Z Transform (H.1)
Definition

20170110

Based on
Complex Analysis for Mathematics and Engineering
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Z - Transform

$$X(z) = \sum_{k=-\infty}^{+\infty} x[k]\, z^{-k}$$

$$z = |r|\, e^{j2\pi F} = |r|\, e^{j\Omega}$$

$$x[n] \longleftrightarrow X(z)$$

One Sided   z-transform

$$X(z) = \sum_{k=0}^{+\infty} x[k]\, z^{-k}$$

Inverse z-Transform

$$X(z) = Z\left[\{x_n\}_{n=0}^{\infty}\right] = \sum_{n=0}^{\infty} x_n z^{-n} = \sum_{n=0}^{\infty} x[n] z^{-n}$$

$$x_n = x[n] = Z^{-1}\left[X(z)\right] = \frac{1}{2\pi i}\int_C X(z) z^{n-1}\, dz$$

Admissible Form of z-transform

$$X(z) = \sum_{k=0}^{\infty} x[n]\, z^{-n}$$

admissible z-transform

if $X(z)$ is a rational function

$$X(z) = \frac{P(z)}{Q(z)} = \frac{b_0 + b_1 z^1 + b_2 z^2 + \cdots + b_{p-1} z^{p-1} + b_p z^p}{a_0 + a_1 z^1 + a_2 z^2 + \cdots + a_{q-1} z^{q-1} + a_q z^q}$$

$P(z)$: a polynomial of degree p

$Q(z)$: a polynomial of degree q

D: Simply connected domain

C: Simple closed contour (CCW) in D

if $f(z)$ is analytic inside C and on C
   except at the points $z_1, z_2, \ldots, z_k$ in C
then

$$\frac{1}{2\pi i} \int_C f(z)\, dz = \sum_{j=1}^{k} \text{Res}\left(f(z), z_j\right)$$

$$\oint_c f(z)\,dz = 2\pi i \sum_{k=1}^{n} \text{Res}\left(f(z), z_k\right)$$

finite number k of
Singular points z_k
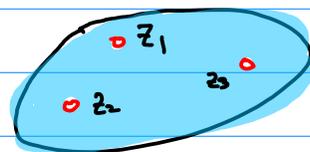
$$\oint_c f(z)\,dz = 0$$

if $f(z)$ is continuous in D and
$f(z) = F'(z)$; $F(z)$ is an antiderivative of $f(z)$
fundamental theorem of calculus

$$\oint_c f(z)\,dz = 0$$

if $f(z)$ is analytic within and on C

no Singularity
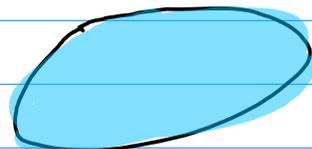
$$f(z) = \sum_{n=n_1}^{\infty} a_n^{\{m\}} (z - z_m)^n$$

$$a_n^{\{m\}} = \frac{1}{2\pi i} \oint_C \frac{f(z')}{(z'-z_m)^{n+1}} \, dz'$$

$$= \sum_k \text{Res}\left(\frac{f(z)}{(z-z_m)^{n+1}}, z_k\right) \quad z_k \text{ within } C$$

$$= \frac{1}{n!} f^{(n)}(z_m) \qquad n \geq 0$$
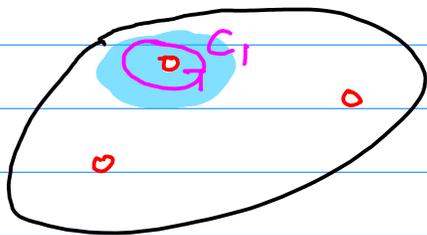
$a_n^{\{0\}}$ expansion at z_0
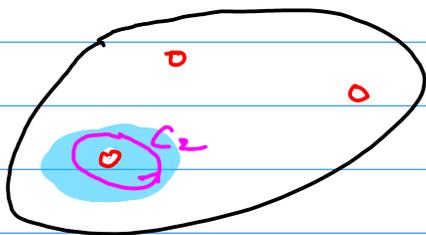
$a_n^{\{1\}}$ expansion at z_1
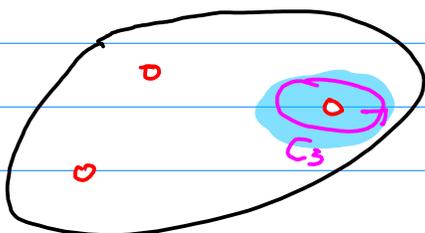
$a_n^{\{2\}}$ expansion at z_2

