# Logic Background (1B)

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10/8/19

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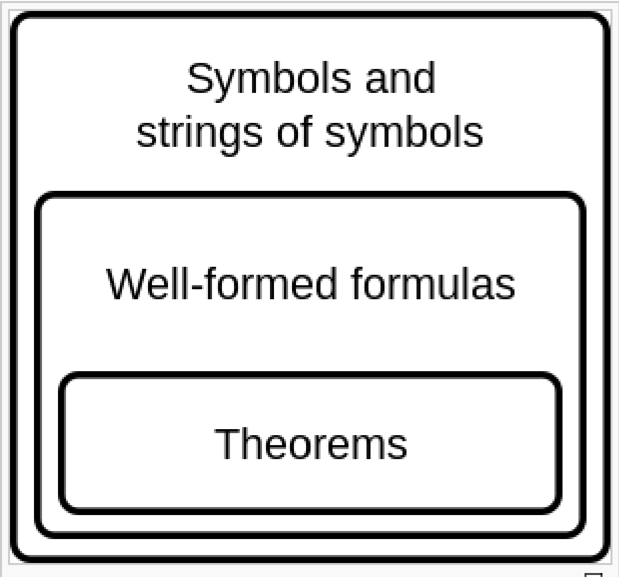
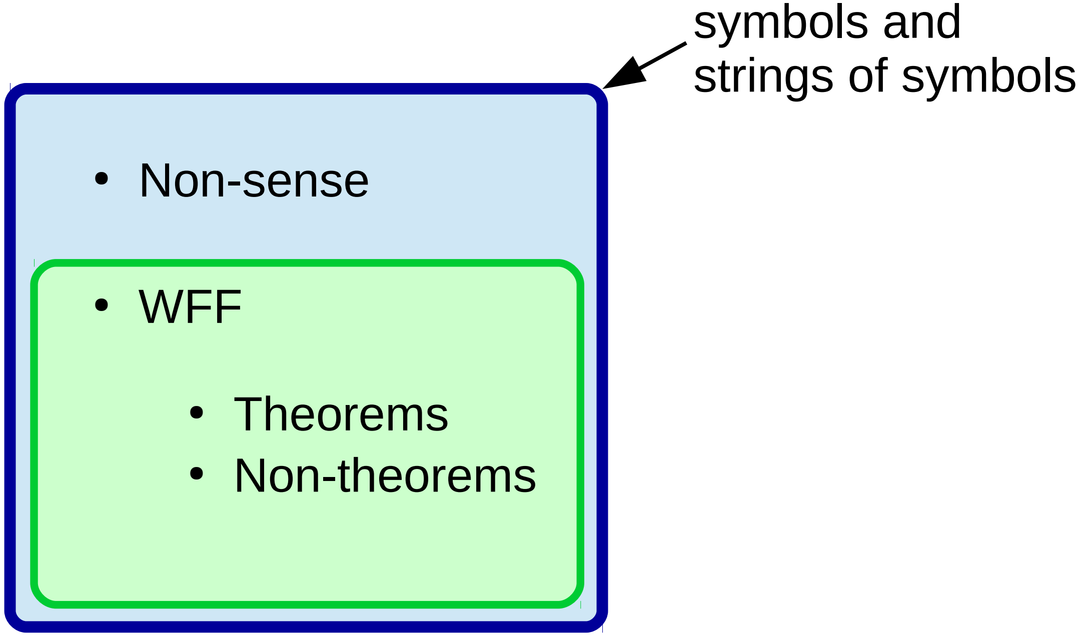
# Symbols and Formal Language




This diagram shows the syntactic entities that may be constructed from formal languages. The symbols and strings of symbols may be broadly divided into nonsense and well-formed formulas. A formal language can be thought of as identical to the set of its well-formed formulas. The set of well-formed formulas may be broadly divided into theorems and non-theorems.




