Hybrid CORDIC 2.A Sine/Cosine Generator

20171007

The details moved to

https://en.wikiversity.org/wiki/Butterfly_Hardware_Implementations

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Wilson ROM based Sine/Cosine Generation

[24] Fu & Willson Sine / Cosine Generation

ROM-based

For high resolution, ROM size grows exponentially

quater-wave symmetry

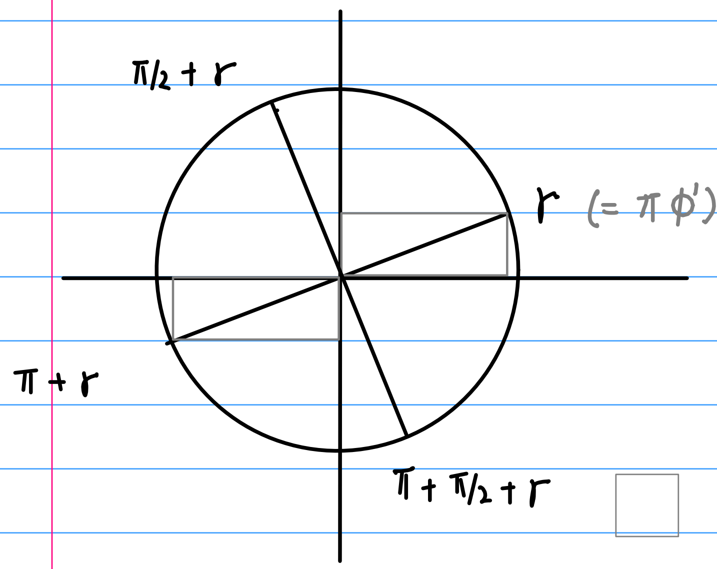
$$\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$$

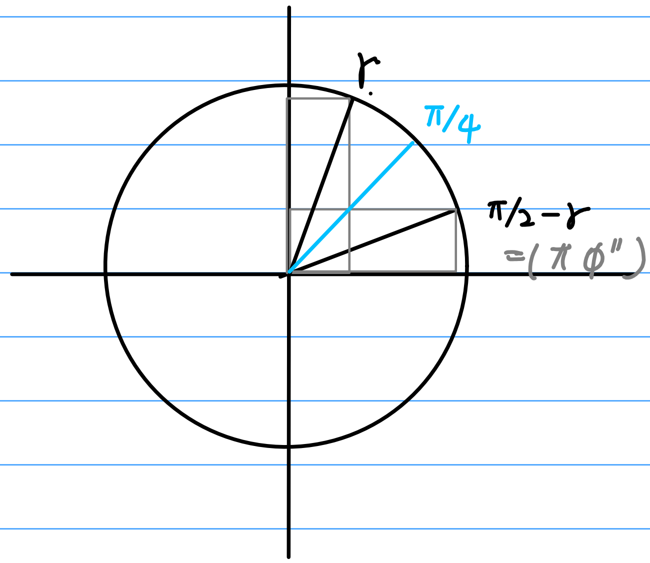
ϕ $[0, 2\pi] \longrightarrow [0, \frac{\pi}{4}]$

conditionally interchanging inputs X_0 & Y_0
Conditionally interchanging and negating outputs X & Y

$$X = X_0 \cos\phi - Y_0 \sin\phi$$
$$Y = Y_0 \cos\phi + X_0 \sin\phi$$

Madisetti VLSI arch





for frequency synthesis

argument: signed normalized by π angle $[-1, 1]$

binary representation of a radian angle required

$[-1, 1] \longrightarrow [0, \pi/4] \longrightarrow$ sine/cosine generator

ϕ

$\theta = \pi\phi$

① a phase accumulator ϕ $[-1, 1]$

② a radian converter $\phi \to \theta$

③ a sine/cosine generator $\sin\theta$, $\cos\theta$

④ an output stage $\sin\theta$, $\cos\theta$

$\downarrow$ $\downarrow$

$\sin\pi\phi$ $\cos\pi\phi$

.