$$a_n^{\{m\}} = \text{Res}\left(f(z), z_m\right)$$

the residue of $f(z)$ at z_m using C_m

assumed that
there are several (m) singularities (poles) of $f(z)$ in a region

but that
C is taken to enclose only the pole z_m : C_m
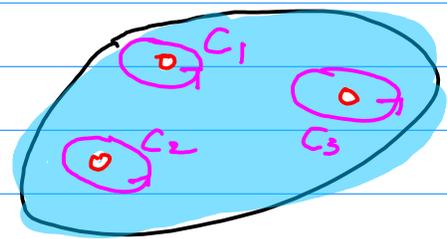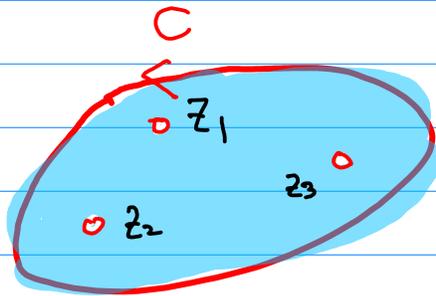
Laurent's Theorem

f : analytic within the annular domain D

$$r < |z - z_0| < R$$

then

$$f(z) = \sum_{k=-\infty}^{+\infty} a_k (z - z_0)^k \quad , \text{ valid for } \quad r < |z - z_0| < R$$

$$a_k = \frac{1}{2\pi i} \oint_C \frac{f(s)}{(s - z_0)^{k+1}} \, ds, \qquad k = 0, \pm 1, \pm 2, \cdots$$

C : a simple closed curve
that lies entirely within D
that encloses z_0

$$a_{-1} = \frac{1}{2\pi i} \oint_C f(s) \, ds \qquad \Longrightarrow \qquad \oint_C f(s) \, ds = 2\pi i \cdot a_{-1}$$

$$a_{-1} = \frac{1}{2\pi i} \oint_C f(s) \, ds = \text{Res}(f(z), z_0)$$

$$= \begin{cases} \displaystyle\lim_{z \to z_0} (z - z_0) f(z) & (\text{simple}) \\[2em] \displaystyle\frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} (z - z_0)^n f(z) & (\text{order } n) \end{cases}$$

Cauchy's Residue Theorem

$f(z)$: analytic on and within c

except a finite number of singular points

$z_1, z_2, \cdots, z_n$ within C

then

$$\int_c f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname{Res}(f(z), z_k)$$

D: a simply connected domain

C: a simple closed contour in D
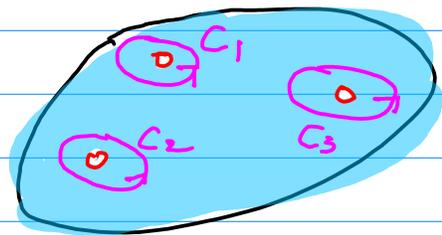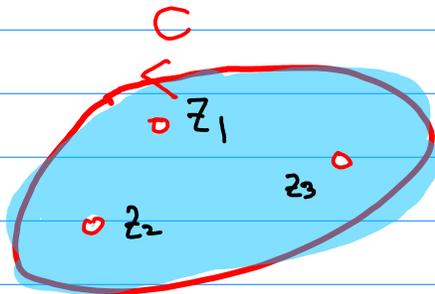
$\boxed{z_1}$	$f(z) = \sum_{k=-\infty}^{+\infty} a_k (z-z_1)^k$	$a_{-1}^{\{1\}} = \frac{1}{2\pi i} \oint_{C_1} f(s)\,ds = \operatorname{Res}(f(z), z_1)$
$\boxed{z_2}$	$f(z) = \sum_{k=-\infty}^{+\infty} a_k (z-z_2)^k$	$a_{-1}^{\{2\}} = \frac{1}{2\pi i} \oint_{C_2} f(s)\,ds = \operatorname{Res}(f(z), z_2)$
$\boxed{z_3}$	$f(z) = \sum_{k=-\infty}^{+\infty} a_k (z-z_3)^k$	$a_{-1}^{\{3\}} = \frac{1}{2\pi i} \oint_{C_3} f(s)\,ds = \operatorname{Res}(f(z), z_3)$

