Trigonometry (4A)

- Trigonometric Identities
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1/2/15

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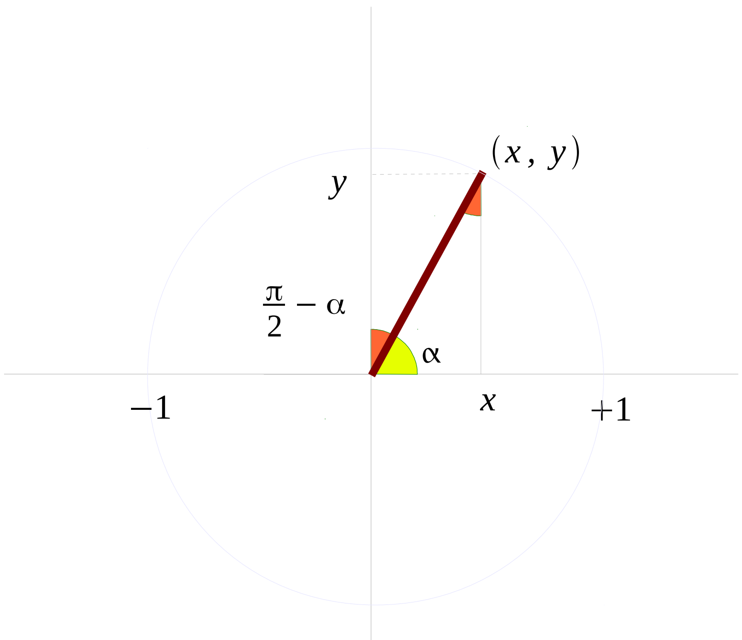
Co-function Identities



$$\sin\alpha = y \Rightarrow \cos\left(\frac{\pi}{2} - \alpha\right)$$

$$\cos\alpha = x \Rightarrow \sin\left(\frac{\pi}{2} - \alpha\right)$$

$$\tan\alpha = y/x \Rightarrow \cot\left(\frac{\pi}{2} - \alpha\right)$$

$$\cos\left(\frac{\pi}{2} - \alpha\right) = \sin\alpha$$

$$\sin\left(\frac{\pi}{2} - \alpha\right) = \cos\alpha$$

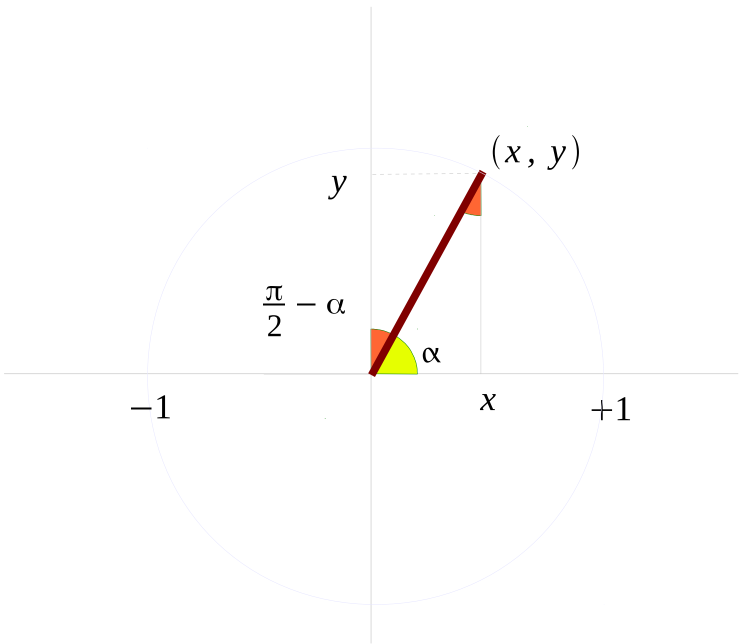
$$\tan\left(\frac{\pi}{2} - \alpha\right) = \cot\alpha$$

$$\cot\left(\frac{\pi}{2} - \alpha\right) = \tan\alpha$$

$$csc\left(\frac{\pi}{2} - \alpha\right) = sec\,\alpha$$

$$sec\left(\frac{\pi}{2} - \alpha\right) = sec\,\alpha$$

Angle Sum and Difference Identities (1)



$$\sin(\alpha + \beta) = \sin\alpha \cdot \cos\beta + \cos\alpha \cdot \sin\beta$$

$$\sin(\alpha - \beta) = \sin\alpha \cdot \cos\beta - \cos\alpha \cdot \sin\beta$$

$$\sin(60^\circ + 30^\circ) = 1$$

$+$ $\sin(60^\circ) = \frac{\sqrt{3}}{2}$ $\quad$ $\sin(30^\circ) = \frac{1}{2}$

$+$ $\cos(60^\circ) = \frac{1}{2}$ $\quad$ $\cos(30^\circ) = \frac{\sqrt{3}}{2}$

$$\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2} + \frac{1}{2}\cdot\frac{1}{2} = 1$$

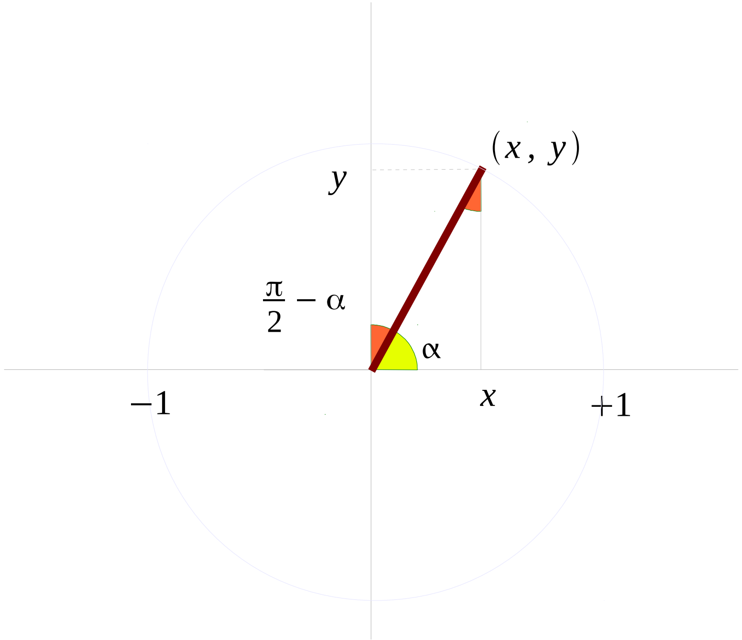
$$\sin(60^\circ - 30^\circ) = \frac{1}{2}$$