output stage $\quad \sin\theta \rightarrow \sin\pi\phi \quad [-\pi, +\pi]$

$\cos\theta \rightarrow \cos\pi\phi$

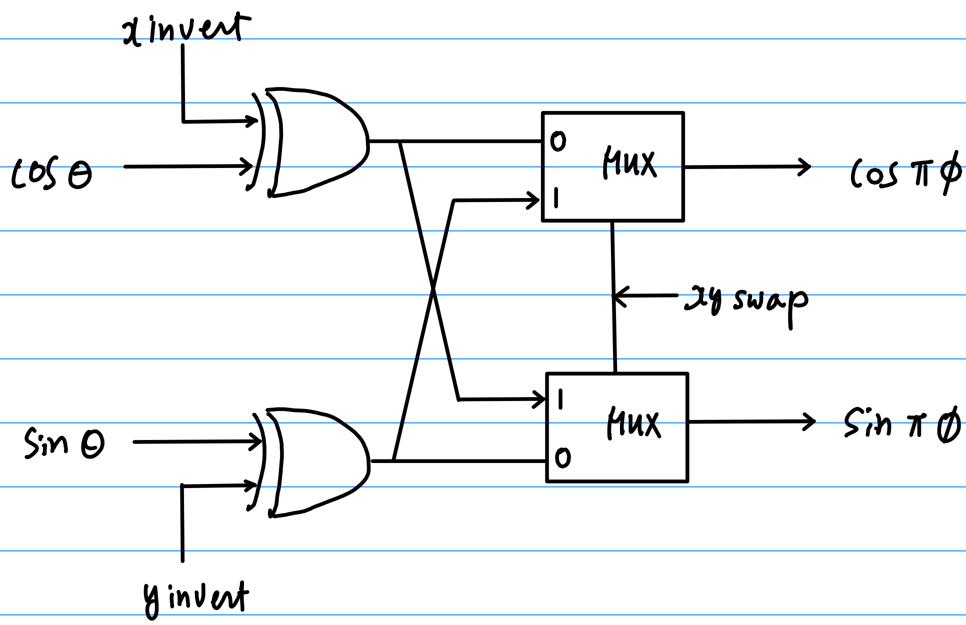
negation / interchange

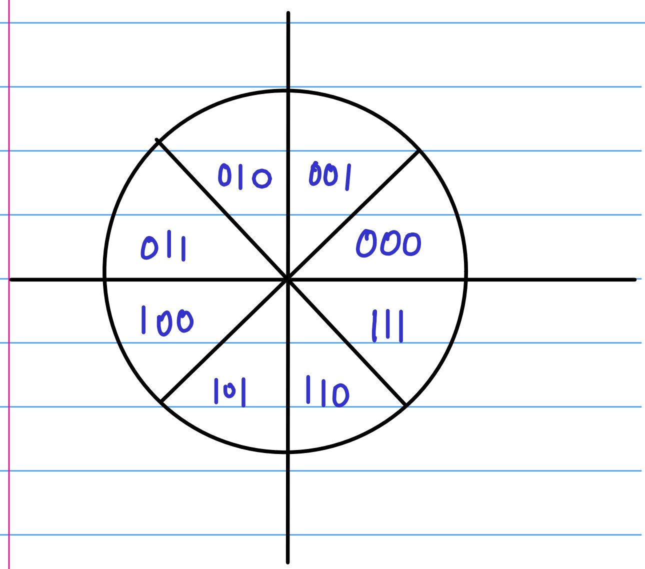
MSB_1, MSB_2, MSB_3 → x invert, y invert, xy swap

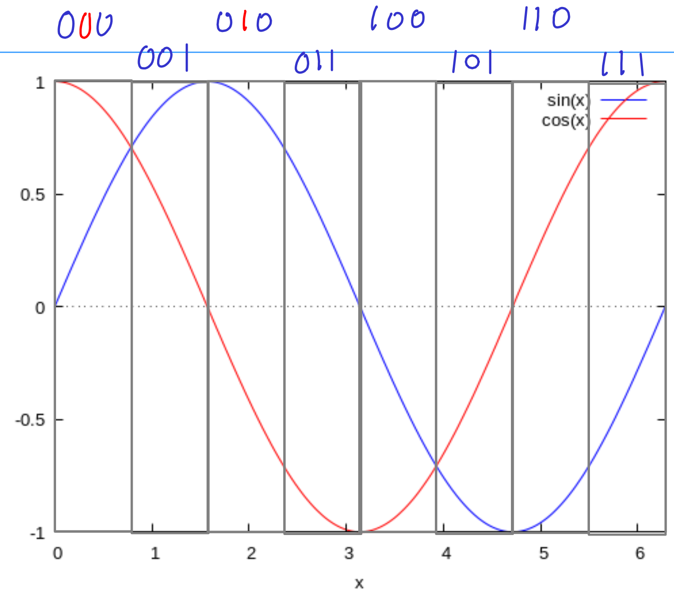
x invert, y invert: the negation of $\cos\theta = X_{N+1}$, $\sin\theta = Y_{N+1}$

xy swap: Interchange

negate before swap







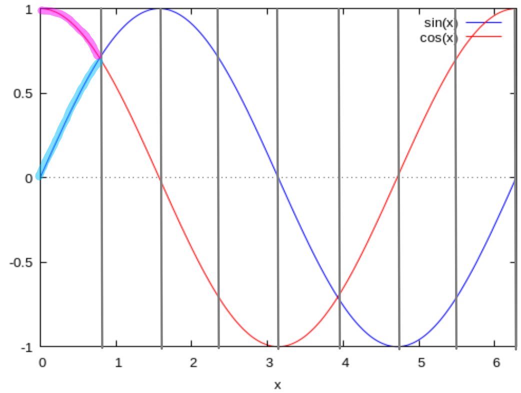
	cos Xinv	sin. Yinv	swap	$\cos \pi \theta$	$\sin \pi \phi$
000	0	0	0	$\cos \theta$	$\sin \theta$
001	0	0	1	$\sin \theta$	$\cos \theta$
010	0	1	1	$-\sin \theta$	$\cos \theta$
011	1	0	0	$-\cos \theta$	$\sin \theta$
100	1	1	0	$-\cos \theta$	$-\sin \theta$
101	1	1	1	$-\sin \theta$	$-\cos \theta$
110	1	0	1	$\sin \theta$	$-\cos \theta$
111	0	1	0	$\cos \theta$	$-\sin \theta$