http://en.wikipedia.org/wiki/

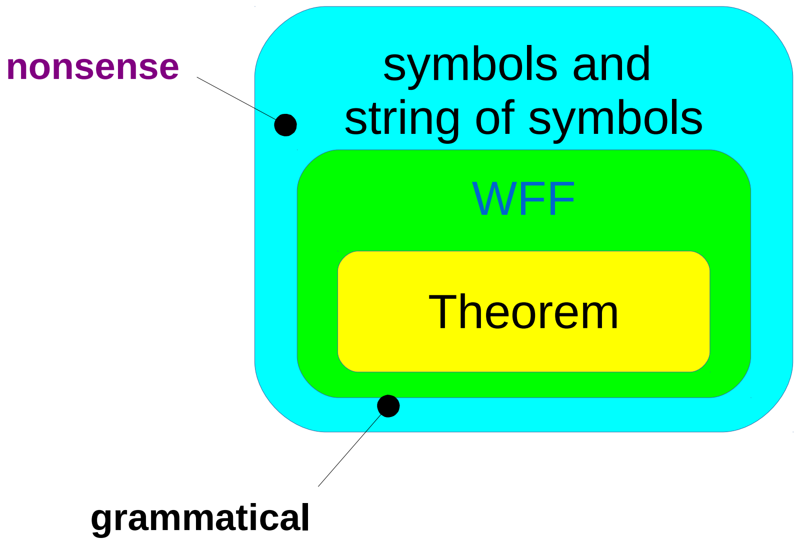
# Syntactic entities from formal languages

the **syntactic entities**
constructed from **formal languages**.

The **symbols** and **strings of symbols**
- **nonsense**
- **well-formed formulas**.

A **formal language**
the **set of its well-formed formulas**.

**The set of well-formed formulas**
- **theorems**
- **non-theorems**.




http://en.wikipedia.org/wiki/

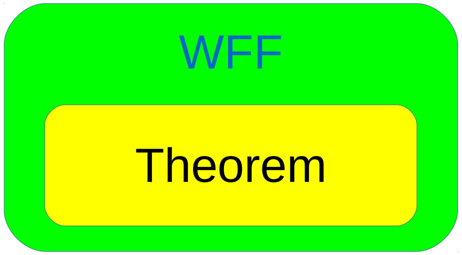
# Well-formedness

**Well-formedness**
the quality of a clause, word, or other linguistic element that conforms to the **grammar** of the **language** of which it is a part.

**Well-formed words** or **phrases** are **grammatical**, meaning they obey all relevant rules of **grammar**.

a form that violates some **grammar rule** is **ill-formed** and does not constitute part of the language.

**WFF** is a **word**
a finite sequence of symbols from a given alphabet which is part of a **formal language**.




**grammatical**

http://en.wikipedia.org/wiki/

# Theorem

In mathematics, a **theorem** is
a statement that has been proven on

- other **theorems**
  previously established statements
- **axioms**
  generally accepted statements

A **theorem** is a
**logical consequence** of the **axioms**.

Theorem

**proofs**

sequences of **formulas**
with certain **properties**

http://en.wikipedia.org/wiki/

# Proof

The **proof** of a **mathematical theorem** is
a **logical argument**
for the **theorem statement** given
in accord with the **rules**
of a **deductive system**.

the **proof** of a **theorem**
is often interpreted as
**justification** of the **truth**
of the **theorem** statement.

In **formal logic**, **proofs** can be represented
by sequences of **formulas** with certain **properties**,
and the **final** **formula** in the sequence is what is proven.

