

pb2test3.mw

```
> restart:with(DEtools):with(plots):assume(n,integer):
```

Problem :  $u_{xx} = u_{tt} + 10$  and  $u(0, t) = 0$  and  $u(1, t) = 0$  , ICs  $u(x, 0) = 0$  and  $u_t(x, 0) = x(1 - x)$

By using  $u(x, t) = w(x, t) + v(x)$  we can break the problem up into two problems.

$w_{xx} + v_{xx} = w_{tt} + 10$  with  $w(x, 0) + v(x) = 0$  ,  $w_t(x, 0) = x(1 - x)$  ,  $w(0, t) + v(0) = 0$  ,  
 $w(1, t) + v(1) = 0$

Problem 1 :  $v_{xx} = 10$   $v(0) = 0$   $v(1) = 0$  gives the solution  $v(x) = 5 \cdot x^2 - 5 \cdot x$

```
> v:=x->5*x^2-5*x;
```

$$v := x \rightarrow 5x^2 - 5x \quad (1)$$

Problem 2: Find the solution to  $w_{xx} = w_{tt}$  with  $w(x, 0) = -v(x)$  ,  $w_t(x, 0) = x(1 - x)$  ,  $w(0, t) = 0$  ,  
 $w(1, t) = 0$

$w(x, t) = X(x) \cdot T(t)$  substitute into PDE  $\frac{X''}{X} = \frac{T''}{T} = \text{constant}$

Case I:  $\text{constant} > 0$   $X(x) = 0$  trivial solution.

Case II:  $\text{constant} = 0$  again  $X(x) = 0$  trivial solution

Case III:  $\text{constant} < 0$

$X'' - \lambda^2 X = 0$  leads to  $X(x) = c_1 \cos(\lambda \cdot x) + c_2 \cdot \sin(\lambda \cdot x)$  using the BCs,  $X(0) = 0$  and  $X(1) = 0$  the equation becomes  $X(x) = 0 = \sin(\lambda)$  from which we deduce that  $\lambda_n = n \cdot \pi$

```
> lambda:=n*Pi;
```

$$\lambda := n \sim \pi \quad (2)$$

```
> X:=(n,x)->sin(lambda*x);
```

$$X := (n, x) \rightarrow \sin(\lambda x) \quad (3)$$

$$T_n(t) = a_n \cdot \cos(\lambda_n \cdot t) + b_n \cdot \sin(\lambda_n \cdot t)$$

```
> T:=(n,t)->(a(n)*cos(lambda*t)+b(n)*sin(lambda*t));
```

$$T := (n, t) \rightarrow a(n) \cos(\lambda t) + b(n) \sin(\lambda t) \quad (4)$$

$$w(x, t) = \sum_{n=1}^{\infty} \sin(\lambda_n \cdot x) \cdot (a_n \cdot \cos(\lambda_n \cdot t) + b_n \cdot \sin(\lambda_n \cdot t))$$

use the initial conditions to find the coefficients  $w(x, 0) = -v(x)$

$$-5 \cdot x^2 + 5 \cdot x = \sum_{n=1}^{\infty} \sin(\lambda_n \cdot x) \cdot a_n$$

$$a_n = \frac{\int_0^1 (-5 \cdot x^2 + 5 \cdot x) \cdot \sin(\lambda_n \cdot x) \, dx}{\int_0^1 \sin^2(\lambda_n \cdot x) \, dx}$$

>

> `a:=n->int(-v(x)*sin(lambda*x),x=0..1)/(int(sin(lambda*x)^2,x=0..1));`

$$a := n \rightarrow \frac{\int_0^1 (-v(x) \sin(\lambda x)) \, dx}{\int_0^1 \sin(\lambda x)^2 \, dx} \quad (5)$$