Madisetti & Willson, DDS Freq synthesizer

0

$\cos\theta$

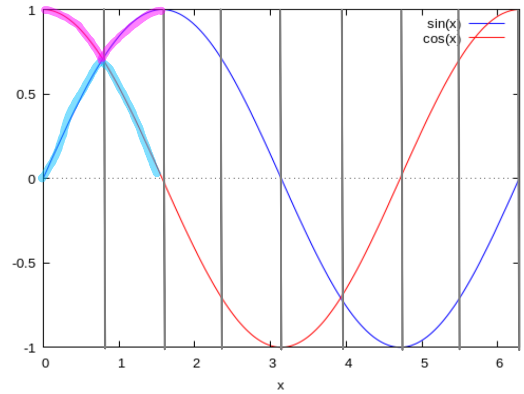
$\sin\theta$



1

$\sin\theta$

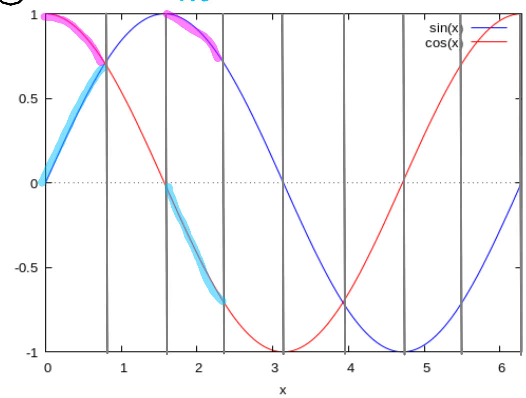
$\cos\theta$



2

$-\sin\theta$

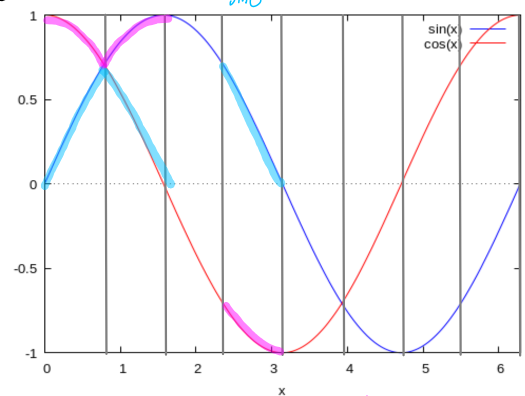
$\cos\theta$



3

$-\cos\theta$

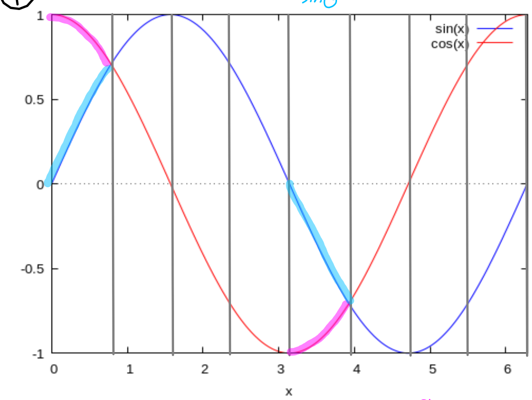
$\sin\theta$



4

$-\cos\theta$

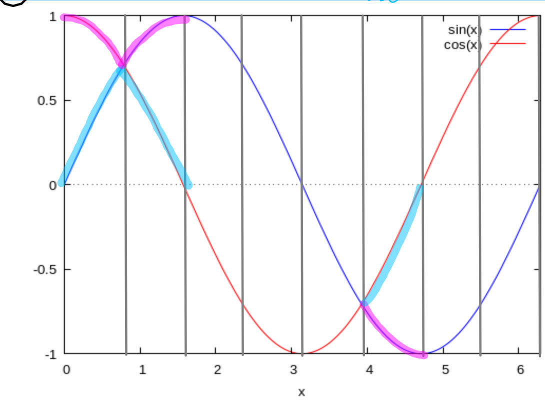
$-\sin\theta$



5

$-\sin\theta$

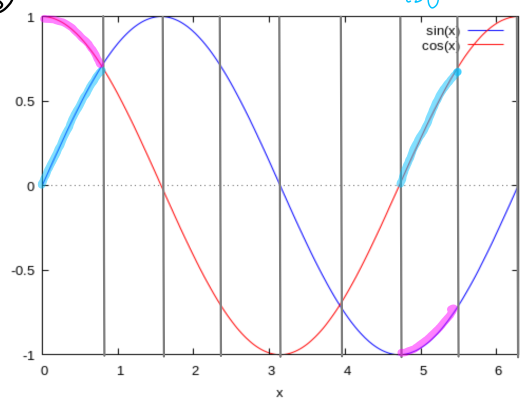
$-\cos\theta$



6

$\sin\theta$

