pb2test3.mw

```
> restart:with(DEtools):with(plots):assume(n,integer):
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Problem : $u_{xx} = u_{tt} + 10$ and $u(0, t) = 0$ and $u(1, t) = 0$, ICs $u(x, 0) = 0$ and $u_t(x, 0) = x(1 - x)$

By using $u(x, t) = w(x, t) + v(x)$ we can break the problem up into two problems.

$w_{xx} + v_{xx} = w_{tt} + 10$ with $w(x, 0) + v(x) = 0$, $w_t(x, 0) = x(1 - x)$, $w(0, t) + v(0) = 0$, $w(1, t) + v(1) = 0$

Problem 1 : $v_{xx} = 10$ $v(0) = 0$ $v(1) = 0$ gives the solution $v(x) = 5 \cdot x^2 - 5 \cdot x$

```
> v:=x->5*x^2-5*x;
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$$v := x \rightarrow 5\,x^2 - 5\,x \tag{1}$$

Problem 2: Find the solution to $w_{xx} = w_{tt}$ with $w(x, 0) = -v(x)$, $w_t(x, 0) = x(1 - x)$, $w(0, t) = 0$, $w(1, t) = 0$

$w(x, t) = X(x) \cdot T(t)$ substitute into PDE $\frac{X''}{X} = \frac{T''}{T} = constant$

Case I: constant > 0 $X(x) = 0$ trivial solution.

Case II: $constant = 0$ again $X(x) = 0$ trivial solution

Case III: $constant < 0$

$X'' - \lambda^2 X = 0$ leads to $X(x) = c_1 \cos(\lambda \cdot x) + c_2 \cdot \sin(\lambda \cdot x)$ using the BCs, $X(0) = 0$ and $X(1) = 0$ the equation becomes $X(x) = 0 = \sin(\lambda)$ from which we deduce that $\lambda_n = n \cdot \pi$

```
> lambda:=n*Pi;
```

$$\lambda := n{\sim}\,\pi \tag{2}$$

```
> X:=(n,x)->sin(lambda*x);
```

$$X := (n, x) \rightarrow \sin(\lambda\,x) \tag{3}$$

Next find to the solution for $\frac{T''}{T} = -\lambda^2$, using the lambda from above the solution is another sine cosine pair.

$T_n(t) = a_n \cdot \cos(\lambda_n \cdot t) + b_n \cdot \sin(\lambda_n \cdot t)$

```
> T:=(n,t)->(a(n)*cos(lambda*t)+b(n)*sin(lambda*t));
```

$$T := (n, t) \to a(n) \cos(\lambda t) + b(n) \sin(\lambda t) \quad (4)$$

Each product $w_n(x, t) = X_n(x) \cdot T_n(t)$ is a solution of the pde $w_{xx} = w_{tt}$ $w(0, t) = 0$, $w(1, t) = 0$ a sum of these products is also a solution. Using the Fourier Series approach the solution is presented as.

$$w(x, t) = \sum_{n=1}^{\infty} \sin(\lambda_n \cdot x) \cdot (a_n \cdot \cos(\lambda_n \cdot t) + b_n \cdot \sin(\lambda_n \cdot t))$$

The coefficients a_n and b_n are found by using the initial conditions for the homogeneous problem $w_{xx} = w_{tt}$. Use the initial condition $w(x, 0) = -v(x)$ to find a_n. The process is to set $t = 0$ and then $w(x, 0) = f(x) - v(x)$, in this problem $f(x) = 0$. Then each side is multiplied by the eigenfunction $X_n(x)$ and integrated of the length of the interval. Using orthogonality the resulting equation will allow us to solve for a_n as shown below.

$$w(x, 0) = f(x) - v(x) = -5 \cdot x^2 + 5 \cdot x = \sum_{n=1}^{\infty} \sin(\lambda_n \cdot x) \cdot a_n$$

$$a_n = \frac{\int_0^1 (-5 \cdot x^2 + 5 \cdot x) \cdot \sin(\lambda_n \cdot x) \, dx}{\int_0^1 \sin^2(\lambda_n \cdot x) \, dx}$$

```
>
> a:=n->int(-v(x)*sin(lambda*x),x=0..1)/(int(sin(lambda*x)^2,x=0..1));
```

$$a := n \to \frac{\int_0^1 (-v(x) \sin(\lambda x)) \, dx}{\int_0^1 \sin(\lambda x)^2 \, dx} \quad (5)$$

```
> a(1);
```

$$-\frac{20 (-1 + (-1)^{n\sim})}{n\sim^3 \pi^3} \quad (6)$$

The b_n coefficients are found in the same manner as the a_n except the second boundary condition is used.