Theorem

**proofs**

sequences of **formulas**
with certain **properties**

http://en.wikipedia.org/wiki/

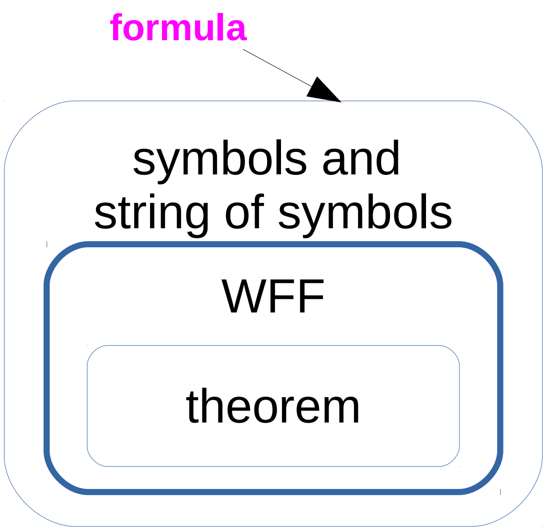
# Formula

A formula is a **syntactic formal object**
that can be informally given a **semantic meaning**.

a formula is **a string of symbols φ**
for which it makes sense to ask "**is φ true?**",
once any free variables in φ have been **instantiated**.

A key use of formula is
- **propositional logic**
- **predicate logics** such as **first-order logic**.

A **formal language can be considered to be identical to the set** containing all and only its formula**.**




**formal logic**

**grammatical**

http://en.wikipedia.org/wiki/

# Predicate

**predicate** (*plural* **predicates**)

1. (*grammar*) The part of the sentence (or clause) which states something about the subject or the object of the sentence. [quotations ▼]

   *In "The dog barked very loudly", the subject is "the dog" and the* ***predicate*** *is "barked very loudly".*

2. (*logic*) A term of a statement, where the statement may be true or false depending on whether the thing referred to by the values of the statement's variables has the property signified by that (predicative) term. [quotations ▼]

   *A nullary* ***predicate*** *is a proposition. Also, an instance of a* ***predicate*** *whose terms are all constant — e.g., P(2,3) — acts as a proposition.*

   *A* ***predicate*** *can be thought of as either a relation (between elements of the domain of discourse) or as a truth-valued function (of said elements).*

   *A* ***predicate*** *is either valid, satisfiable, or unsatisfiable.*

   *There are two ways of binding a* ***predicate****'s variables: one is to assign constant values to those variables, the other is to quantify over those variables (using universal or existential quantifiers). If all of a* ***predicate****'s variables are bound, the resulting formula is a proposition.*

3. (*computing*) An operator or function that returns either true or false.

http://en.wikipedia.org/wiki/

# Predicate in mathematics

In mathematics, a **predicate** is commonly understood to be a Boolean-valued function $P: X \to \{\text{true}, \text{false}\}$, called the predicate on $X$. However, predicates have many different uses and interpretations in mathematics and logic, and their precise definition, meaning and use will vary from theory to theory. So, for example, when a theory defines the concept of a relation, then a predicate is simply the characteristic function or the indicator function of a relation. However, not all theories have relations, or are founded on set theory, and so one must be careful with the proper definition and semantic interpretation of a predicate.

http://en.wikipedia.org/wiki/

# First-order Logic

**First-order logic** (**predicate logic**, **first-order predicate calculus**)

a collection of **formal systems** used in mathematics,
philosophy, linguistics, and computer science.

First-order logic uses **quantified variables** over non-logical objects and
allows the use of **sentences** that contain **variables**

unlike propositions such as Socrates is a man one
can have expressions in the form
"there exists X such that X is Socrates and X is a man"
and there exists is a **quantifier** while X is a **variable**.

This distinguishes it from propositional logic,
which does not use **quantifiers** or **relations**;
propositional logic is the foundation of first-order logic.

http://en.wikipedia.org/wiki/

# Propositional logic

**Propositional logic**

- consists of a set of atomic propositional symbols
- e.g. Socrates, Father, etc
- often referred to by letters p, q, r etc.
- these letters are not variables
- propositional logic has no means of binding variables.
- these symbols are joined together by logical operators (or connectives) to form sentences.
- can only talk about specifics
- e.g. "Socrates is a man"

https://www.quora.com/What-is-the-difference-between-predicate-logic-first-order-logic-second-order-logic-and-higher-order-logic

# 1st-order logic

**First-order Predicate Logic**

- is an extension of propositional logic
- allows **quantification** over variables.
- can also talk more generally
- e.g. "all men are mortal"
- **variables** to range over atomic symbols in the domain.
- doesn't allow variables to be bound to predicate symbols

https://www.quora.com/What-is-the-difference-between-predicate-logic-first-order-logic-second-order-logic-and-higher-order-logic

# 2nd-order logic

A **second order logic** (such as second order predicate logic)

- allow variables to be bound to predicate symbols
- can write sentences such as: $\forall p.p(Socrates)$.