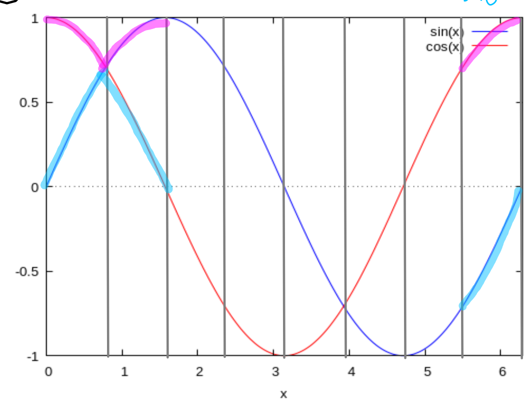
$-\cos\theta$



7

$\cos\theta$

$-\sin\theta$

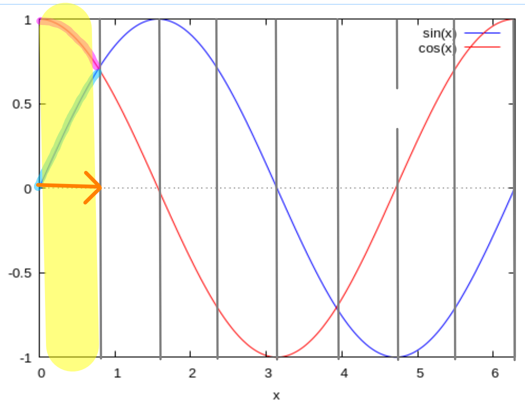


Madisetti & Willson, DDS Freq synthesizer

$\begin{cases} \cos\phi \\ \sin\phi \end{cases}$

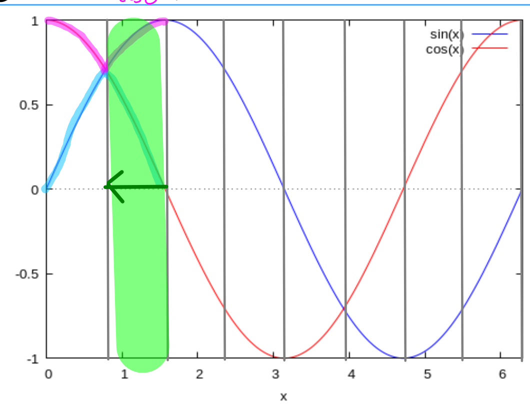
0

$\cos\theta$
$\sin\theta$



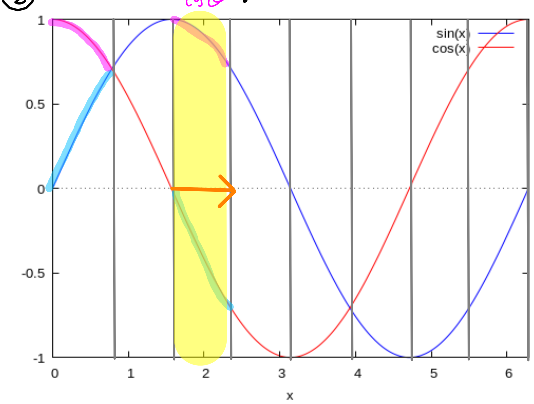
1

$\sin\theta$
$\cos\theta$



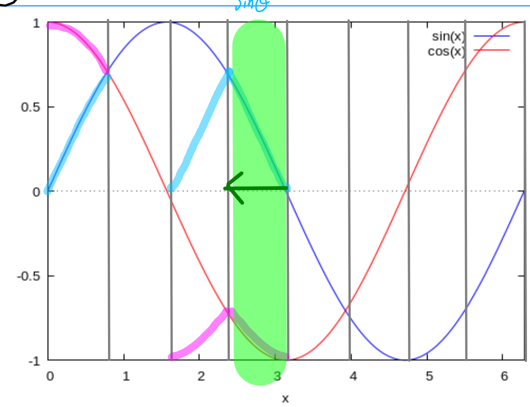
2

$-\sin\theta$
$\cos\theta$



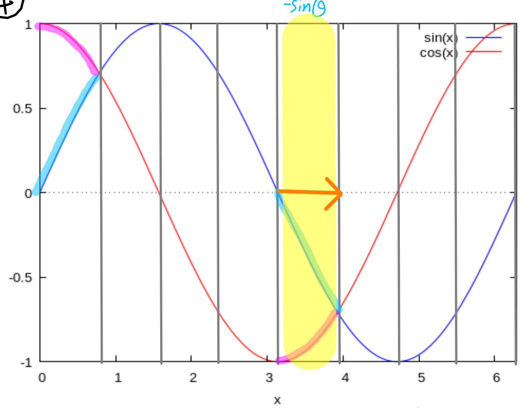
3

$-\cos\theta$
$\sin\theta$