$\boxed{z_1}$
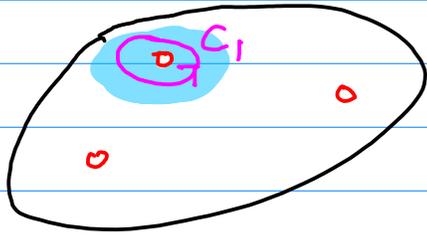
$$f(z) = \sum_{k=-\infty}^{+\infty} a_k (z-z_1)^k$$

$$a_{-1}^{(1)} = \frac{1}{2\pi i} \oint_{C_1} f(s)\, ds = \text{Res}(f(z), z_1)$$

$\boxed{z_2}$
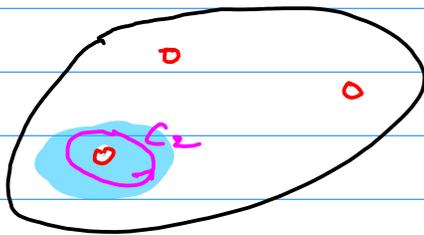
Laurent series expansion at z_2

$$f(z) = \sum_{k=-\infty}^{+\infty} a_k (z-z_2)^k$$

$$a_{-1}^{(2)} = \frac{1}{2\pi i} \oint_{C_2} f(s)\, ds = \text{Res}(f(z), z_2)$$

$\boxed{z_3}$
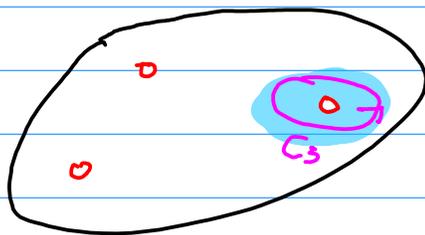
Laurent series expansion at z_3

$$f(z) = \sum_{k=-\infty}^{+\infty} a_k (z-z_3)^k$$

$$a_{-1}^{(3)} = \frac{1}{2\pi i} \oint_{C_3} f(s)\, ds = \text{Res}(f(z), z_3)$$
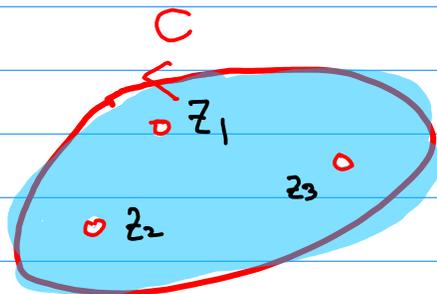
$$\int_C f(z)\, dz = 2\pi i \sum_{k=1}^{n} \text{Res}(f(z), z_k)$$

C

z_1

z_3

z_2
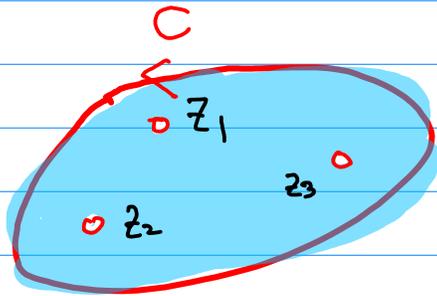
$$\int_c f(z)\,dz = 2\pi i \ \operatorname{Res}(f(z), z_1)$$
$$+ \ 2\pi i \ \operatorname{Res}(f(z), z_2)$$
$$+ \ 2\pi i \ \operatorname{Res}(f(z), z_3)$$

