac{N_x - M_y}{M}\right)dy}$$

$$\mu(y)M(x,y)dx + \mu(y)N(x,y)dy = 0$$

Verifying Exact Equations

$$\mu(x)M(x, y)dx + \mu(x)N(x, y)dy = 0$$

$$\frac{d\mu}{dx} = \left(\frac{M_y - N_x}{N} \right) \mu = P(x)\mu$$

$$\mu(x) = ce^{\int \left(\frac{M_y - N_x}{N} \right) dx}$$

$$\frac{\partial}{\partial y} [\mu(x)M(x, y)] = \cancel{\mu_y M} + \mu M_y$$

$$\frac{\partial}{\partial x} [\mu(x)N(x, y)] = \mu_x N + \mu N_x$$

$$\mu_x N + \mu N_x = \left(\frac{M_y - N_x}{N} \right) \mu N + \mu N_x$$

$$\mu_x N + \mu N_x = \mu M_y$$

$$\frac{\partial}{\partial y} [\mu(x)M(x, y)] = \frac{\partial}{\partial x} [\mu(x)N(x, y)]$$

$$\mu(y)M(x, y)dx + \mu(y)N(x, y)dy = 0$$

$$\frac{d\mu}{dy} = \left(\frac{N_x - M_y}{M} \right) \mu = P(y)\mu$$

$$\mu(y) = ce^{\int \left(\frac{N_x - M_y}{M} \right) dy}$$

$$\frac{\partial}{\partial y} [\mu(y)M(x, y)] = \mu_y M + \mu M_y$$

$$\frac{\partial}{\partial x} [\mu(y)N(x, y)] = \cancel{\mu_x N} + \mu N_x$$

$$\mu_y M + \mu M_y = \left(\frac{N_x - M_y}{M} \right) \mu M + \mu M_y$$

$$\mu_y M + \mu M_y = \mu N_x$$

$$\frac{\partial}{\partial y} [\mu(y)M(x, y)] = \frac{\partial}{\partial x} [\mu(y)N(x, y)]$$

Substitution Method

Substitution Method (1)

A General Form of First Order Differential Equations

$$\frac{dy}{dx} = g(x, y)$$

$$y' = g(x, y)$$

$$y = h(x, u) \quad u = \Phi(x)$$

$$z = f(x(t), y(t)) \rightarrow$$

$$\frac{dz}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}$$

$$\frac{dy}{dx} = \frac{\partial h}{\partial x} \frac{dx}{dx} + \frac{\partial h}{\partial u} \frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{\partial h}{\partial x} + \frac{\partial h}{\partial u} \frac{du}{dx}$$

$$\frac{dy}{dx} = g(x, y) = g(x, h(x, u))$$

$$g(x, h(x, u)) = \frac{\partial h}{\partial x} + \frac{\partial h}{\partial u} \frac{du}{dx}$$

$$\frac{dy}{dx} = h_x(x, u) + h_u(x, u) \frac{du}{dx}$$

$$\frac{dy}{dx} = g(x, y) = g(x, h(x, u))$$

$$g(x, h(x, u)) = h_x(x, y) + h_u(x, u) \frac{du}{dx}$$

Substitution Method (2)

A General Form of First Order Differential Equations

$$\frac{dy}{dx} = s\left(\frac{y}{x}\right)$$

$$y' = s\left(\frac{y}{x}\right)$$

$$y = h(x, u) \quad u = \Phi(x)$$

$$y = h(x, u) \leftarrow u = \Phi(x)$$

$$\frac{dy}{dx} = \frac{\partial h}{\partial x} \frac{dx}{dx} + \frac{\partial h}{\partial u} \frac{du}{dx}$$

$$y = ux \quad u = \frac{y}{x}$$

$$\frac{dy}{dx} = \frac{\partial h}{\partial x} + \frac{\partial h}{\partial u} \frac{du}{dx}$$

$$y' = u + xu'$$

$$\frac{dy}{dx} \leftarrow \frac{\partial h}{\partial x} + \frac{\partial h}{\partial u} \frac{du}{dx}$$