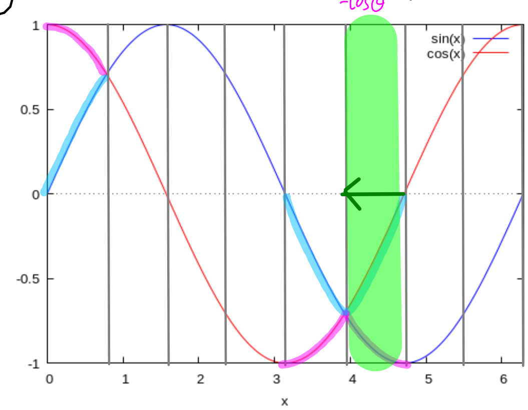
4

$-\cos\theta$
$-\sin\theta$



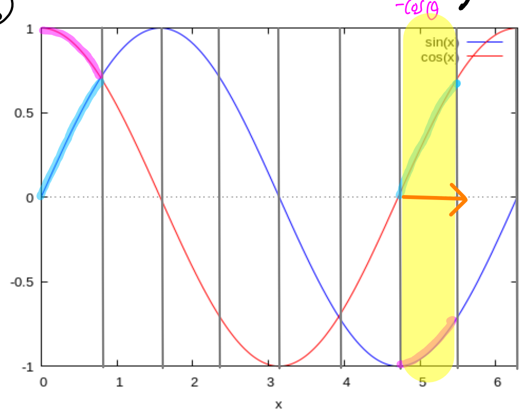
5

$-\sin\theta$
$-\cos\theta$



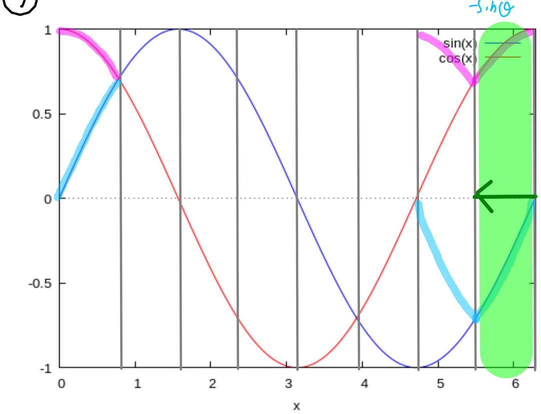
6

$\sin\theta$
$-\cos\theta$



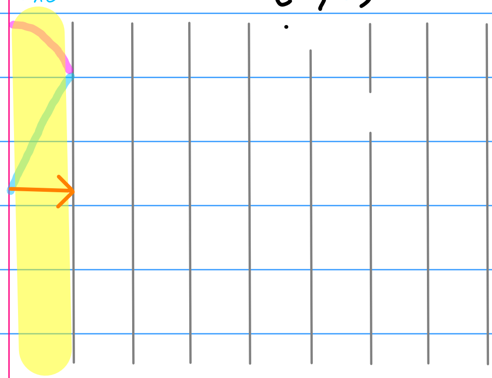
7

$\cos\theta$
$-\sin\theta$

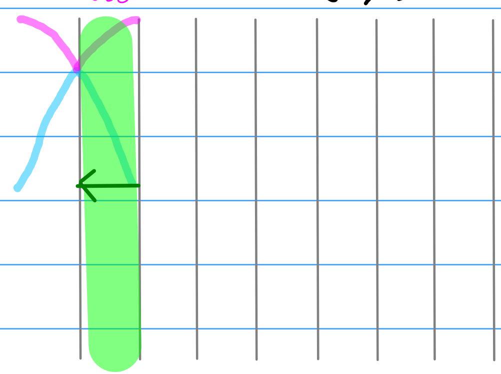


sin ϕ

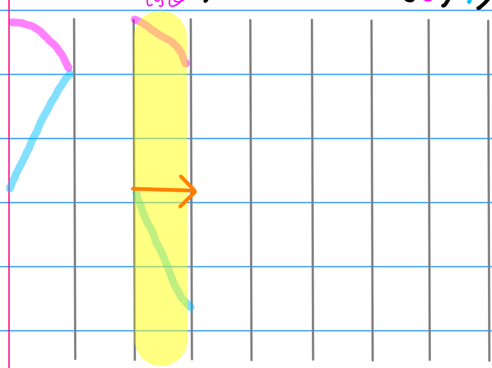
0 $\cos\theta$ / $\sin\theta$ + + (0, 0)



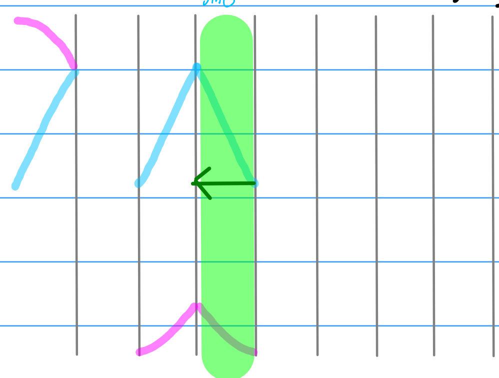
1 $\sin\theta$) $\cos\theta$ + + (0, 0)