$+$ $\sin(60^\circ) = \frac{\sqrt{3}}{2}$ $\quad$ $\sin(30^\circ) = \frac{1}{2}$

$-$ $\cos(60^\circ) = \frac{1}{2}$ $\quad$ $\cos(30^\circ) = \frac{\sqrt{3}}{2}$

$$\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2} - \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{2}$$

Angle Sum and Difference Identities (2)



$$\cos(\alpha + \beta) = \cos\alpha \cdot \cos\beta - \sin\alpha \cdot \sin\beta$$

$$\cos(\alpha - \beta) = \cos\alpha \cdot \cos\beta + \sin\alpha \cdot \sin\beta$$

$$\cos(30^\circ + 60^\circ) = 0$$

$-$ $\sin(60^\circ) = \frac{\sqrt{3}}{2}$ —— $\sin(30^\circ) = \frac{1}{2}$

$+$ $\cos(60^\circ) = \frac{1}{2}$ —— $\cos(30^\circ) = \frac{\sqrt{3}}{2}$

$$\frac{\sqrt{3}}{2} \cdot \frac{1}{2} - \frac{1}{2} \cdot \frac{\sqrt{3}}{2} = 0$$

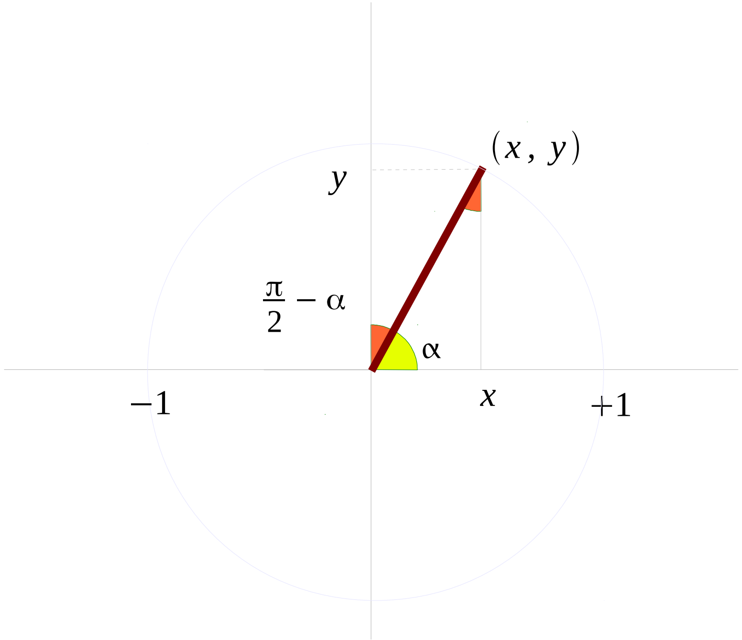
$$\cos(30^\circ - 60^\circ) = \frac{\sqrt{3}}{2}$$

$+$ $\sin(60^\circ) = \frac{\sqrt{3}}{2}$ —— $\sin(30^\circ) = \frac{1}{2}$

$+$ $\cos(60^\circ) = \frac{1}{2}$ —— $\cos(30^\circ) = \frac{\sqrt{3}}{2}$

$$\frac{\sqrt{3}}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}$$

Angle Sum and Difference Identities (3)



$$\tan(\alpha + \beta) = \frac{\tan(\alpha) + \tan(\beta)}{1 - \tan(\alpha)\tan(\beta)}$$

$$\tan(\alpha - \beta) = \frac{\tan(\alpha) - \tan(\beta)}{1 + \tan(\alpha)\tan(\beta)}$$

$$\tan(30° + 60°) = +\infty$$

$\tan(30°) = \frac{1}{\sqrt{3}}$ —— $\tan(60°) = \sqrt{3}$

$\tan(60°) = \sqrt{3}$ —— $\tan(30°) = \frac{1}{\sqrt{3}}$

$$\frac{\frac{1}{\sqrt{3}} + \sqrt{3}}{1 - \frac{1}{\sqrt{3}}\cdot\sqrt{3}} = +\infty$$

$$\tan(30° - 60°) = -\frac{1}{\sqrt{3}}$$

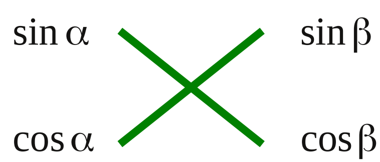
$\tan(30°) = \frac{1}{\sqrt{3}}$ —— $\tan(60°) = \sqrt{3}$

$\tan(60°) = \sqrt{3}$ —— $\tan(30°) = \frac{1}{\sqrt{3}}$

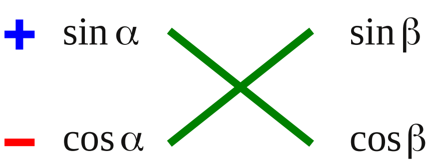
$$\frac{\frac{1}{\sqrt{3}} - \sqrt{3}}{1 + \frac{1}{\sqrt{3}}\cdot\sqrt{3}} = -\frac{1}{\sqrt{3}}$$

Angle Sum and Difference Identities (4)

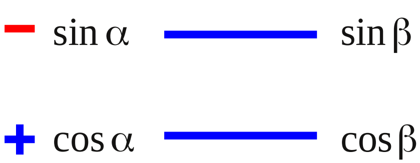
$$\sin(\alpha + \beta) = \sin\alpha \cdot \cos\beta + \cos\alpha \cdot \sin\beta$$



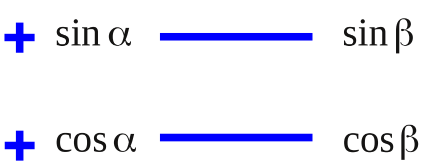
$$\sin(\alpha - \beta) = \sin\alpha \cdot \cos\beta - \cos\alpha \cdot \sin\beta$$



$$\cos(\alpha + \beta) = \cos\alpha \cdot \cos\beta - \sin\alpha \cdot \sin\beta$$



$$\cos(\alpha - \beta) = \cos\alpha \cdot \cos\beta + \sin\alpha \cdot \sin\beta$$



$$\frac{\sin(\alpha + \beta)}{\cos(\alpha + \beta)} = \frac{\sin\alpha \cdot \cos\beta + \cos\alpha \cdot \sin\beta}{\cos\alpha \cdot \cos\beta - \sin\alpha \cdot \sin\beta}$$

$$\frac{\sin(\alpha - \beta)}{\cos(\alpha - \beta)} = \frac{\sin\alpha \cdot \cos\beta - \cos\alpha \cdot \sin\beta}{\cos\alpha \cdot \cos\beta + \sin\alpha \cdot \sin\beta}$$

$$\tan(\alpha + \beta) = \frac{\tan(\alpha) + \tan(\beta)}{1 - \tan(\alpha)\tan(\beta)}$$

$$\tan(\alpha - \beta) = \frac{\tan(\alpha) - \tan(\beta)}{1 + \tan(\alpha)\tan(\beta)}$$