$$\frac{dy}{dx} = g(x, y) = g(x, h(x, u))$$

$$y' = g(x, ux) = s(u)$$

$$y \leftarrow h(x, u)$$

$$g(x, h(x, u)) = \frac{\partial h}{\partial x} + \frac{\partial h}{\partial u} \frac{du}{dx}$$

$$s(u) = u + xu'$$

$$s(u) - u = xu'$$

$$\frac{du}{s(u) - u} = \frac{dx}{x}$$

Substitution Method (3)

a new literal a function of x



$$u = \Phi(x)$$

contains x and y literals
(y is also a function of x)

a new literal u is introduced
using old literals x and y :
a new function of x

$$u = \frac{y}{x}$$

a old literal a function of x and u



$$y = h(x, u)$$

the old literal y can be replaced by
the new literal u and the old literal x :
a new function of u and x

$$y = ux$$

Substitution Method (4)

(1) replace y'

$$\frac{dy}{dx} \leftarrow \frac{\partial h}{\partial x} + \frac{\partial h}{\partial u} \frac{du}{dx}$$

$$y' = u + xu' \leftarrow y = ux \leftarrow u = \frac{y}{x}$$

(2) replace y

$$g(x, y) \leftarrow g(x, h(x, u))$$

$$y' = g(x, ux) = s(u) \leftarrow \frac{dy}{dx} = s\left(\frac{y}{x}\right)$$

Find $y = f(x)$ in

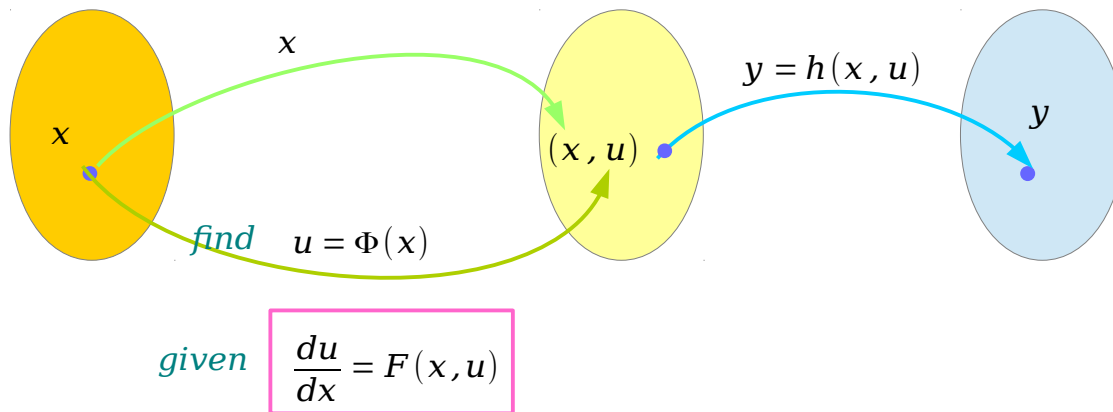
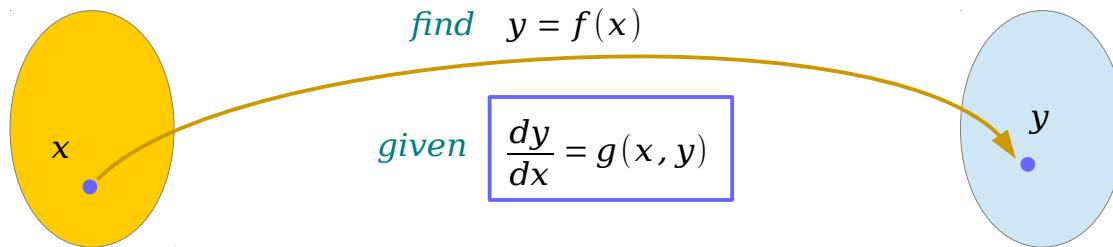
$$\frac{dy}{dx} = g(x, y)$$



Find $u = \Phi(x)$ in

$$\frac{du}{dx} = F(x, u)$$

Substitution Method (5)



$$\frac{dy}{dx} = \frac{\partial h}{\partial x} + \frac{\partial h}{\partial u} \frac{du}{dx}$$