A **higher order logic** allows

- predicates to accept arguments which are themselves predicates.
- **Second order logic** cannot be reduced to **first-order logic**.

https://www.quora.com/What-is-the-difference-between-predicate-logic-first-order-logic-second-order-logic-and-higher-order-logic

# First-Order Logic (1)

The definition of a **formula** comes in several parts.

1. the set of **terms** is defined recursively.
**terms**, informally, are **expressions**
that represent **objects** from the domain of discourse.

Any **variable** is a **term**.
Any **constant** **symbol** from the signature is a **term**
an expression of the form $\mathbf{f(t_1,\ldots,t_n)}$,
where **f** is an **n-ary function** symbol, and $t_1,\ldots,t_n$ are terms, is again a **term**.

2. the **atomic formulas**.

If $t_1$ and $t_2$ are **terms** then $t_1 = t_2$ is an **atomic formula**
If **R** is an **n-ary relation** symbol,
and $t_1,\ldots,t_n$ are **terms**, then $\mathbf{R(t_1,\ldots,t_n)}$ is an **atomic formula**

**formula**

1. the set of **terms**
   a **variable**
   a **constant**
   $\mathbf{f(t_1,\ldots,t_n)}$,

2. the **atomic formulas**.
   $t_1 = t_2$
   $\mathbf{R(t_1,\ldots,t_n)}$

3. the set of **formulas**
   $\neg \phi$
   $( \phi \wedge \psi ), ( \phi \vee \psi )$
   $\exists x \phi$
   $\forall x \phi$

https://en.wikipedia.org/wiki/Well-formed_formula

# First-Order Logic (2)

3. the set of **formulas** is defined to be the smallest set containing the set of **atomic formulas** such that the following holds:

¬ ϕ is a **formula** when ϕ is a **formula**
( ϕ ∧ ψ ) and ( ϕ ∨ ψ ) are **formulas** when ϕ and ψ are **formulas**;
∃ x ϕ is a **formula** when x is a **variable** and ϕ is a **formula**;
∀ x ϕ is a **formula** when x is a **variable** and ϕ is a **formula**
(alternatively, ∀x ϕ could be defined as an abbreviation for ¬∃x ¬ϕ).

If a formula has no occurrences of ∃ x or ∀ x , for any variable x, then it is called **quantifier-free**.

An **existential formula** is a formula starting with a sequence of existential quantification followed by a quantifier-free formula.

**formula**

1. the set of **terms**
   a **variable**
   a **constant**
   $\mathbf{f(t_1,...,t_n)}$,

2. the **atomic formulas**.
   $\mathbf{t_1 = t_2}$
   $\mathbf{R(t_1,...,t_n)}$

3. the set of **formulas**
   ¬ ϕ
   ( ϕ ∧ ψ ), ( ϕ ∨ ψ )
   ∃ x ϕ
   ∀ x ϕ

https://en.wikipedia.org/wiki/Well-formed_formula

# Atomic sentences

A **sentence** is usually defined as
a **formula** without **free variables**.