C'

z_1

z_3

z_2
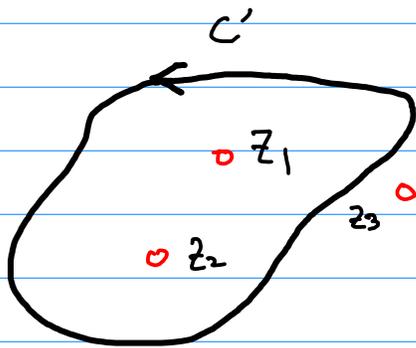
$$\int_{c'} f(z)\,dz = 2\pi i \ \operatorname{Res}(f(z), z_1)$$
$$+ \ 2\pi i \ \operatorname{Res}(f(z), z_2)$$

C''

z_1

z_3

z_2

$$\int_{c''} f(z)\,dz = 2\pi i \ \operatorname{Res}(f(z), z_1)$$

C'''

z_1

z_3

z_2
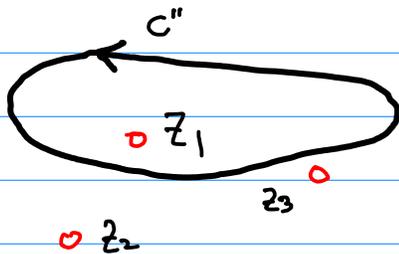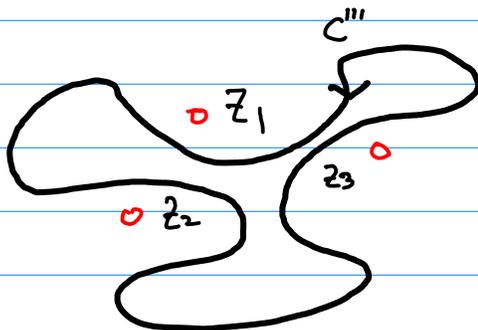
$$\int_{c''} f(z)\,dz = 0$$