(1) replace y'

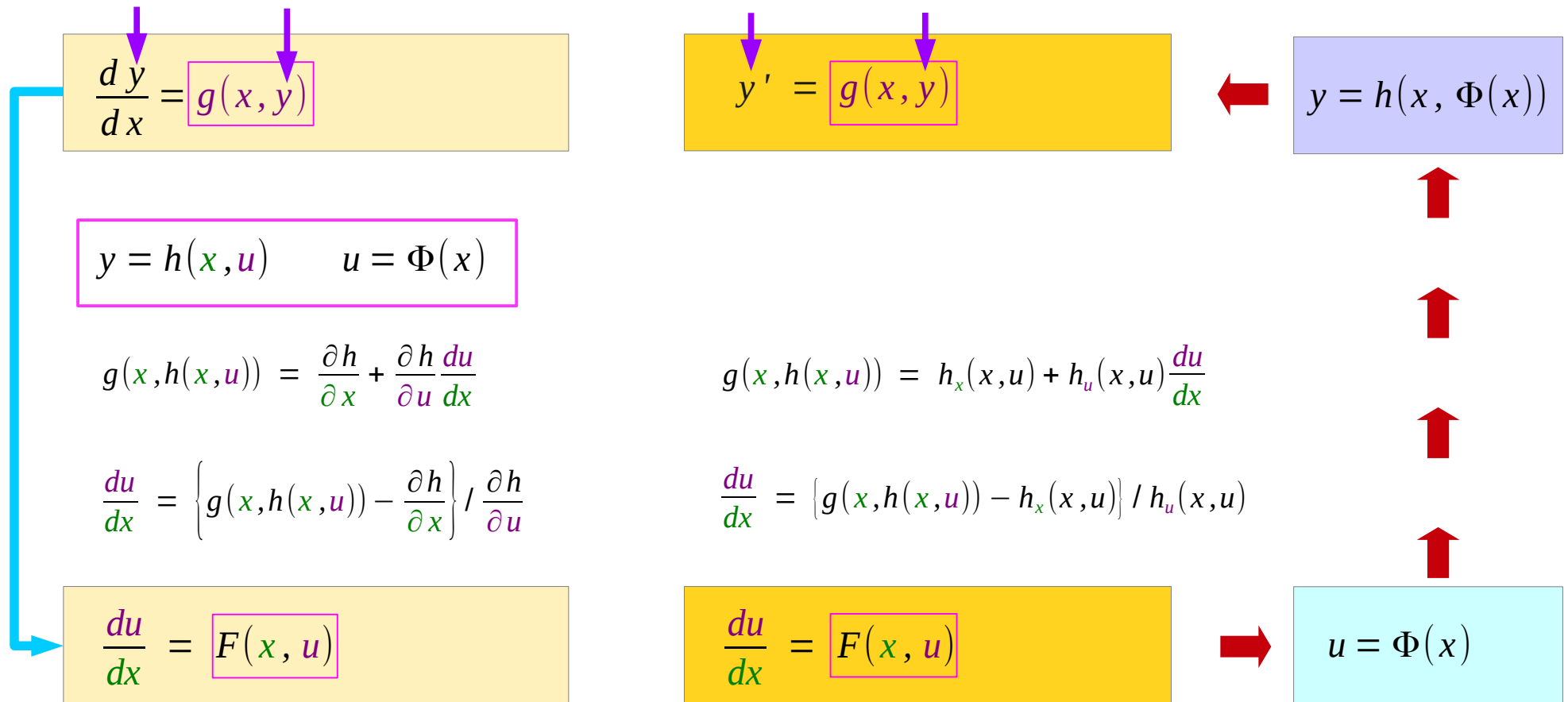
$$\frac{dy}{dx} \leftarrow \frac{\partial h}{\partial x} + \frac{\partial h}{\partial u} \frac{du}{dx}$$

(2) replace y

$$g(x, y) \leftarrow g(x, h(x, u))$$

Substitution Method (6)