An **atomic formula** is
a **formula** without **connectives**.

examples)

an **atomic formula** is P(x)
where x is a certain individual variable.
an **atomic sentence** is P(c)
where c is a certain predicate constant.

| | **atomic** | |
|---|---|---|
| | P(x) | P(x)∧Q(x) |
| **sentence** | P(c) | P(c)∧Q(c) |

**formula** :
P(x), P(x)∧Q(x), P(c), P(c)∧Q(c)

**sentence** :
P(c), P(c)∧Q(c)

**atomic formula** :
P(x), P(c)

**atomic sentence** :
P(c)

https://www.quora.com/What-is-the-difference-between-an-atomic-sentence-and-an-atomic-formula-in-first-order-logic

# Model and evaluation

There is a problem with **formuals** containing **free variables**:

to know whether they are **true**,
we need not only a **model**
(i.e. some **interpretation** of **predicate** and **functional constants**)
but also **evaluate** these variables.

this means that many such **formulas** are contingent
upon their **free variables** which can be undesirable.

https://www.quora.com/What-is-the-difference-between-an-atomic-sentence-and-an-atomic-formula-in-first-order-logic

# Semantic interpretation of an atomic formula

The precise semantic interpretation of an atomic formula and an atomic sentence will vary from theory to theory.

- In propositional logic, atomic formulas are called propositional variables.[3] In a sense, these are nullary (i.e. 0-arity) predicates.
- In first-order logic, an atomic formula consists of a predicate symbol applied to an appropriate number of terms.

an **atomic formula** is P(x)

where x is a certain individual variable.

an **atomic sentence** is P(c)

where c is a certain predicate constant.

any **variable** is a **term**.
any **constant** is a **term**
an **n-ary function** expression $\mathbf{f(t_1,...,t_n)}$ is a **term**
where $\mathbf{t_1},...,\mathbf{t_n}$ are terms

http://en.wikipedia.org/wiki/

# Formal Language Interpretation

A formal language consists of a fixed collection of *sentences* (also called *words* or *formulas*, depending on the context) composed from a fixed set of *letters* or *symbols*. The inventory from which these letters are taken is called the *alphabet* over which the language is defined. To distinguish the strings of symbols that are in a formal language from arbitrary strings of symbols, the former are sometimes called *well-formed formulæ* (wff). The essential feature of a formal language is that its syntax can be defined without reference to interpretation. For example, we can determine that (*P* or *Q*) is a well-formed formula even without knowing whether it is true or false.

**Example** [ edit ]

A formal language $\mathcal{W}$ can be defined with the alphabet α = { △, □ }, and with a word being in $\mathcal{W}$ if it begins with △ and is composed solely of the symbols △ and □.

A possible interpretation of $\mathcal{W}$ could assign the decimal digit '1' to △ and '0' to □. Then △□△ would denote 101 under this interpretation of $\mathcal{W}$.

Alphabet

Letters / Symbols

Sentences / Formulas

Well Formed Formula

Syntax without interpretation

http://en.wikipedia.org/wiki/

# Formal Language Expressions

The formal language used to create **expressions** consists of symbols

**Symbols**

- **constants**
  - **logical symbols**
  - **non-logical symbols**
- **variables**

https://en.wikipedia.org/wiki/Well-formed_formula

# Logical Constants

| | |
|---|---|
| T | true |
| F | false |
| ¬ | not |
| ∧ | and |
| ∨ | or |
| → | implies |
| ∀ | for all |
| ∃ | there exists |
| = | equals |

https://en.wikipedia.org/wiki/Well-formed_formula

# Non-logical Symbols

In case of a language of **first-order logic**
the **non-logical symbols**
- **predicates**
- **individual constants**

in an **interpretation**, symbols that may stand for
- **predicates**
- **individual constants**
- **variables**
- **functions**

the **logical symbols**
- **logical connectives**
- **quantifiers**
- **variables** that stand for **statements**

https://en.wikipedia.org/wiki/Well-formed_formula

# Non-logical Symbols

A **non-logical symbol** only has meaning or semantic content when one is assigned to it by means of an **interpretation**

A **sentence** containing a **non-logical symbol** lacks meaning except under an **interpretation**

A **sentence** is said to be **true** or **false** under an **interpretation**

The **logical constants** have the same meaning in all **interpretations**

https://en.wikipedia.org/wiki/Well-formed_formula

# Symbols

A **logical symbol** is a fundamental concept in logic, tokens of which may be marks or a configuration of marks which form a particular pattern.[citation needed] Although the term "symbol" in common use refers at some times to the idea being symbolized, and at other times to the marks on a piece of paper or chalkboard which are being used to express that idea; in the formal languages studied in mathematics and logic, the term "symbol" refers to the idea, and the marks are considered to be a token instance of the symbol. [dubious – discuss] In logic, symbols build literal utility to illustrate ideas.