Product to Sum (1)

$$+\ \sin(\alpha + \beta) = \sin\alpha \cdot \cos\beta + \cos\alpha \cdot \sin\beta$$
$$+\ \sin(\alpha - \beta) = \sin\alpha \cdot \cos\beta - \cos\alpha \cdot \sin\beta$$

$$\sin(\alpha + \beta) + \sin(\alpha - \beta) = 2\sin\alpha \cdot \cos\beta$$

$$\sin\alpha \cdot \cos\beta = \tfrac{1}{2}\{\sin(\alpha + \beta) + \sin(\alpha - \beta)\}$$

$$+\ \sin(\alpha + \beta) = \sin\alpha \cdot \cos\beta + \cos\alpha \cdot \sin\beta$$
$$-\ \sin(\alpha - \beta) = \sin\alpha \cdot \cos\beta - \cos\alpha \cdot \sin\beta$$

$$\sin(\alpha + \beta) - \sin(\alpha - \beta) = 2\cos\alpha\sin\beta$$

$$\cos\alpha \cdot \sin\beta = \tfrac{1}{2}\{\sin(\alpha + \beta) - \sin(\alpha - \beta)\}$$

$$+\ \cos(\alpha + \beta) = \cos\alpha \cdot \cos\beta - \sin\alpha \cdot \sin\beta$$
$$+\ \cos(\alpha - \beta) = \cos\alpha \cdot \cos\beta + \sin\alpha \cdot \sin\beta$$

$$\cos(\alpha + \beta) + \cos(\alpha - \beta) = 2\cos\alpha \cdot \cos\beta$$

$$\cos\alpha \cdot \cos\beta = \tfrac{1}{2}\{+\cos(\alpha + \beta) + \cos(\alpha - \beta)\}$$

$$-\ \cos(\alpha + \beta) = \cos\alpha \cdot \cos\beta - \sin\alpha \cdot \sin\beta$$
$$+\ \cos(\alpha - \beta) = \cos\alpha \cdot \cos\beta + \sin\alpha \cdot \sin\beta$$

$$-\cos(\alpha + \beta) + \cos(\alpha - \beta) = 2\sin\alpha\sin\beta$$

$$\sin\alpha \cdot \sin\beta = \tfrac{1}{2}\{-\cos(\alpha + \beta) + \cos(\alpha - \beta)\}$$

Product to Sum (2)

$$\sin(\alpha \pm \beta) = \square\,\alpha \cdot \square\,\beta \pm \square\,\alpha\ \square\,\beta$$

$$\sin(\alpha + \beta) = \sin\alpha \cdot \cos\beta + \cos\alpha \cdot \sin\beta$$

$$\sin(\alpha - \beta) = \sin\alpha \cdot \cos\beta - \cos\alpha \cdot \sin\beta$$

$$\cos(\alpha \pm \beta) = \square\,\alpha \cdot \square\,\beta \mp \square\,\alpha \cdot \square\,\beta$$

$$\cos(\alpha + \beta) = \cos\alpha \cdot \cos\beta - \sin\alpha \cdot \sin\beta$$

$$\cos(\alpha - \beta) = \cos\alpha \cdot \cos\beta + \sin\alpha \cdot \sin\beta$$

$$\square\,\alpha \cdot \square\,\beta = \tfrac{1}{2}\{\sin(\square) + \sin(\square)\}$$

$$\sin\alpha \cdot \cos\beta = \tfrac{1}{2}\{+\sin(\alpha + \beta) + \sin(\alpha - \beta)\}$$

$$\cos\alpha \cdot \sin\beta = \tfrac{1}{2}\{+\sin(\alpha + \beta) - \sin(\alpha - \beta)\}$$

$$\square\,\alpha \cdot \square\,\beta = \tfrac{1}{2}\{+\cos(\square) + \cos(\square)\}$$

$$\cos\alpha \cdot \cos\beta = \tfrac{1}{2}\{+\cos(\alpha + \beta) + \cos(\alpha - \beta)\}$$

$$\sin\alpha \cdot \sin\beta = \tfrac{1}{2}\{-\cos(\alpha + \beta) + \cos(\alpha - \beta)\}$$

Angle sum and difference identities

Sine	$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$ [8][9]
Cosine	$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$ [9][10]
Tangent	$\tan(\alpha \pm \beta) = \dfrac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$ [9][11]
Arcsine	$\arcsin\alpha \pm \arcsin\beta = \arcsin\left(\alpha\sqrt{1-\beta^2} \pm \beta\sqrt{1-\alpha^2}\right)$ [12]
Arccosine	$\arccos\alpha \pm \arccos\beta = \arccos\left(\alpha\beta \mp \sqrt{(1-\alpha^2)(1-\beta^2)}\right)$ [13]
Arctangent	$\arctan\alpha \pm \arctan\beta = \arctan\left(\dfrac{\alpha \pm \beta}{1 \mp \alpha\beta}\right)$ [14]

http://en.wikipedia.org/wiki/Derivative

Double Angle Formula

Double-angle formulae[18][19]			
$\sin 2\theta = 2\sin\theta\cos\theta$ $= \frac{2\tan\theta}{1+\tan^2\theta}$	$\cos 2\theta = \cos^2\theta - \sin^2\theta$ $= 2\cos^2\theta - 1$ $= 1 - 2\sin^2\theta$ $= \frac{1-\tan^2\theta}{1+\tan^2\theta}$	$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$	$\cot 2\theta = \frac{\cot^2\theta - 1}{2\cot\theta}$

http://en.wikipedia.org/wiki/Derivative

Triple-angle formulae

Triple-angle formulae[16][20]			
$\sin 3\theta = -\sin^3\theta + 3\cos^2\theta\sin\theta$ $= -4\sin^3\theta + 3\sin\theta$	$\cos 3\theta = \cos^3\theta - 3\sin^2\theta\cos\theta$ $= 4\cos^3\theta - 3\cos\theta$	$\tan 3\theta = \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$	$\cot 3\theta = \dfrac{3\cot\theta - \cot^3\theta}{1 - 3\cot^2\theta}$