A General Form of First Order Differential Equations



Homogeneous First Order ODEs

Homogeneous Functions

A *homogeneous function of degree α*

$$f(tx, ty) = t^\alpha f(x, y)$$

$$f(x, y) = x^2 + y^2$$

$$\begin{aligned} f(tx, ty) &= (tx)^2 + (ty)^2 \\ &= t^2(x^2 + y^2) \\ &= t^2 f(x, y) \end{aligned}$$

A *homogeneous Equations of degree α*

$$M(x, y)dx + N(x, y)dy = 0$$

$$\begin{aligned} M(tx, ty) &= t^\alpha M(x, y) \\ N(tx, ty) &= t^\alpha N(x, y) \end{aligned}$$

$$M(x, y) = M(x, x \cdot y/x) = x^\alpha M(1, y/x)$$

$$M(x, y) = M(y \cdot x/y, y) = y^\alpha M(x/y, 1)$$

$$N(x, y) = N(x, x \cdot y/x) = x^\alpha N(1, y/x)$$

$$N(x, y) = N(y \cdot x/y, y) = y^\alpha N(x/y, 1)$$

Homogeneous Equations (1)

A homogeneous Equations of degree α

$$M(x, y)dx + N(x, y)dy = 0$$

$$M(x, y) = x^\alpha M(1, y/x)$$

$$N(x, y) = x^\alpha N(1, y/x)$$

$$u = y/x \quad y = ux$$

$$x^\alpha M(1, y/x)dx + x^\alpha N(1, y/x)dy = 0$$

$$M(1, u)dx + N(1, u)dy = 0$$

$$dy = \frac{\partial}{\partial x}(ux)dx + \frac{\partial}{\partial u}(ux)du$$

$$dy = udx + xdu$$

$$M(1, u)dx + N(1, u)(udx + xdu) = 0$$

A homogeneous Equations of degree α

$$M(x, y)dx + N(x, y)dy = 0$$

$$M(x, y) = y^\alpha M(x/y, 1)$$

$$N(x, y) = y^\alpha N(x/y, 1)$$

$$v = x/y \quad x = vy$$

$$y^\alpha M(x/y, 1)dx + y^\alpha N(x/y, 1)dy = 0$$

$$M(v, 1)dx + N(v, 1)dy = 0$$

$$dx = \frac{\partial}{\partial y}(vy)dy + \frac{\partial}{\partial v}(vy)dv$$

$$dx = vdy + ydv$$

$$M(v, 1)(vdy + ydv) + N(v, 1)dy = 0$$

Homogeneous Equations (2)