Symbols of a formal language need not be symbols *of anything*. For instance there are logical constants which do not refer to any idea, but rather serve as a form of punctuation in the language (e.g. parentheses). Symbols of a formal language must be capable of being specified without any reference to any interpretation of them.

A symbol or string of symbols may comprise a well-formed formula if it is consistent with the formation rules of the language.

A formal symbol as used in first-order logic may be a variable (member from a universe of discourse), a constant, a function (mapping to another member of universe) or a predicate (mapping to T/F).

Formal symbols are usually thought of as purely syntactic structures, composed into larger structures using a formal grammar, though sometimes they may be associated with an interpretation or model (a formal semantics).

http://en.wikipedia.org/wiki/

# Interpretations for proposition logic

The formal language for propositional logic consists of formulas built up from propositional symbols (also called sentential symbols, sentential variables, and propositional variables) and logical connectives. The only non-logical symbols in a formal language for propositional logic are the propositional symbols, which are often denoted by capital letters. To make the formal language precise, a specific set of propositional symbols must be fixed.

The standard kind of interpretation in this setting is a function that maps each propositional symbol to one of the truth values true and false. This function is known as a *truth assignment* or *valuation* function. In many presentations, it is literally a truth value that is assigned, but some presentations assign truthbearers instead.

1. the set of **terms**
   a **variable**
   a **constant**
   $\mathbf{f(t_1,\ldots,t_n)}$,

2. the **atomic formulas**.
   $\mathbf{t_1 = t_2}$
   $\mathbf{R(t_1,\ldots,t_n)}$

3. the set of **formulas**
   $\neg\, \phi$
   $(\phi \wedge \psi)$, $(\phi \vee \psi)$
   $\exists\, x\, \phi$
   $\forall\, x\, \phi$

http://en.wikipedia.org/wiki/

# Interpretations for first-order logic

An example of interpretation $\mathcal{I}$ of the language **L** described above is as follows.

- Domain: A chess set
- Individual constants: a: The white King b: The black Queen c: The white King's pawn
- F(x): x is a piece
- G(x): x is a pawn
- H(x): x is black
- I(x): x is white
- J(x, y): x can capture y

In the interpretation $\mathcal{I}$ of L:

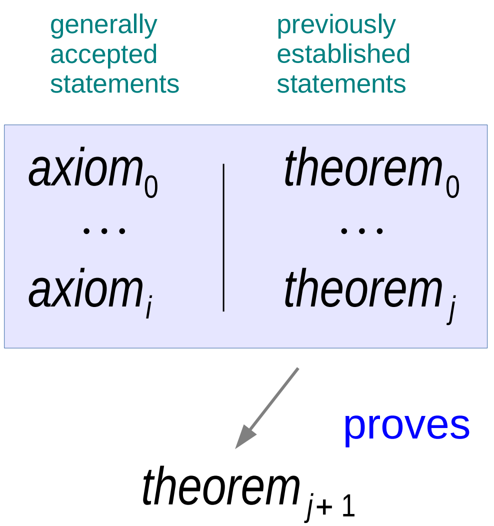
- the following are true sentences: F(a), G(c), H(b), I(a) J(b, c),
- the following are false sentences: J(a, c), G(a).

http://en.wikipedia.org/wiki/

# Formal System

A **formal system** is broadly defined as
any well-defined system of abstract thought
based on the model of mathematics.