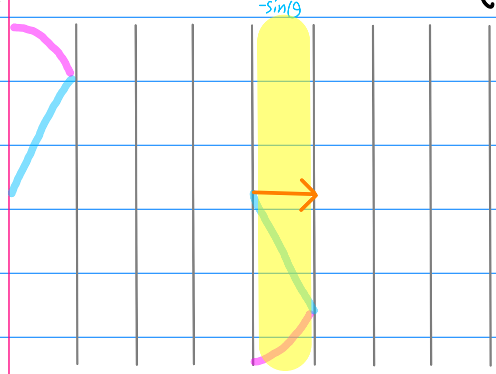
2 $-\sin\theta$) $\cos\theta$ + − (0, 1)



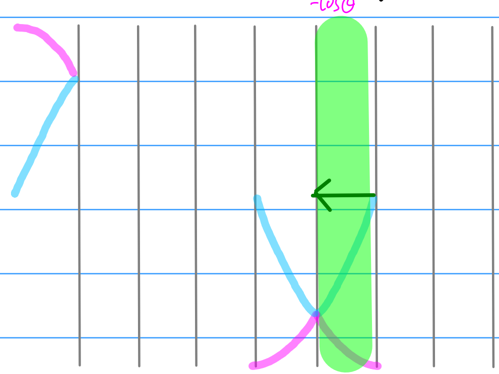
3 $-\cos\theta$ $\sin\theta$ − + (1, 0)



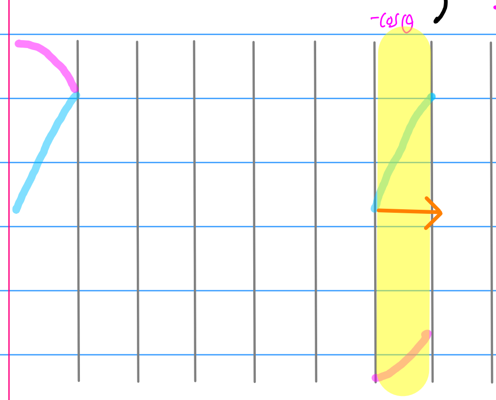
4 $-\cos\theta$ $-\sin\theta$ − − (1, 1)



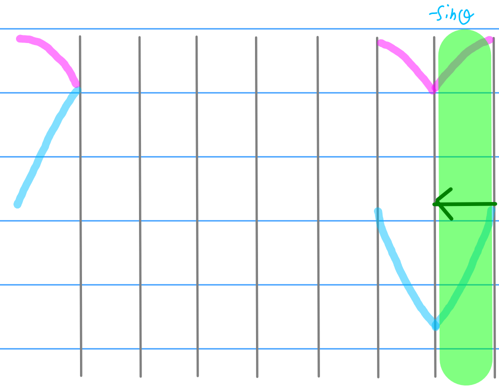
5 $-\sin\theta$) $-\cos\theta$ − − (1, 1)



6 $\sin\theta$) $-\cos\theta$ − + (1, 0)



7 $\cos\theta$ $-\sin\theta$ + − (0, 1)



	Xinv	Yinv	swap	$\cos\pi\phi$	$\sin\pi\phi$
0 0 0	0	0	0	$\cos\theta$	$\sin\theta$
0 0 1	0	0	1	$\sin\theta$	$\cos\theta$
0 1 0	0	1	1	$-\sin\theta$	$\cos\theta$
0 1 1	1	0	0	$-\cos\theta$	$\sin\theta$
1 0 0	1	1	0	$-\cos\theta$	$-\sin\theta$
1 0 1	1	1	1	$-\sin\theta$	$-\cos\theta$
1 1 0	1	0	1	$\sin\theta$	$-\cos\theta$
1 1 1	0	1	0	$\cos\theta$	$-\sin\theta$

0 0	0 0
0 1	1 0
1 1	1 1
1 0	0 1

0	0	0	0
0	1	1	0
1	1	1	1
1	0	0	1

Madisetti & Willson, DDS Freq synthesizer

$$\theta = \sum_{k=1}^{N} b_k \theta_k$$

b_k sign + N bit — (N+1) bit fractional b

$b_k \in \{0, 1\}$

$\theta_k = 2^{-k}$

θ is constrained to be positive $\boxed{b_0 = 0}$

$$\theta = \sum_{k=1}^{N} b_k 2^{-k} = \phi_0 + \sum_{k=2}^{N+1} r_k 2^{-k}$$

$r_k \in \{-1, +1\}$ signed digits

ϕ_0 constant

(+) subrotation by 2^{-k}

2 equal (+) half rotations by 2^{-k-1}

(0) subrotation

2 equal opposite half rotations by $\pm 2^{-k-1}$

Binary Representation

$b_k = 1$: rotation by 2^{-k}

$b_k = 0$: zero rotation

k-th rotation

- fixed rotation by 2^{-k-1}
- $\begin{cases} \text{pos rotation} \leftarrow b_k = 1 \\ \text{neg rotation} \leftarrow b_k = 0 \end{cases}$