A homogeneous Equations of degree α

$$M(x, y)dx + N(x, y)dy = 0$$

$$x^\alpha M(1, y/x)dx + x^\alpha N(1, y/x)dy = 0$$

$$M(1, u)dx + N(1, u)dy = 0$$

$$u = y/x \quad y = ux$$

$$dy = udx + xdu$$

$$M(1, u)dx + N(1, u)(udx + xdu) = 0$$

$$[M(1, u) + uN(1, u)]dx + xN(1, u)du = 0$$

$$\frac{dx}{x} + \frac{N(1, u)du}{[M(1, u) + uN(1, u)]} = 0$$

A homogeneous Equations of degree α

$$M(x, y)dx + N(x, y)dy = 0$$

$$y^\alpha M(x/y, 1)dx + y^\alpha N(x/y, 1)dy = 0$$

$$M(v, 1)dx + N(v, 1)dy = 0$$

$$v = x/y \quad x = vy$$

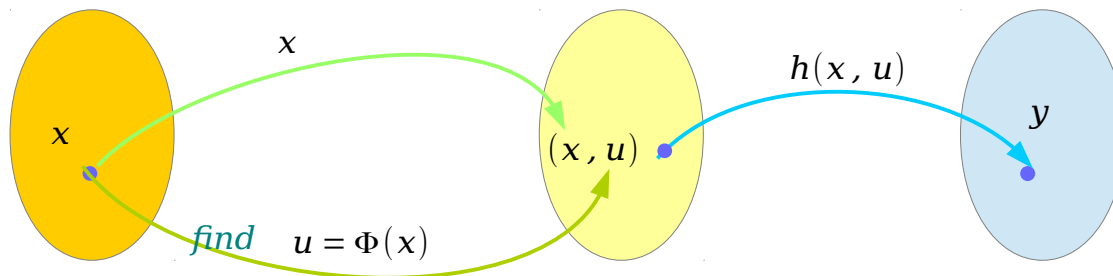
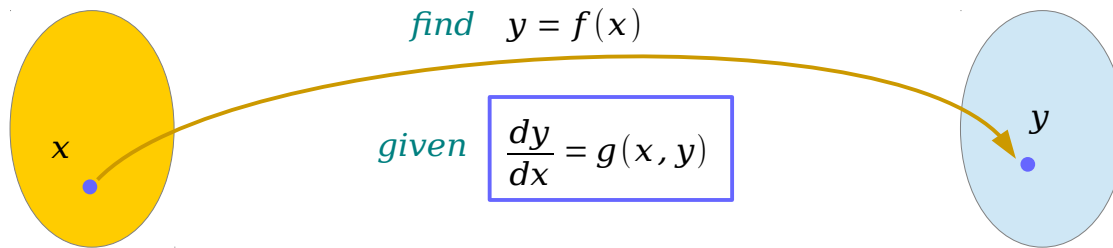
$$dx = vdy + ydv$$

$$M(v, 1)(vdy + ydv) + N(v, 1)dy = 0$$

$$[vM(v, 1) + N(v, 1)]dy + yM(v, 1)dv = 0$$

$$\frac{dy}{y} + \frac{M(v, 1)dv}{[vM(v, 1) + N(v, 1)]} = 0$$

Homogeneous Equations (3)



given $\frac{du}{dx} = F(x, u)$

$$\frac{dy}{dx} = \frac{\partial h}{\partial x} + \frac{\partial h}{\partial u} \frac{du}{dx}$$

$$\frac{dy}{dx} dx = \frac{\partial h}{\partial x} dx + \frac{\partial h}{\partial u} \frac{du}{dx} dx$$

$$dy = u dx + x du$$

$$M(x, y)dx + N(x, y)dy = 0$$

$$M(x, y) = x^\alpha M(1, y/x)$$

$$N(x, y) = x^\alpha N(1, y/x)$$

$$u = \Phi(x) = y/x$$

$$y = h(x, u) = ux$$

$$dy = u dx + x du$$

$$\frac{dx}{x} + \frac{N(1, u) du}{[M(1, u) + u N(1, u)]} = 0$$

Homogeneous Equations (4)