In mathematics, a **theorem** is a statement that has been proven on the basis of previously established statements, such as other theorems, and generally accepted statements, such as axioms.




a **tautology** (from the Greek word ταυτολογία) is
a formula which is true in every possible interpretation.

An **axiom**, or postulate, is
a premise or starting point of reasoning.

As classically conceived,
an axiom is a premise so evident
as to be accepted as true without controversy.

http://en.wikipedia.org/wiki/

# Propositional Calculus and WFF

The WFF of **propositional logic**

(1) An **atomic proposition** is A is a **wff**

(2) If A and B, and C are **wffs**,
then so are $\neg$A, (A $\wedge$ B), (A $\vee$ B), (A $\rightarrow$ B), and (A $\leftrightarrow$ B).

(3) If A is a **wff**, then so is **(**A**)**.

http://en.wikipedia.org/wiki/

The WFF of **propositional logic**

(1) True and False are **wff**s.

(2) Each propositional **constant** (i.e. specific proposition), and
each propositional **variable** (i.e. a variable representing propositions) are **wff**s.

(3) Each **atomic formula** (i.e. a specific predicate with variables) is a **wff**.

(4) If A and B are **wff**s, then so are ¬A, (A ∧ B), (A ∨ B), (A → B), and (A ↔ B).

(5) If x is a variable (representing objects of the universe of discourse), and
A is a wff, then so are ∃**x** A and ∀x A .

http://en.wikipedia.org/wiki/

# First Order Logic and WFF

Not all **strings** can represent **propositions** of the predicate logic.
Those which produce a **proposition**
when their symbols are **interpreted**
must follow the rules given below,
and they are called **wffs** of the first order predicate logic.

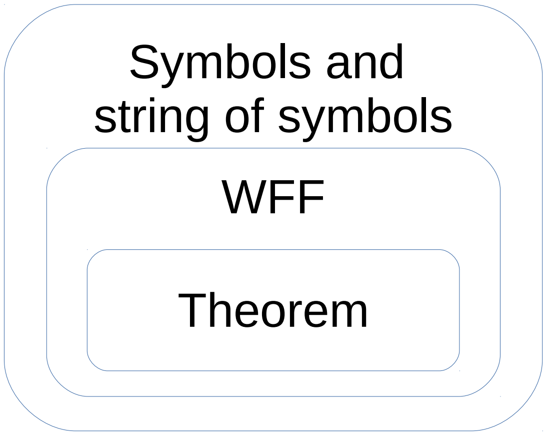
A predicate name followed by a list of variables such as P(x, y), where P is a predicate name, and x and y are variables, is called an atomic formula.

http://en.wikipedia.org/wiki/

Although the term "**formula**" may be used for **written marks** (for instance, on a piece of paper or chalkboard), it is more precisely understood **as the sequence** being expressed, with the **marks** being **a token instance of formula**.

It is **not necessary** for the existence of a formula that there be any actual **tokens** of it.

A **formal language** may thus have an infinite number of formulas regardless whether each formula has a **token instance**. Moreover, a single formula may have more than one **token instance**, if it is written more than once.




http://en.wikipedia.org/wiki/

# WFF and Interpretation

**Formulas** are quite often interpreted as **propositions** (as, for instance, in propositional logic).

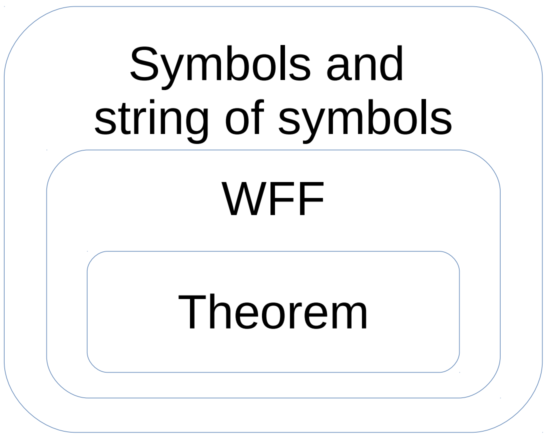
However formulas are **syntactic entities**, and as such must be specified in a formal language without regard to any interpretation of them.