http://en.wikipedia.org/wiki/Derivative

Half-angle formulae

Half-angle formulae[21][22]			
$\sin\frac{\theta}{2} = \operatorname{sgn}\left(2\pi-\theta+4\pi\left\lfloor\frac{\theta}{4\pi}\right\rfloor\right)\sqrt{\frac{1-\cos\theta}{2}}$ $\left(\text{or}\ \ \sin^2\frac{\theta}{2} = \frac{1-\cos\theta}{2}\right)$	$\cos\frac{\theta}{2} = \operatorname{sgn}\left(\pi+\theta+4\pi\left\lfloor\frac{\pi-\theta}{4\pi}\right\rfloor\right)\sqrt{\frac{1+\cos\theta}{2}}$ $\left(\text{or}\ \ \cos^2\frac{\theta}{2} = \frac{1+\cos\theta}{2}\right)$	$\tan\frac{\theta}{2} = \csc\theta - \cot\theta = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \frac{\sin\theta}{1+\cos\theta} = \frac{1-\cos\theta}{\sin\theta}$ $\tan\frac{\eta+\theta}{2} = \frac{\sin\eta+\sin\theta}{\cos\eta+\cos\theta}$ $\tan\left(\frac{\theta}{2}+\frac{\pi}{4}\right) = \sec\theta+\tan\theta$ $\sqrt{\frac{1-\sin\theta}{1+\sin\theta}} = \frac{1-\tan(\theta/2)}{1+\tan(\theta/2)}$ $\tan\frac{1}{2}\theta = \frac{\tan\theta}{1+\sqrt{1+\tan^2\theta}}$ $\text{for}\quad \theta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$	$\cot\frac{\theta}{2} = \csc\theta+\cot\theta = \pm\sqrt{\frac{1+\cos\theta}{1-\cos\theta}} = \frac{\sin\theta}{1-\cos\theta} = \frac{1+\cos\theta}{\sin\theta}$

http://en.wikipedia.org/wiki/Derivative

Power-reduction formula

Sine	Cosine	Other
$\sin^2\theta = \frac{1-\cos 2\theta}{2}$	$\cos^2\theta = \frac{1+\cos 2\theta}{2}$	$\sin^2\theta\cos^2\theta = \frac{1-\cos 4\theta}{8}$
$\sin^3\theta = \frac{3\sin\theta-\sin 3\theta}{4}$	$\cos^3\theta = \frac{3\cos\theta+\cos 3\theta}{4}$	$\sin^3\theta\cos^3\theta = \frac{3\sin 2\theta-\sin 6\theta}{32}$
$\sin^4\theta = \frac{3-4\cos 2\theta+\cos 4\theta}{8}$	$\cos^4\theta = \frac{3+4\cos 2\theta+\cos 4\theta}{8}$	$\sin^4\theta\cos^4\theta = \frac{3-4\cos 4\theta+\cos 8\theta}{128}$
$\sin^5\theta = \frac{10\sin\theta-5\sin 3\theta+\sin 5\theta}{16}$	$\cos^5\theta = \frac{10\cos\theta+5\cos 3\theta+\cos 5\theta}{16}$	$\sin^5\theta\cos^5\theta = \frac{10\sin 2\theta-5\sin 6\theta+\sin 10\theta}{512}$

http://en.wikipedia.org/wiki/Derivative

Product-to-sum

Product-to-sum[24]
$\cos\theta\cos\varphi = \dfrac{\cos(\theta-\varphi)+\cos(\theta+\varphi)}{2}$
$\sin\theta\sin\varphi = \dfrac{\cos(\theta-\varphi)-\cos(\theta+\varphi)}{2}$
$\sin\theta\cos\varphi = \dfrac{\sin(\theta+\varphi)+\sin(\theta-\varphi)}{2}$
$\cos\theta\sin\varphi = \dfrac{\sin(\theta+\varphi)-\sin(\theta-\varphi)}{2}$
$\tan\theta\tan\varphi = \dfrac{\cos(\theta-\varphi)-\cos(\theta+\varphi)}{\cos(\theta-\varphi)+\cos(\theta+\varphi)}$
$\prod_{k=1}^{n}\cos\theta_k = \dfrac{1}{2^n}\sum_{e\in S}\cos(e_1\theta_1+\cdots+e_n\theta_n)$ where $S=\{1,-1\}^n$

http://en.wikipedia.org/wiki/Derivative

Sum-to-product

Sum-to-product[25]
$\sin\theta \pm \sin\varphi = 2\sin\left(\frac{\theta \pm \varphi}{2}\right)\cos\left(\frac{\theta \mp \varphi}{2}\right)$
$\cos\theta + \cos\varphi = 2\cos\left(\frac{\theta + \varphi}{2}\right)\cos\left(\frac{\theta - \varphi}{2}\right)$
$\cos\theta - \cos\varphi = -2\sin\left(\frac{\theta + \varphi}{2}\right)\sin\left(\frac{\theta - \varphi}{2}\right)$

http://en.wikipedia.org/wiki/Derivative

Euler's Formula

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

$$e^{i(A+B)} = \cos(A+B) + i\sin(A+B)$$

$$e^{iA}e^{iB} = (\cos(A) + i\sin(A))(\cos(B) + i\sin(B))$$

$$= [\cos(A)\cos(B) - \sin(A)\sin(B)]$$

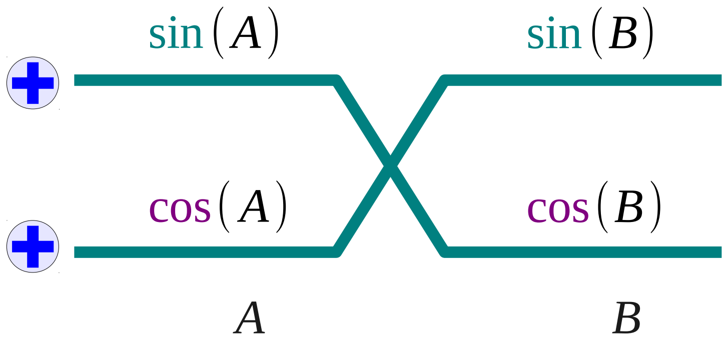
$$+ \ i[\cos(A)\sin(B) + \sin(A)\cos(B)]$$

$$\sin(A+B) = \sin(A)\cos(B) + \cos(A)\sin(B)$$

$$\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$$

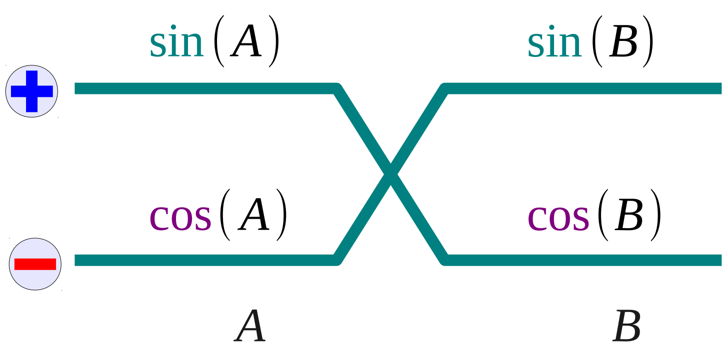
Sin(angle sum and difference)