> `a(1);`

$$-\frac{20(-1 + (-1)^n)}{n^3 \pi^3} \quad (6)$$

The  $b_n$  coefficient

$$w_t(x, t) = \sum_{n=1}^{\infty} \sin(\lambda_n \cdot x) \cdot (-a_n \cdot \lambda_n \cdot \sin(\lambda_n \cdot t) + b_n \cdot \lambda_n \cdot \cos(\lambda_n \cdot t))$$

use the initial conditions to find the coefficients  $w(x, 0) = x \cdot (1 - x)$

$$-5 \cdot x^2 + 5 \cdot x = \sum_{n=1}^{\infty} \sin(\lambda_n \cdot x) \cdot b_n \cdot \lambda_n$$

$$b_n = \frac{1}{\lambda_n} \frac{\int_0^1 (x \cdot (1 - x)) \cdot \sin(\lambda_n \cdot x) \, dx}{\int_0^1 \sin^2(\lambda_n \cdot x) \, dx}$$

> `g := x -> x * (1 - x)`

`g := x -> x (1 - x)` (7)

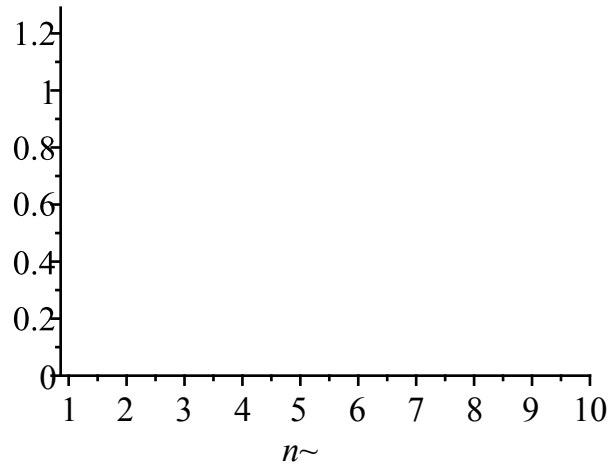
> `b:=n->int(g(x)*sin(lambda*x),x=0..1)/(lambda*int(sin(lambda*x)^2,x=0..1));`

$$b := n \rightarrow \frac{\int_0^1 g(x) \sin(\lambda x) dx}{\lambda \left( \int_0^1 \sin(\lambda x)^2 dx \right)} \quad (8)$$

```
> b(1);
```

$$-\frac{4(-1 + (-1)^{n\sim})}{n\sim^4 \pi^4} \quad (9)$$

```
> plot([a(n),b(n)],n=1..10);
```



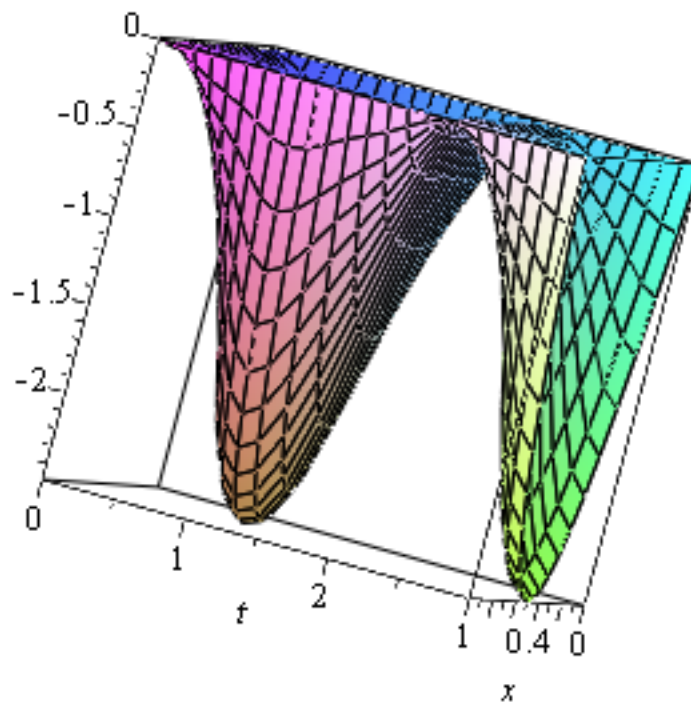
```
> u:=(x,t)->v(x)+sum(X(n,x)*T(n,t),n=1..4);
```

$$u := (x, t) \rightarrow v(x) + \sum_{n=1}^4 X(n, x) T(n, t)$$

(10)

```
> plot3d(u(x,t),x=0..1,t=0..3,axes=box,title="Constant applied force");
```

Constant applied force



```
> animate(u(x,t),x=0..1,t=0..5,frames=200);
```