Combining all the fixed rotations

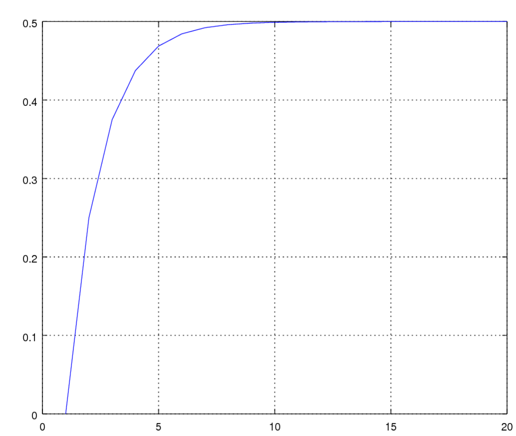
$\rightarrow$ initial fixed rotation

	b_1 2^{-1}	b_2 2^{-2}	b_3 2^{-3}		b_N 2^{-N}
fixed $\Rightarrow$	$+2^{-2}$	$+2^{-3}$	$+2^{-4}$		$+2^{-N-1}$
	$(b_1=1)$ $+2^{-2}$ $(b_1=0)$ -2^{-2}	$(b_2=1)$ $+2^{-3}$ $(b_2=0)$ -2^{-3}	$(b_3=1)$ $+2^{-4}$ $(b_3=0)$ -2^{-4}		$(b_N=1)$ $+2^{-N-1}$ $(b_N=0)$ -2^{-N-1}

initial fixed rotation

$$\phi_0 = \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{N+1}}$$

$$= \frac{\frac{1}{2^2}\left(1-\frac{1}{2^N}\right)}{\left(1-\frac{1}{2}\right)} = \frac{1}{2}\left(1-\frac{1}{2^N}\right) = \frac{1}{2} - \frac{1}{2^{N+1}}$$



Signed Digit Recoding

the rotation after recoding

— a fixed initial rotation ϕ_0

a sequence of ⊕/⊖ rotations

$b_k = 1 \quad + 2^{-k-1}$ rotation

$b_k = 0 \quad - 2^{-k-1}$ rotation

$r_k = (2b_{k-1} - 1)$

$2 \cdot 1 - 1 = +1 \qquad b_{k-1} = 1 \longrightarrow r_k = +1$

$2 \cdot 0 - 1 = -1 \qquad b_{k-1} = 0 \longrightarrow r_k = -1$

The recoding need not be explicitly performed

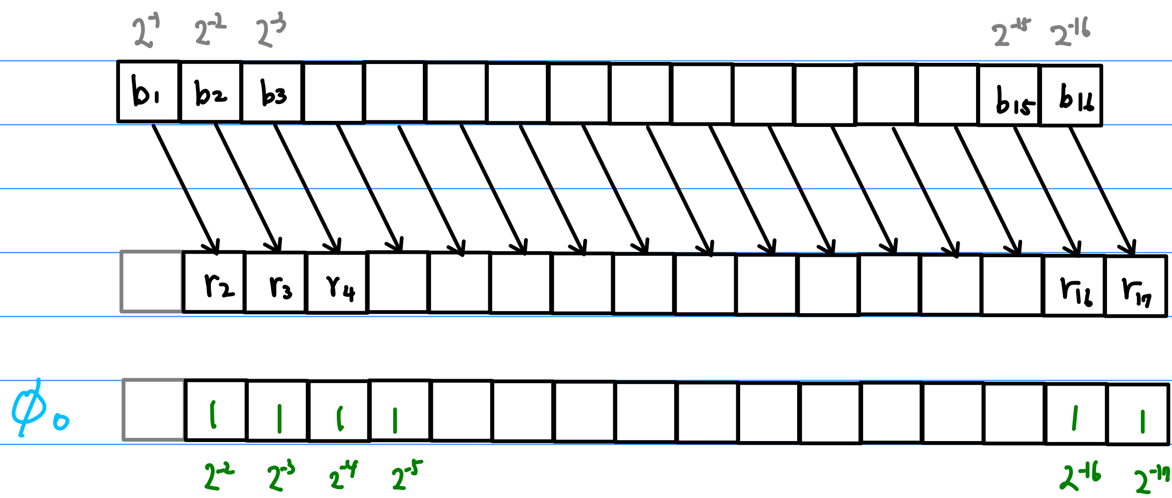
Simply replacing $b_k = 0$ with -1

This recoding maintains

a constant scaling factor K

$$\theta = \sum_{k=1}^{N} b_k 2^{-k} = \phi_0 + \sum_{k=2}^{N+1} r_k 2^{-k}$$

Binary Representation $\{b_k\}$



Signed Digit Recoding $\{r_k\}$

The Scaling K.
The initial rotation ϕ_0

rotation starting point
$$(X_0, Y_0) = (K\cos\phi_0,\ K\sin\phi)$$

— fixed
— no error buildup

— rotation direction
immediately obtained from the binary representation
→ no need for comparison

the subangles $\boxed{\theta_k = 2^{-k}}$ used in recoding
the subangles $\boxed{\theta_k = \tan^{-1}(2^{-k})}$ used in CORDIC

$\tan\theta_k$ multipliers used
in the first few subrotation stages
cannot be implemented
as a simple shift-and-add operations

→ ROM implementation

reduced chip area
higher operating speed.

