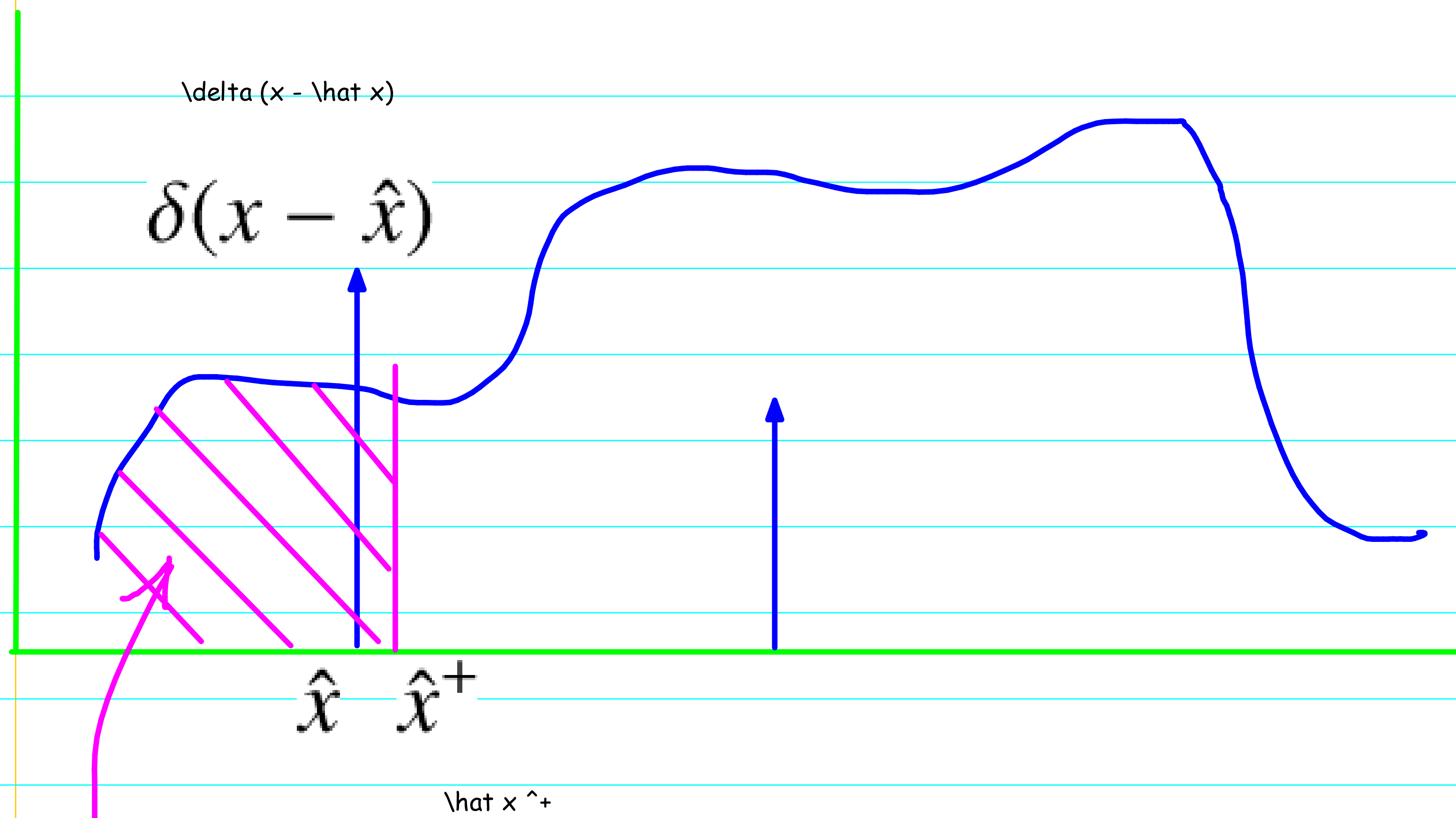
Thu Dec 1, 2011 8:50 AM



$$P(X < \hat{x}^+) = P(X \le \hat{x}^+) = \int_{-\infty}^{\hat{x}^+} f(y)dy$$

P(X < \hat x ^+) = P (X \le \hat x ^+) = \int_{-\infty}^{\hat x ^+} f (y) dy

$$P(X < \hat{x}^-) = P(X \le \hat{x}^-) = \int_{-\infty}^{\hat{x}^-} f(y)dy$$

P(X < \hat x ^-) = P (X \le \hat x ^-) = \int_{-\infty}^{\hat x ^-} f (y) dy

$$P(X < \hat{x}) \ne P(X \le \hat{x})\ ?$$

P(X < \hat x) \ne P (X \le \hat x) \ ?