all are functions of x

$$y = f(x) \quad \longrightarrow \quad y(x)$$

$$u = \Phi(x) \quad \longrightarrow \quad u(x)$$

$$u = y/x \quad \longrightarrow \quad u(x) = y(x)/x$$

$$y = ux \quad \longrightarrow \quad y(x) = u(x)x$$

$$y = h(x, u) = h(x, \Phi(x))$$

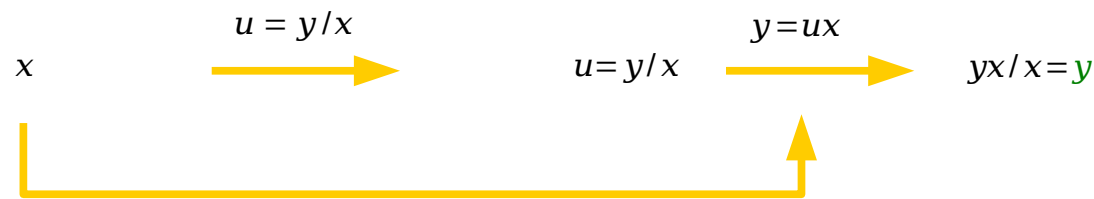
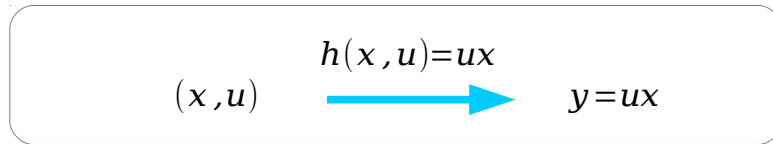
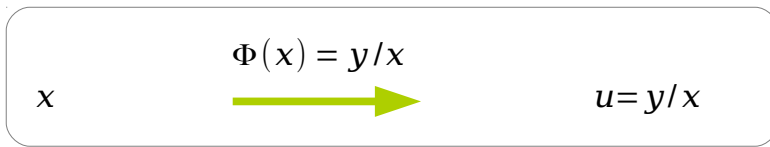
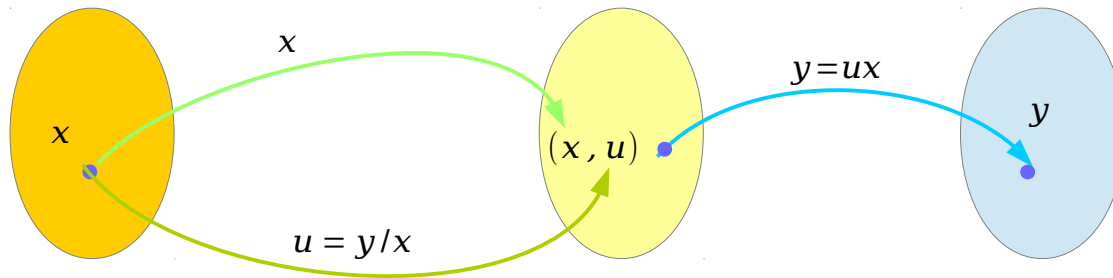
$$= ux \quad = \Phi(x)x$$

$$= \frac{y}{x}x$$

$$y = ux$$

$$dy = u dx + x du$$

Homogeneous Equations (5)



$$M(x, y)dx + N(x, y)dy = 0$$

$$M(x, y) = x^\alpha M(1, y/x)$$

$$N(x, y) = x^\alpha N(1, y/x)$$

$$u = \Phi(x) = y/x$$

$$y = h(u, x) = ux$$

$$dy = u dx + x du$$

$$\frac{dx}{x} + \frac{N(1, u) du}{[M(1, u) + u N(1, u)]} = 0$$

Bernoulli's First Order ODEs

Bernoulli's Equations (1)

Bernoulli's Equation

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

$$\frac{dy}{dx} + P(x)y = Q(x)y^0$$

$$\frac{dy}{dx} + P(x)y = Q(x) \quad n = 0$$

Linear Equation

$$\frac{dy}{dx} + P(x)y = Q(x)y^1$$

$$\frac{dy}{dx} + [P(x) - Q(x)]y = 0 \quad n = 1$$

Linear Equation

Bernoulli's Equation

$$y' + P(x)y = Q(x)y^n$$

$$y' + P(x)y = Q(x)y^0$$

$$y' + P(x)y = Q(x) \quad n = 0$$

Linear Equation

$$y' + P(x)y = Q(x)y^1$$

$$y' + [P(x) - Q(x)]y = 0 \quad n = 1$$

Linear Equation

Bernoulli's Equations (2)

Bernoulli's Equation

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

$$\frac{1}{y^n} \frac{dy}{dx} + P(x) \frac{y}{y^n} = Q(x)$$

$$y^{-n} \frac{dy}{dx} + P(x)y^{1-n} = Q(x)$$

$$u = y^{1-n} \quad \frac{du}{dx} = (1-n) \boxed{y^{-n} \frac{dy}{dx}}$$

$$\frac{1}{(1-n)} \frac{du}{dx} + P(x)u = Q(x)$$

$$\frac{du}{dx} + (1-n)P(x)u = (1-n)Q(x)$$

Linear Equation

Bernoulli's Equation

$$y' + P(x)y = Q(x)y^n$$

$$\frac{1}{y^n} y' + P(x) \frac{y}{y^n} = Q(x)$$

$$y^{-n} y' + P(x)y^{1-n} = Q(x)$$

$$u = y^{1-n} \quad u' = (1-n) \boxed{y^{-n} y'}$$

$$\frac{1}{(1-n)} u' + P(x)u = Q(x)$$

$$u' + (1-n)P(x)u = (1-n)Q(x)$$

Linear Equation

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