$\sin(A+B)$



$\sin(A)\cos(B) - \cos(A)\sin(B)$

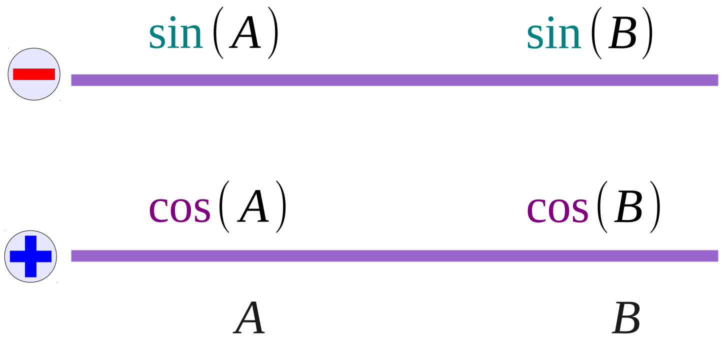
$\sin(A-B)$



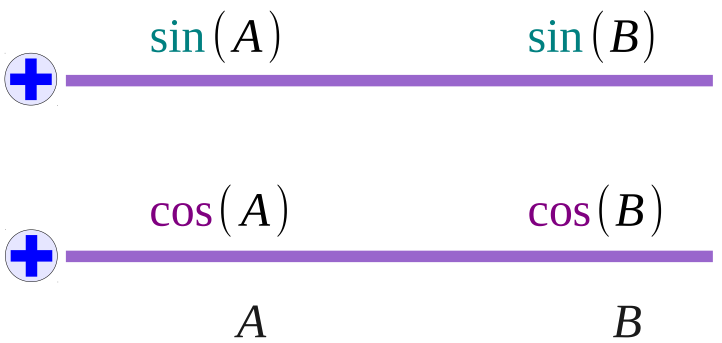
$\sin(A)\cos(B) - \cos(A)\sin(B)$

Cos(angle sum and difference)

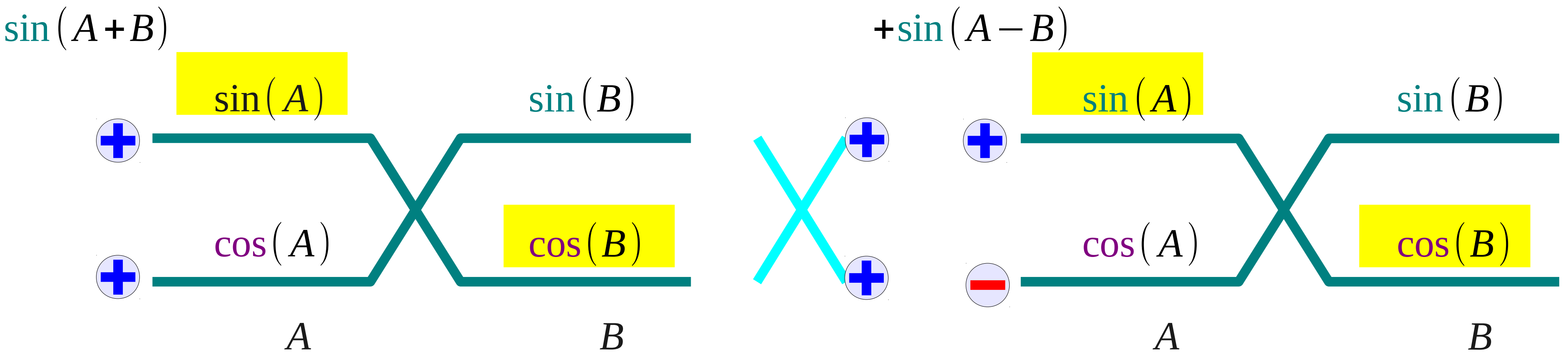
$\cos(A+B)$ $\qquad$ $\cos(A)\cos(B) - \sin(A)\sin(B)$