An **interpreted formula** may be

- the name of something,
- an adjective,
- an adverb,
- a preposition,
- a phrase,
- a clause,
- an imperative sentence,
- a string of sentences,
- a string of names, etc.

A formula may even turn out to be **nonsense**, if the symbols of the language are specified so that it does.

Furthermore, a formula need not be given any interpretation.




http://en.wikipedia.org/wiki/

# Proposition

**proposition** (*countable and uncountable*, *plural* **propositions**)

1. (*uncountable*) The act of offering (an idea) for consideration.
2. (*c* Appendix:Glossary ea or a plan offered. [quotations ▼]
3. (*countable*, *business settings*) The terms of a transaction offered.
4. (*countable*, *US*, *politics*) In some states, a proposed statute or constitutional amendment to be voted on by the electorate.
5. (*countable*, *logic*) The content of an assertion that may be taken as being true or false and is considered abstractly without reference to the linguistic sentence that constitutes the assertion.
6. (*countable*, *mathematics*) An assertion so formulated that it can be considered true or false.
7. (*countable*, *mathematics*) An assertion which is provably true, but not important enough to be called a theorem.
8. A statement of religious doctrine; an article of faith; creed. [quotations ▼]

   *the **propositions** of Wyclif and Huss*

9. (*poetry*) The part of a poem in which the author states the subject or matter of it.

http://en.wikipedia.org/wiki/

# Syntax

**Symbols**

**Formal language**

**Formation rules**

**Propositions**

**Formal theories**

**Formal systems**

http://en.wikipedia.org/wiki/

# Symbols

A symbol is an idea, abstraction or concept, tokens of which may be marks or a configuration of marks which form a particular pattern.

Symbols of a formal language need not be symbols of anything.

For instance there are logical constants which do not refer to any idea, but rather serve as a form of punctuation in the language (e.g. parentheses).

A symbol or string of symbols may comprise a **well-formed formula** if the formulation is **consistent** with the formation **rules** of the language.

Symbols of a formal language must be capable of being specified without any reference to any interpretation of them.

http://en.wikipedia.org/wiki/

# Formal Language



A formal language is a **syntactic entity**
which consists of a set of finite strings of symbols
which are its **words** (usually called its well-formed formulas).

Which strings of symbols are words is determined
by fiat by the creator of the language,
usually by specifying a set of formation rules.

Such a language can be defined
without reference to any meanings of any of its expressions;

it can exist before any interpretation is assigned to it –
that is, before it has any meaning.

does not describe their semantics (i.e. what they mean).

http://en.wikipedia.org/wiki/

# Formation rules

Formation rules are a **precise description** of **which** strings of symbols are the well-formed formulas of a formal language.

It is synonymous with the set of strings over the alphabet of the formal language which constitute well formed formulas.

However, it does not describe their semantics (i.e. what they mean).

http://en.wikipedia.org/wiki/

# Propositions

A proposition is a **sentence**
expressing something **true** or **false**.

A proposition is identified ontologically
as an idea, concept or abstraction
whose token instances are
patterns of symbols, marks, sounds, or strings of words.

Propositions are considered to be
syntactic entities and
also truthbearers.

# Formal Theories



A formal theory is a **set of sentences** in a formal language.

# Formal Systems



A formal system (a **logical calculus**, a **logical system**)
consists of a **formal language**
together with a **deductive apparatus** (also called a deductive system).

The deductive apparatus may consist of
a set of **transformation rules** (also called inference rules)
or a set of **axioms**, or have both.

A formal system is used
to derive one expression
from one or more other expressions.

Formal systems, like other syntactic entities
may be defined without any interpretation given to it
(as being, for instance, a system of arithmetic).

**References**

[1] http://en.wikipedia.org/
[2]