$$P(X < \hat{x}) = \int_{-\infty}^{\hat{x}} f(y)dy := \alpha$$

P(X < \hat x) = \int_{-\infty}^{\hat x} f (y) dy := \alpha

$$P(X \le \hat{x}) = \int_{-\infty}^{\hat{x}} f(y)dy := \beta$$

\le \hat x) = \int_{-\infty}^{\hat x} f (y) dy := \beta

so if alpha is different from beta then the two integrals are different; what does that mean ? In other words, how do you define the integration when the upper bound of the integral sits right at the Dirac delta, i.e., x hat.

Note that the integral of a Dirac delta is not defined with the upper bound that stops at the Dirac delta; it is defined as follows:

$$\int_{-\infty}^{+\infty} \delta(x - \hat{x})dx = 1 = \int_{\hat{x}^-}^{\hat{x}^+} \delta(x - \hat{x})dx$$

\int_{-\infty}^{+\infty} \delta (x - \hat x) dx = 1 = \int_{\hat x^-}^{\hat x ^+} \delta (x - \hat x) dx

and another property of the Dirac delta is (part of its definition):

$$\delta(x - \hat{x}) = 0 \text{ for } x \ne \hat{x}$$

\delta(x - \hat x) = 0 \text{ for } x \ne \hat x

and $\delta(0)$ is not defined !

Is it possible to define the integration when the upper limit is right on the Dirac delta as follows ?

$$\int_{-\infty}^{\hat{x}} \delta(x - \hat{x})dx = \frac{1}{2}$$

\displaystyle \int_{-\infty}^{\hat x} \delta (x - \hat x) dx = \frac{1}{2}

Did Durrett 2010 and Balakrishnan & Nevzorov 2003 define the integration that they use ? How did they deal with the Dirac delta ?

Did they start with the axiom:

$$P(X < \hat{x}) \neq P(X \leq \hat{x}) \ ?$$

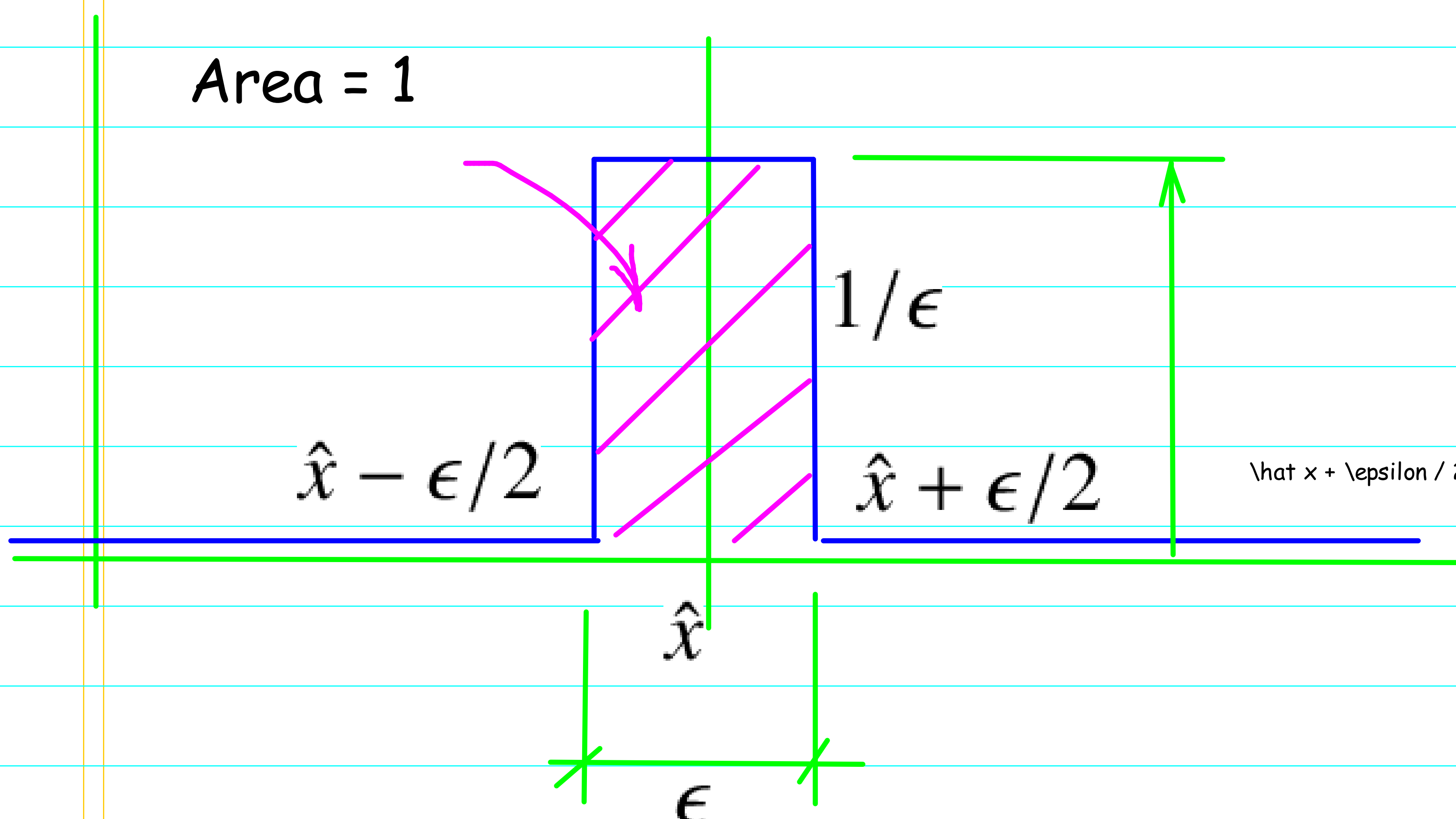
P(X < \hat x) \ne P (X \le \hat x) \ ?

and not relate the probability P to the integration of a pdf ? That would be unlikely ! But the concept of density (or measure) is important in probability ! So it is not possible to talk about probability without talking about pdf.

A possible "engineering" justification of

$$\int_{-\infty}^{\hat{x}} \delta(x - \hat{x})dx = \frac{1}{2}$$

\displaystyle \int_{-\infty}^{\hat x} \delta (x - \hat x) dx = \frac{1}{2}



Think of the Dirac delta as the limit of the above window function when epsilon goes to zero. But mathematically, the value of the Dirac delta is not define at x hat; besides, what does it mean to have infinite probability density at x hat ?

Another way to look at the issue is to reproduce Kolmogorov 1956 p.23 by replacing the < sign by the <= sign, and see whether there is any inconsistencies with the argument of Kolmo. Note that Kolmo also talked about the pdf. So he must have defined the probability with an integration (actually he was the first to use the Lebesgue integration in probability). The difference of the Lebesgue integration and the Riemannian integration (more frequently used in regular engineering as area under the curve, or think of the trapezoidal rule) is more advanced.

Check whether Kolmo talked about left continuity. Yes.

Clearly, Kolmo avoid the integration with upper bound right on x hat, since he used the < sign.

In that case, did he define the probability at x hat ? Could it be defined as follows ?

$$P(X = \hat{x}) := \lim_{\epsilon \downarrow 0} [P(X < \hat{x} + \epsilon) - P(X < \hat{x} - \epsilon)]$$

P(X = \hat x) := \lim_{\epsilon \downarrow 0} [P (X < \hat x + \epsilon) - P (X < \hat x - \epsilon)] but then due to the definition of the Dirac delta, it is clear that:

$$P(X = \hat{x}) = \int_{\hat{x}^-}^{\hat{x}^+} \delta(x - \hat{x})dx = 1$$

P(X = \hat x) = \int_{\hat x^-}^{\hat x ^+} \delta (x - \hat x) dx = 1

Clearly, the above does not make sense; you have the probability of a random variable X equal to an exact value being 1. How could you play "lottery", i.e., generate a random number X, and always expect X to come out as x hat !! ??

Durrett 2010 p.10 defined the Dirac delta in the same way as I did above, i.e., no value of the Dirac delta at x hat.