Different D, Different Laurent Series

Z-transform

$$f(z) = \sum_{n=n_1}^{\infty} a_n^{\{m\}} (z - z_m)^n$$
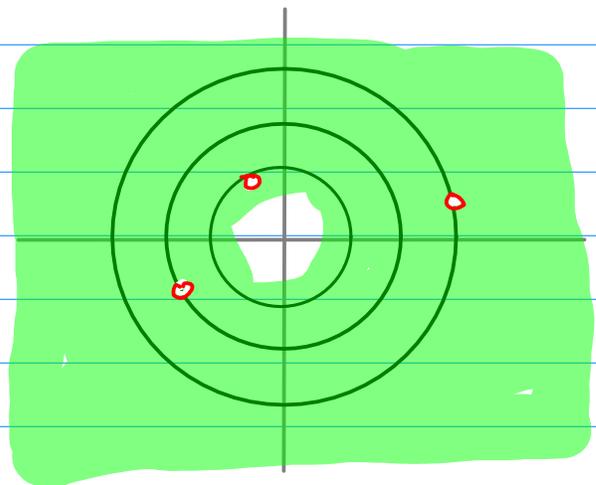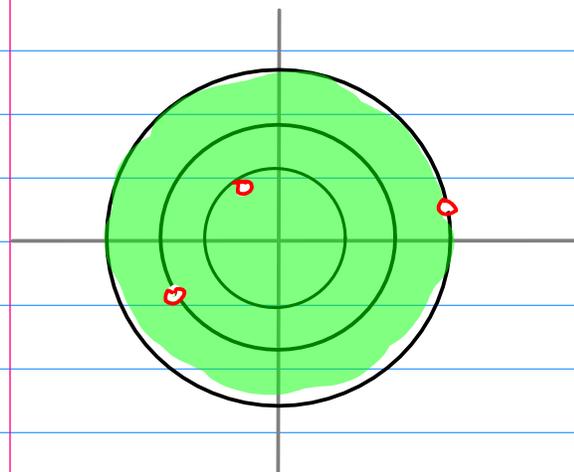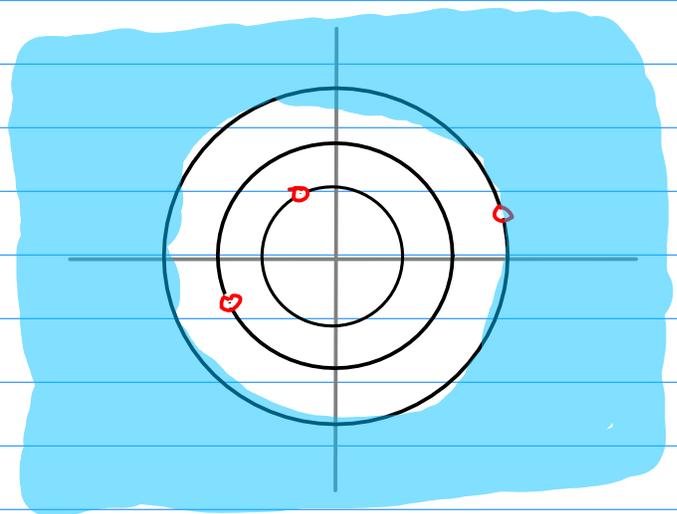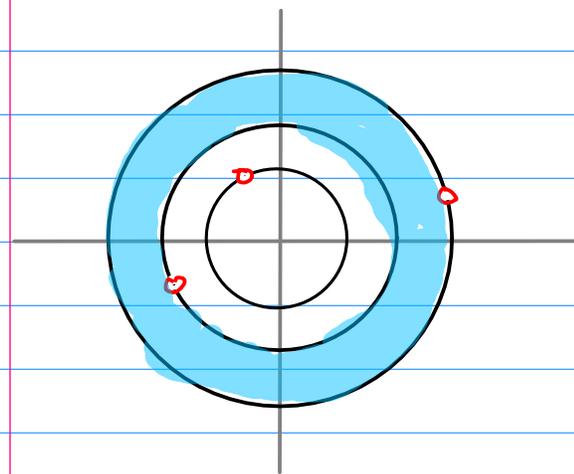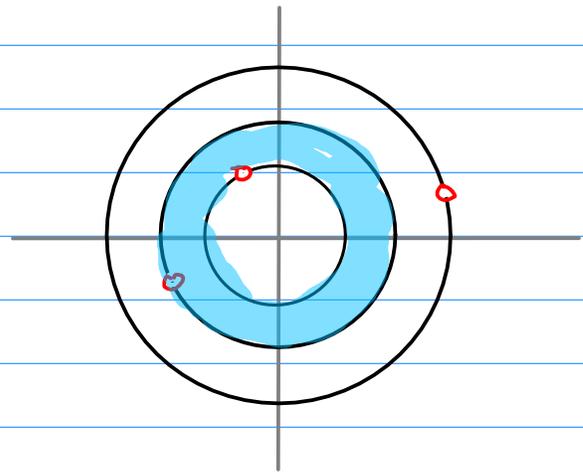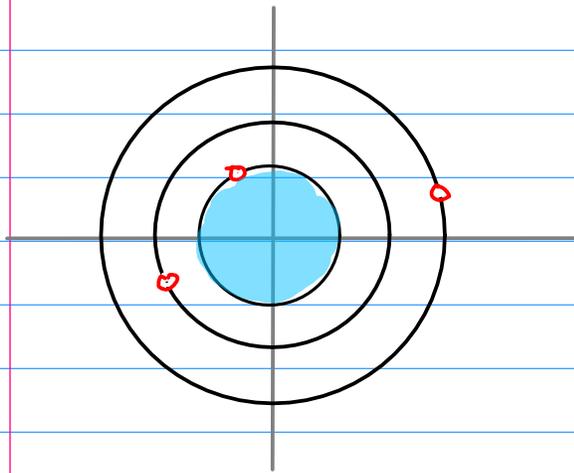
$$a_n^{\{m\}} = \frac{1}{2\pi i} \oint_C \frac{f(z')}{(z'-z_m)^{n+1}} dz'$$

$$= \sum_k \mathrm{Res}\left(\frac{f(z)}{(z-z_m)^{n+1}}, z_k\right)$$

C is in the same region of analyticity of $f(z)$ typically a circle centered on z_m

z_k within C : singularities of $\dfrac{f(z)}{(z-z_m)^{n+1}}$

$n_1 = n_{f,m}$ depends on $f(z)$, z_m

$a_n^{\{m\}}$ depends on $f(z)$, z_m, region of analyticity

Whether $f(z)$ is singular at $z = z_m$ or not
     or at other points between z and z_m
We can expand $f(z)$ about any point z_m
     over powers of $(z - z_m)$.