$\cos(A-B)$ $\qquad$ $\cos(A)\cos(B) + \sin(A)\sin(B)$



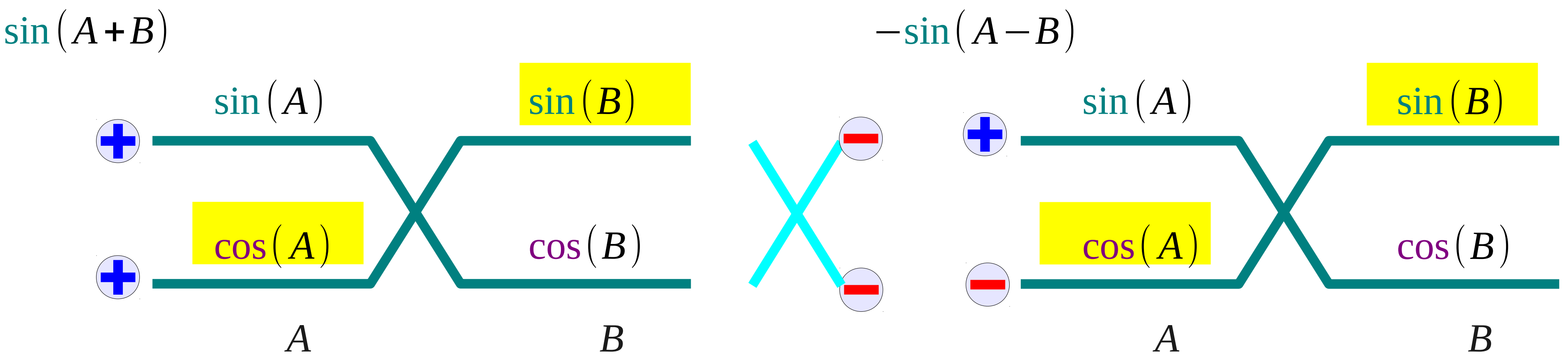
Product to Sum : sin cos



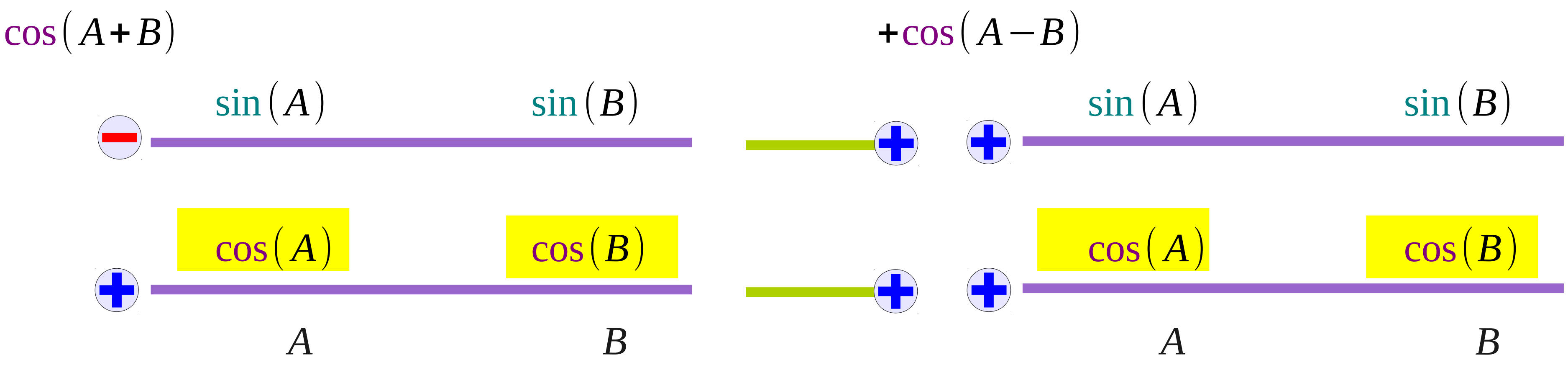
$$\sin(A+B) + \sin(A-B) \Leftarrow 2\sin(A)\cos(B)$$

Product to Sum : cos sin

$$\sin(A+B) - \sin(A-B) \Leftarrow 2\cos(A)\sin(B)$$



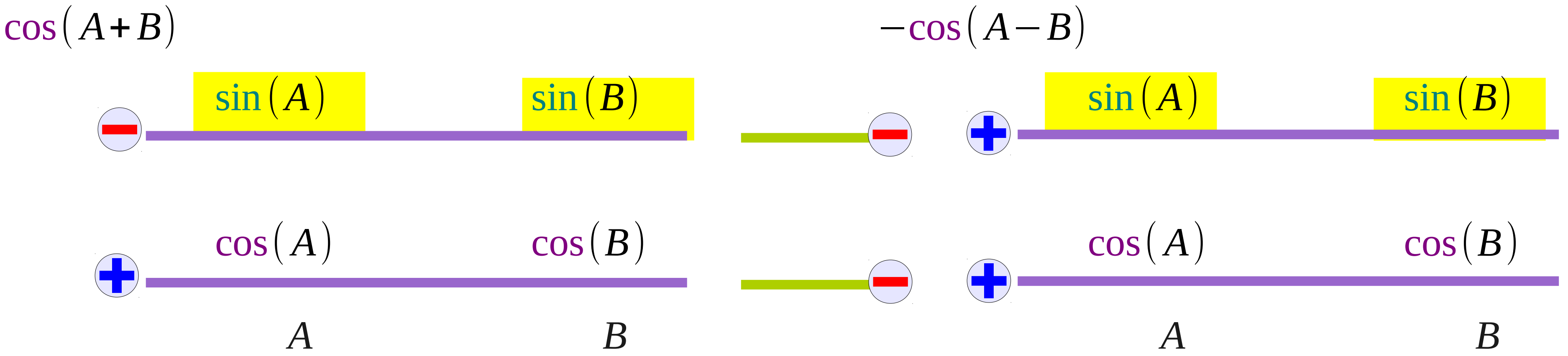
Product to Sum : cos cos



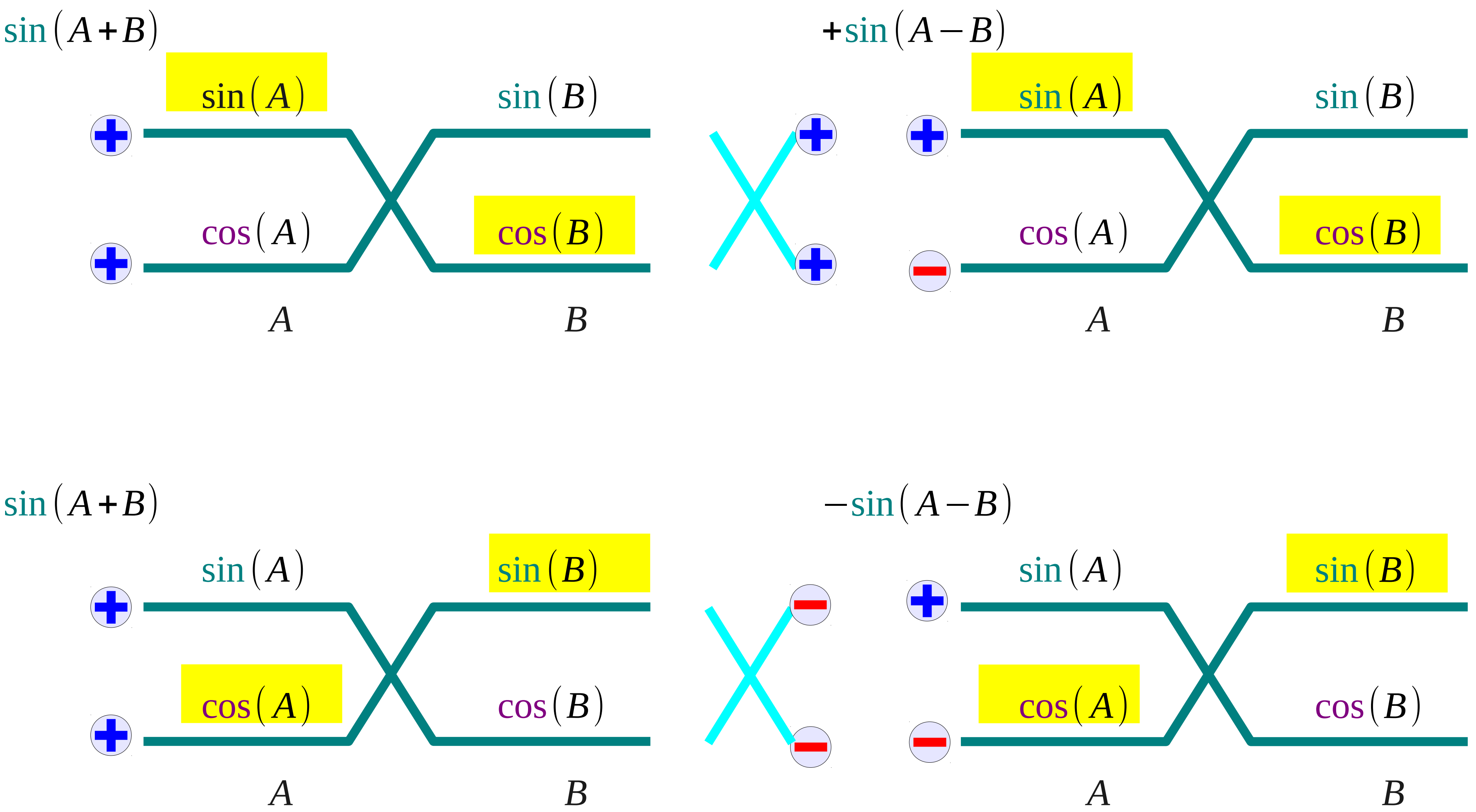
$$\cos(A+B) + \cos(A-B) \Leftarrow 2\cos(A)\cos(B)$$

