Systems of Linear Equations

Young W Lim

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Based on
A First Course in Linear Algebra, R. A. Beezer
http://linear.ups.edu/fcla/front-matter.html

Outline

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System of a Linear Equations

A System of Linear Equations

is a collection of m equations in the variable quantities $x_1, x_2, x_3, \ldots, x_n$ of the form,

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= b_3 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= b_m
\end{aligned}$$

where the values of a_{ij}, x_j , and b_i, $(1 \leq i \leq m,\ 1 \leq j \leq n)$, are from the set of complex numbers, $\mathbb{C}$.

Solution of a System of a Linear Equations

A Solution of a System of Linear Equations

is an ordered list of n complex numbers, $s_1, s_2, s_3, \ldots, s_n$
for n variables, $x_1, x_2, x_3, \ldots, x_n$, such that

if we substitute

s_1 for x_1,
s_2 for x_2,
s_3 for x_3,
$\ldots$,
s_3 for x_n,

then all m equations are true simultaneously, i.e,

for every equation of the system
the left side will equal to the right side

Solution Set of a System of a Linear Equations

The solution set of a System of Linear Equations

is the set which contains every solution to the system,
and nothing more.

Three types of a solution set

- $$\begin{aligned} 2x_1 + 3x_2 &= 3 \\ x_1 - x_2 &= 4 \end{aligned}$$ a single solution
- $$\begin{aligned} 2x_1 + 3x_2 &= 3 \\ 4x_1 + 6x_2 &= 6 \end{aligned}$$ inifintely many solution
- $$\begin{aligned} 2x_1 + 3x_2 &= 3 \\ 4x_1 + 6x_2 &= 10 \end{aligned}$$ no soution

Equivalent Systems

Equivalent Systems

Two systems of linear equations are **equivalent**
if their solution sets are equal.

Equation Operations

Equation Operations

Given a system of linear equations, the following three operations will transform the system into a different one, and each operation is known as an **equation operation**.

1. swap the locations of two equations in the list of equations.
2. multiply each term of an equation by a nonzero quantity.
3. multiply each term of one equation by some quantity, and add these terms to a second equation, on both sides of the equality. leave the first equation the same after this operation, but replace the second equation by the new one.

Equation Operations Preserve Solution Sets

Equation Operations

If we apply one of the three equation operations
to a system of linear equations,
then the original system and the transformed system
are equivalent.

Three Equations and One Solution (1)

solve the following by a sequence of equation operations

$$\begin{aligned} x_1 \quad +2x_2 \quad +2x_3 \quad &= 4 \\ x_1 \quad +3x_2 \quad +3x_3 \quad &= 5 \\ 2x_1 \quad +6x_2 \quad +5x_3 \quad &= 6 \end{aligned}$$

1. $-1 \cdot eq1 + eq2 \rightarrow eq2$
   $-1 \cdot (1, 2, 2, 4) + (1, 3, 3, 5) \rightarrow (0, 1, 1, 1)$

$$\begin{aligned} x_1 \quad +2x_2 \quad +2x_3 \quad &= 4 \\ 0x_1 \quad +1x_2 \quad +1x_3 \quad &= 1 \\ 2x_1 \quad +6x_2 \quad +5x_3 \quad &= 6 \end{aligned}$$

2. $-2 \cdot eq1 + eq3 \rightarrow eq3$
   $-2 \cdot (1, 2, 2, 4) + (2, 6, 5, 6) \rightarrow (0, 2, 1, -2)$

$$\begin{aligned} x_1 \quad +2x_2 \quad +2x_3 \quad &= 4 \\ 0x_1 \quad +1x_2 \quad +1x_3 \quad &= 1 \\ 0x_1 \quad +2x_2 \quad +1x_3 \quad &= -2 \end{aligned}$$

Three Equations and One Solution (2)

3. $-2 \cdot eq2 + eq3 \rightarrow eq3$
   $-2 \cdot (0,1,1,1) + (0,2,1,-2) \rightarrow (0,0,-1,-4)$

$$\begin{array}{rrrr} x_1 & +2x_2 & +2x_3 & =4 \\ 0x_1 & +1x_2 & +1x_3 & =1 \\ 0x_1 & +0x_2 & -1x_3 & =-4 \end{array}$$

4. $-1 \cdot eq3 \rightarrow eq3$
   $-1 \cdot (0,0,-1,-4) \rightarrow (0,0,1,4)$

$$\begin{array}{rrrr} x_1 & +2x_2 & +2x_3 & =4 \\ 0x_1 & +1x_2 & +1x_3 & =1 \\ 0x_1 & +2x_2 & +1x_3 & =4 \end{array}$$