$$f(z) = \sum_{n=n_1}^{\infty} a_n^{\{m\}} (z - z_m)^n$$
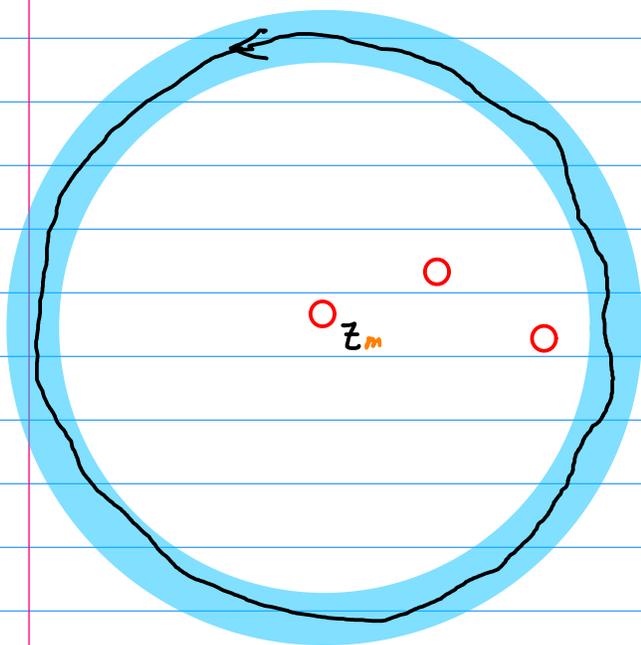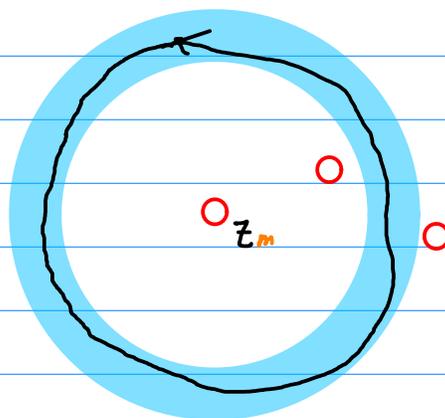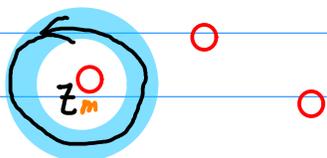
$$a_n^{\{m\}} = \frac{1}{2\pi i} \oint_C \frac{f(z')}{(z'-z_m)^{n+1}} \, dz'$$

$$= \sum_k \text{Res}\left(\frac{f(z)}{(z-z_m)^{n+1}} , z_n \right)$$

$$f(z) = \sum_{n=n_1}^{\infty} a_n^{\{m\}} (z - z_m)^n$$

$$a_n^{\{m\}} = \frac{1}{2\pi i} \oint_C \frac{f(z')}{(z'-z_m)^{n+1}} dz'$$

$$= \sum_k \text{Res}\left(\frac{f(z)}{(z-z_m)^{n+1}}, z_m\right)$$

analytic at z_m

$n_1 \geqslant 0$ Taylor Series

general n_1 $z_m = 0$ MacLaurin Series

Singular at z_m

general n_1 Laurent Series

general n_1 $z_m = 0$ z - Transform

$$f(z) = \sum_{n=n_1}^{\infty} a_n^{\{m\}} (z - z_m)^n$$

$$a_n^{\{m\}} = \frac{1}{2\pi i} \oint_C \frac{f(z')}{(z'-z_m)^{n+1}} \, dz'$$

$$= \sum_k \text{Res}\left(\frac{f(z)}{(z-z_m)^{n+1}} , z_k \right)$$

$$z_m = 0 \qquad a_{-n}^{\{0\}} = h(n) \qquad n \to -n$$

$$H(z) = \sum_{n=n_1}^{\infty} h(-n) \, z^n$$

$$h(n) = \frac{1}{2\pi i} \oint_C \frac{H(z')}{z'^{n+1}} \, dz'$$

$$= \sum_k \text{Res}\left(\frac{H(z)}{z^{n+1}} , z_k \right)$$

$$H(z) = \sum_{n=-\infty}^{\infty} h(n) \, z^{-n}$$

$$h(n) = \frac{1}{2\pi i} \oint_C H(z') \, z'^{n-1} \, dz'$$

$$= \sum_k \text{Res}\left(H(z) \, z^{n-1} , z_k \right)$$

C is in the same region of analyticity of $f(z)$
    typically a circle centered on z_m