Product to Sum : sin sin

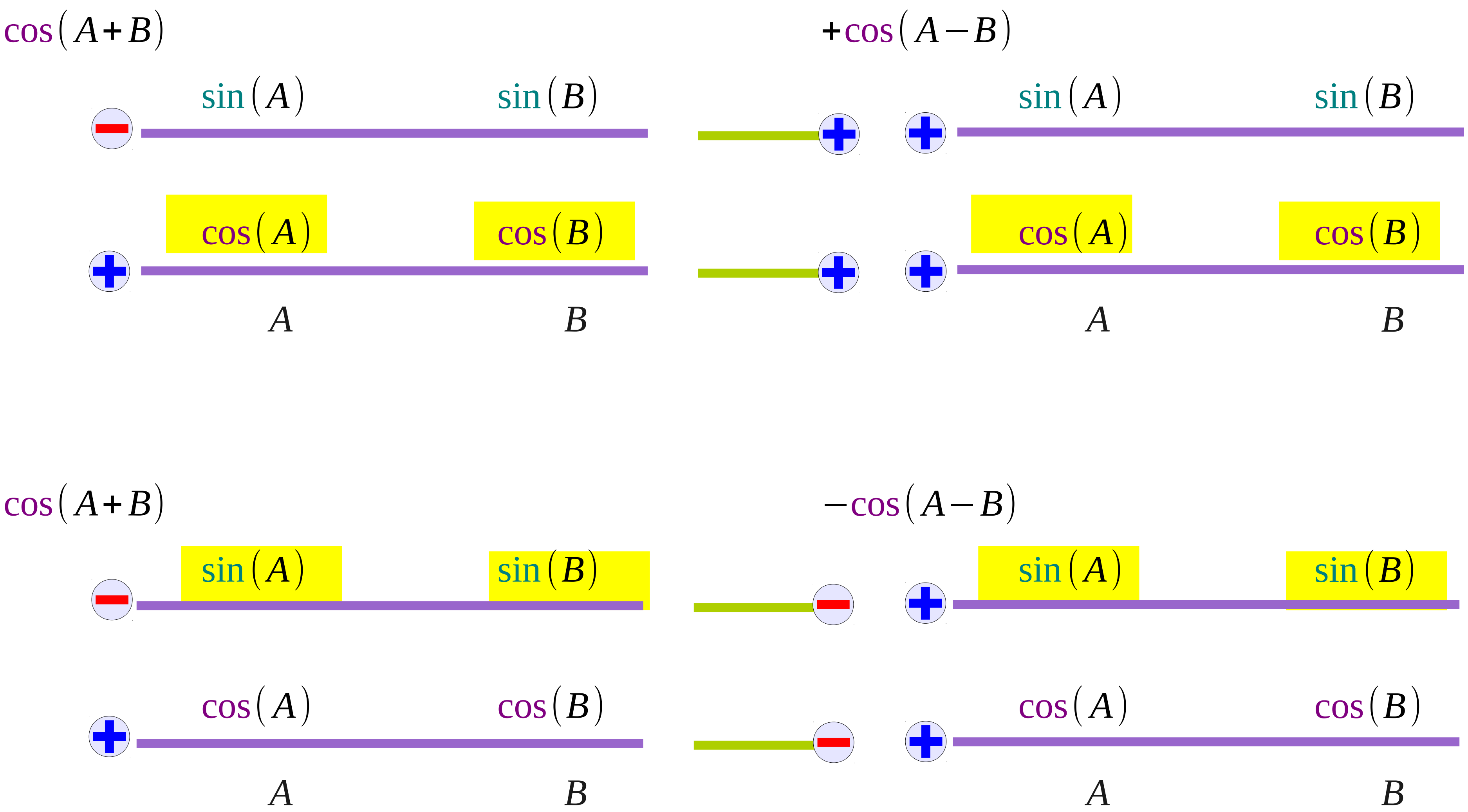
$$\cos(A+B) - \cos(A-B) \Leftarrow -2\sin(A)\sin(B)$$
$$-\cos(A+B) + \cos(A-B) \Leftarrow +2\sin(A)\sin(B)$$