$$w_t(x, t) = \sum_{n=1}^{\infty} \sin(\lambda_n \cdot x) \cdot (-a_n \cdot \lambda_n \cdot \sin(\lambda_n \cdot t) + b_n \cdot \lambda_n \cdot \cos(\lambda_n \cdot t))$$

use the initial conditions to find the coefficients $w(x, 0) = x \cdot (1 - x)$

$$w_t(x,0) = g(x) = x\cdot(1-x) = \sum_{n=1}^{\infty} \sin\left(\lambda_n \cdot x\right)\cdot b_n \cdot \lambda_n$$

$$b_n = \frac{1}{\lambda_n}\,\frac{\int_0^1 (x\cdot(1-x))\cdot\sin\left(\lambda_n\cdot x\right)\,\mathrm{d}x}{\int_0^1 \sin^2\left(\lambda_n\cdot x\right)\,\mathrm{d}x}$$

> $g := x \rightarrow x\cdot(1-x)$

$$g := x \rightarrow x\,(1-x) \tag{7}$$

```
> b:=n->int(g(x)*sin(lambda*x),x=0..1)/(lambda*int(sin(lambda*x)
^2,x=0..1));
```

$$b := n \rightarrow \frac{\int_0^1 g(x)\,\sin(\lambda\,x)\,\mathrm{d}x}{\lambda\left(\int_0^1 \sin(\lambda\,x)^2\,\mathrm{d}x\right)} \tag{8}$$

```
> b(1);
```

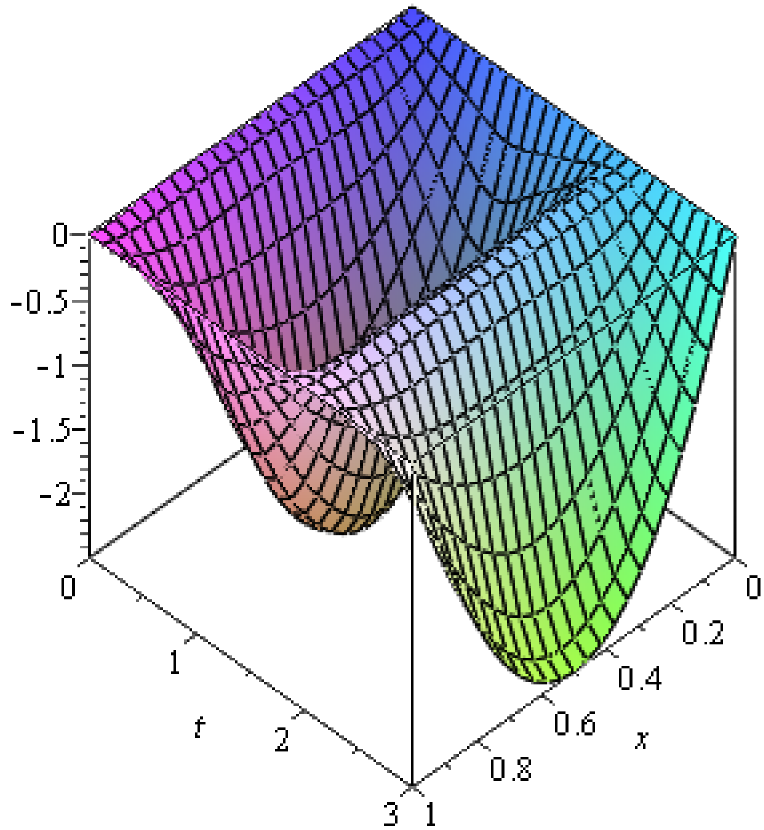
$$-\frac{4\left(-1+(-1)^{n\sim}\right)}{n\sim^4\,\pi^4} \tag{9}$$

```
> u:=(x,t)->v(x)+sum(X(n,x)*T(n,t),n=1..4);
```

$$u := (x,t) \rightarrow v(x) + \sum_{n=1}^{4} X(n,x)\,T(n,t) \tag{10}$$

```
> plot3d(u(x,t),x=0..1,t=0..3,axes=box,title="Constant applied
force");
```

Constant applied force




```
> animate(u(x,t),x=0..1,t=0..5,frames=200);
```