z_k within C : singularities of $\dfrac{f(z)}{(z-z_m)^{n+1}}$

C is in the same region of analyticity of $H(z)$
    typically a circle centered on z_m

generally a circle centered on the origin
may enclose any or all singularities of $H(z)$
often the unit circle

z_k within C : singularities of $H(z) z^{n-1}$

$$H(z) = \sum_{n=-\infty}^{\infty} h(n) \, z^{-n} \qquad\qquad z \in R.O.C$$

$$h(n) = \frac{1}{2\pi i} \oint_C H(z') \, z'^{n-1} \, dz' \qquad\qquad C \text{ in } R.O.C.$$

$$= \sum_{k} Res\left(H(z) \, z^{n-1}, \, z_n\right)$$

①   a power series representation
    of a function $f(z)$ of a complex variable z

②   a transform $H(z)$ of a sequence of 1

$x_c(t)$ continuous

$x_{s,c}(t)$ sampled, continuous
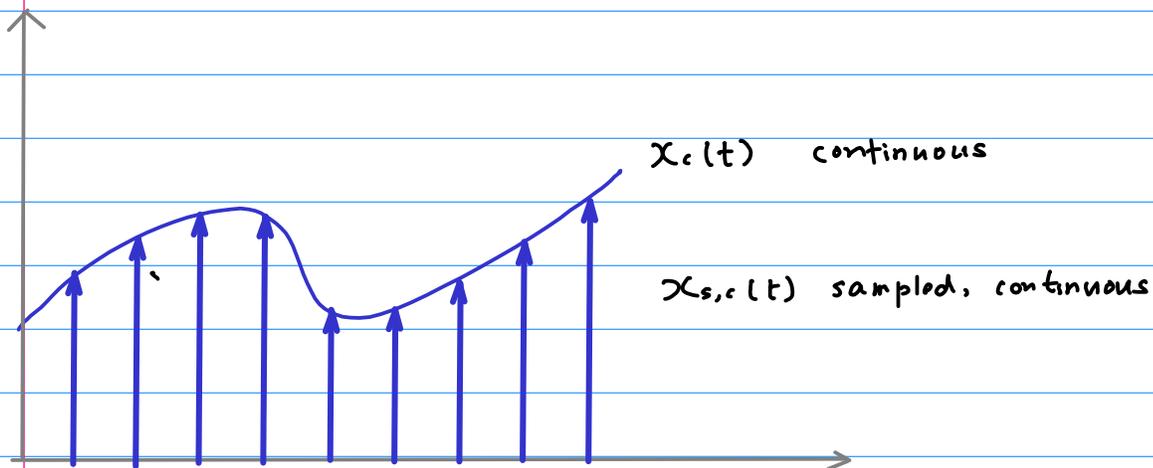
$$x_{s,c}(t) = \sum_{n=-\infty}^{+\infty} x(n)\, \delta_c(t - n\Delta t)$$

$$X_{s,c}(s) = \mathcal{L}\{x_{s,c}(t)\} = \int_{-\infty}^{\infty} \left[\sum_{n=-\infty}^{+\infty} x(n)\, \delta_c(t - n\Delta t) \right] e^{-st}\, dt$$

$$= \sum_{n=-\infty}^{+\infty} x(n) \int_{-\infty}^{\infty} \delta_c(t - n\Delta t)\, e^{-st}\, dt$$

$$= \sum_{n=-\infty}^{+\infty} x(n)\, e^{-s n \Delta t} \qquad e^{s\Delta t} \triangleq z$$

$$X_{s,c}(s) = \left. \sum_{n=-\infty}^{+\infty} x(n)\, z^{-n} \right|_{z = e^{s\Delta t}}$$