Product to Sum



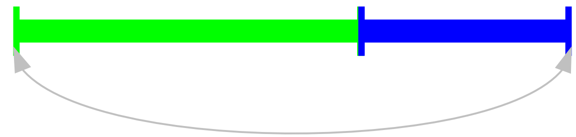
Product to Sum



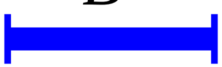
Sum and Difference

A

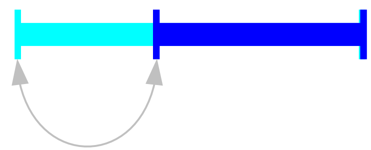
$A+B = X$



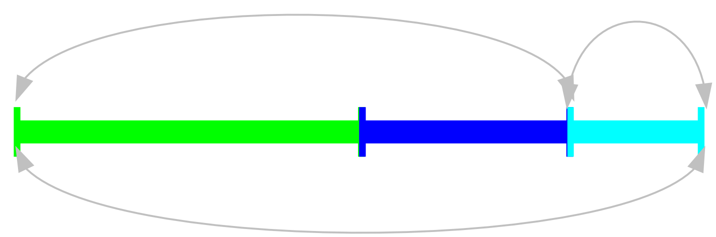
B



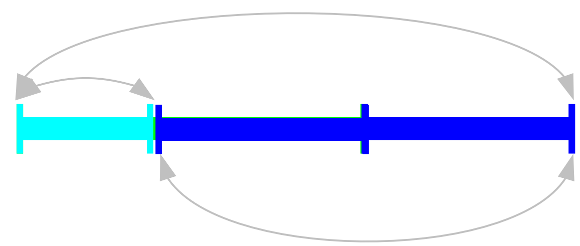
$A-B = Y$



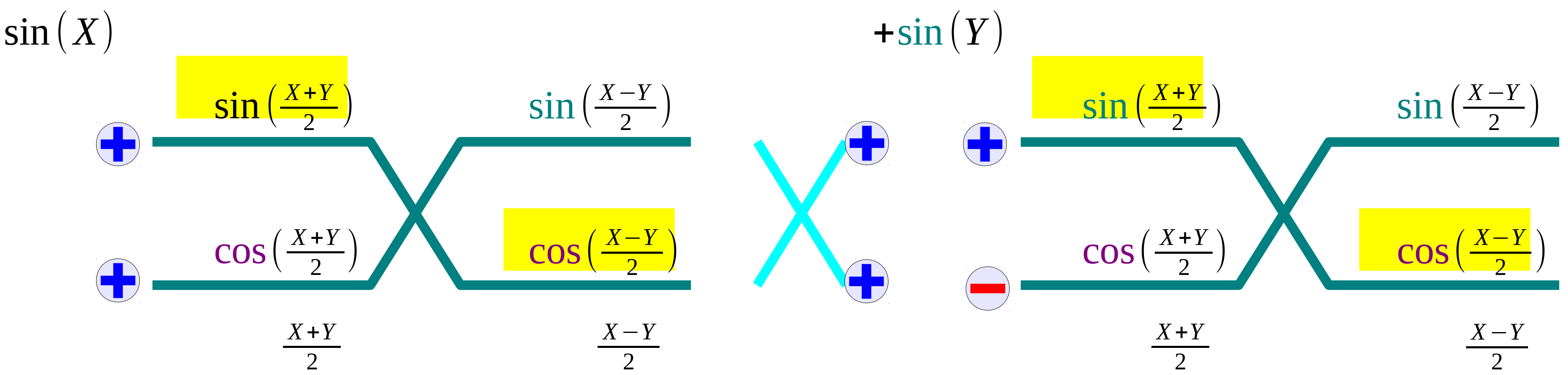
$X+Y = A+B+A-B = 2\mathrm{A}$

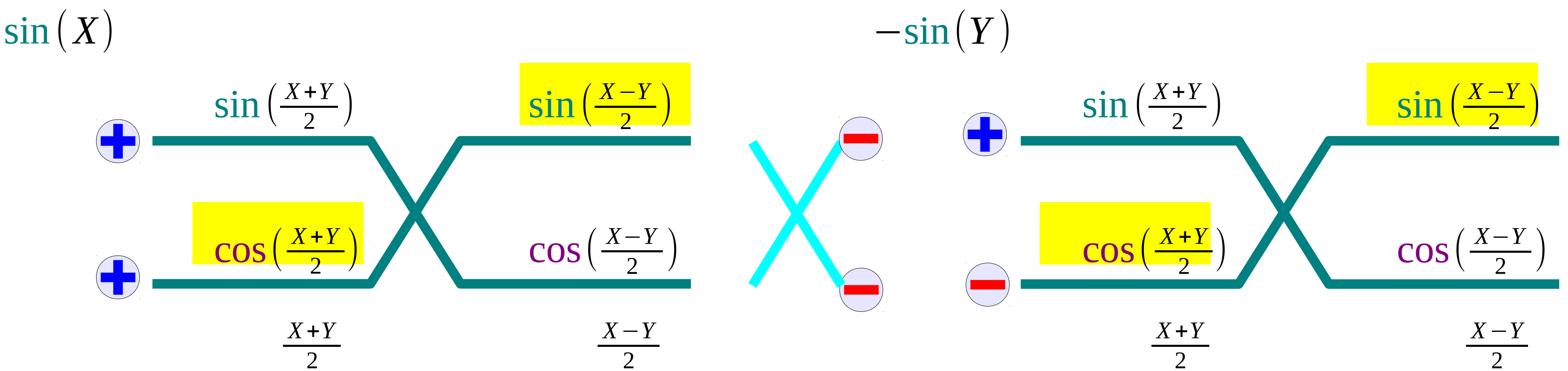


$X-Y = A+B-A+B = 2\mathrm{B}$



Product to Sum





Product to Sum

$\cos(X)$ $\quad + \cos(Y)$

$-\ \sin\left(\frac{X+Y}{2}\right) \sin\left(\frac{X-Y}{2}\right)$ $\quad +$ $\quad +\ \sin\left(\frac{X+Y}{2}\right) \sin\left(\frac{X-Y}{2}\right)$

$+\ \cos\left(\frac{X+Y}{2}\right) \cos\left(\frac{X-Y}{2}\right)$ $\quad +$ $\quad +\ \cos\left(\frac{X+Y}{2}\right) \cos\left(\frac{X-Y}{2}\right)$

$\frac{X+Y}{2}$ $\quad \frac{X-Y}{2}$ $\quad \frac{X+Y}{2}$ $\quad \frac{X-Y}{2}$

$\cos(X)$ $\quad - \cos(Y)$

$-\ \sin\left(\frac{X+Y}{2}\right) \sin\left(\frac{X-Y}{2}\right)$ $\quad -$ $\quad +\ \sin\left(\frac{X+Y}{2}\right) \sin\left(\frac{X-Y}{2}\right)$

$+\ \cos\left(\frac{X+Y}{2}\right) \cos\left(\frac{X-Y}{2}\right)$ $\quad -$ $\quad +\ \cos\left(\frac{X+Y}{2}\right) \cos\left(\frac{X-Y}{2}\right)$

$\frac{X+Y}{2}$ $\quad \frac{X-Y}{2}$ $\quad \frac{X+Y}{2}$ $\quad \frac{X-Y}{2}$

Product-to-Sum & Sum-to-Product

SUM		PRODUCT
$\sin(A+B) + \sin(A-B)$	$\Leftarrow$	$2\sin(A)\cos(B)$
$\sin(A+B) - \sin(A-B)$	$\Leftarrow$	$2\cos(A)\sin(B)$
$\cos(A+B) + \cos(A-B)$	$\Leftarrow$	$2\cos(A)\cos(B)$
$-\cos(A+B) + \cos(A-B)$	$\Leftarrow$	$2\sin(A)\sin(B)$

SUM		PRODUCT
$\sin(X) + \sin(Y)$	$\Rightarrow$	$2\sin\left(\frac{X+Y}{2}\right)\cos\left(\frac{X-Y}{2}\right)$
$\sin(X) - \sin(Y)$	$\Rightarrow$	$2\cos\left(\frac{x+Y}{2}\right)\sin\left(\frac{X-Y}{2}\right)$
$\cos(X) + \cos(Y)$	$\Rightarrow$	$2\cos\left(\frac{X+Y}{2}\right)\cos\left(\frac{X-Y}{2}\right)$
$-\cos(X) + \cos(Y)$	$\Rightarrow$	$2\sin\left(\frac{X+Y}{2}\right)\sin\left(\frac{X-Y}{2}\right)$

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