$$X_{s,c}(s) = X(z) \Big|_{z = e^{s\Delta t}}$$

$$X_{s,c}(s) = \mathcal{L}\{x_{s,c}(t)\} = X(z)\Big|_{z=e^{s\Delta t}}$$

$x_{s,c}(t)$ an impulse train

whose coefficients are given by $x[n] = x_c(n\Delta t)$

Z - transform : a special Laurent series

$z_m = 0$ $a_{-n}^{\{0\}} = h(n)$ $n \to -n$

$$f(z) = \sum_{n=n_1}^{\infty} a_n^{\{m\}} (z - z_m)^n$$

$$a_n^{\{m\}} = \frac{1}{2\pi i} \oint_c \frac{f(z')}{(z'-z_m)^{n+1}} dz'$$

$$= \sum_k \text{Res}\left(\frac{f(z)}{(z-z_m)^{n+1}}, z_k\right)$$

Time Reversal ← Laplace Transform

the transform functions
$X(s) = \int$ over negative powers e^{-st} for $t > 0$
$X(z) = \int$ over negative powers z^{-n} for $n > 0$

the time expansion functions
$x(t) = \int$ over negative powers e^{-st} for $t > 0$
$x[n] = \int$ over negative powers z^{-n} for $n > 0$

Time Reversal ← z^{-1}: unit delay, char eq (modes in z^k)

$x[n]$ → [$h[n]$] → $y[n]$
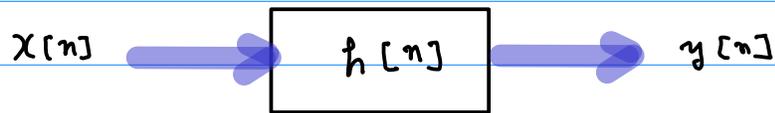
Stable System : $h[n]$ must be absolutely summable

$$|e^{j\omega n}| = 1$$
$$|z^n| \qquad z = 1$$

$$\infty > M_h > \sum_{n=-\infty}^{\infty} |h[n]| \qquad \text{asolutely summable}$$

$$= \sum_{n=-\infty}^{+\infty} |h[n] e^{-j\omega n}|$$

$$\geq \left| \sum_{n=-\infty}^{+\infty} h[n] e^{-j\omega n} \right|$$

$$= \left| H(z) \big|_{z = e^{j\omega}} \right|$$

$$\infty > \left| H(z) \big|_{z = e^{j\omega}} \right|$$

a stable system,
$H(z)$ must converge on the unit circle $|z| = 1$
ROC (Region of Convergence) must include the unit circle

regardless of causality of $h[n]$

$$H(z)\big|_{|z|=1} = H(e^{j\hat{\omega}}) \qquad \text{DTFT of } h[n]$$

discrete	all stable sequence	must have	convergent DTFTs
continuous	all stable signal	must have	convergent CTFTs

$$C \leftarrow \text{unit circle} \qquad z = e^{j\hat{\omega}}$$

$$ZT^{-1} \qquad DTFT^{-1} \qquad \text{identical formulas}$$

$h[n]$ causal

$$H(z) = \sum_{n=-\infty}^{+\infty} h[n]\, z^{-n} = \sum_{n=0}^{+\infty} h[n]\, z^{-n} \quad n \in [0, \infty)$$

for finite values of n,

each term must be finite as long as $z \neq 0$

For the sum to converge,

$h[n]\, z^{-n}$ must vanish as $n \to \infty$

$|z| > r_h \qquad z_h = r_h e^{j\theta}$

z_h^n is the largest magnitude
geometrically increasing component

$n^m z_\ell^n$: the most general term
for impulse responses

$n \to \infty$ z_ℓ^n dominant over n^m for finite m

geometric components — as poles

$$\mathcal{Z}\{ z_l^n\, u[n]\} = \frac{1}{1 - \left(\frac{z_e}{z}\right)} = \frac{z}{z - z_e}$$

ROC of a causal sequence $h[n]$
     outside the radius of the largest magnitude pole of $H(z)$

ROC of a causal signal $h(t)$
     to the right of the rightmost pole of $H_c(s)$

if $h[n]$ is a stable, causal sequence,
     the unit circle must be   included in the ROC