Architecture

1. phase accumulator $\phi \in [-1, +1]$
2. radian converter $\phi \to \theta \in [0, \frac{\pi}{4}]$
3. sine/cosine generator $\sin(\theta)$ $\cos(\theta)$
4. output stage $\sin(\pi\phi)$ $\cos(\pi\phi)$

Overflowing 2's complement accumulator

normalized by π angle ϕ

need radian angle $\theta \in [0, \frac{\pi}{4}]$

$0 < \theta < 1$ rad

N-bit binary representation of θ

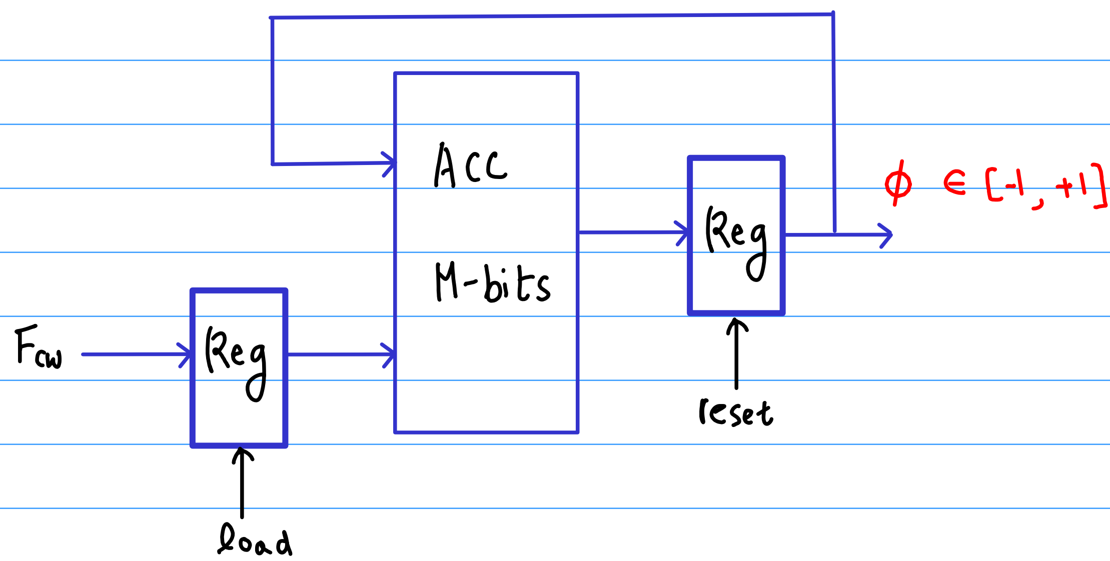
controls the direction of subrotation

N-bit precision of $\cos\theta$ & $\sin\theta$

Output stage

$$\theta \rightarrow \pi\phi$$
$$\sin\theta \rightarrow \sin\pi\phi$$
$$\cos\theta \rightarrow \cos\pi\phi$$

phase accumulator



M-bit adder

repeatedly increments the phase angle by F_{cw} at each clock cycle

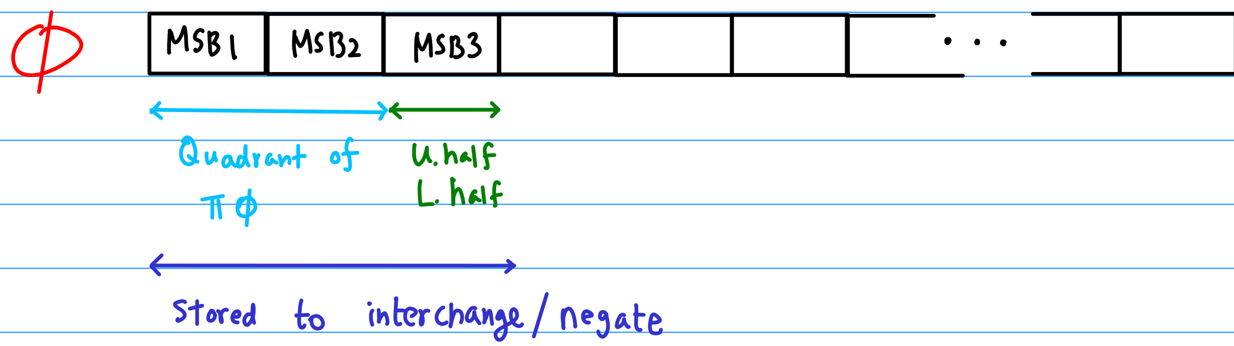
frequency control word

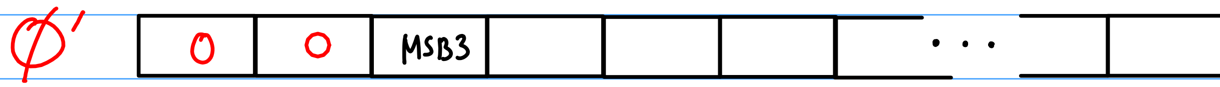
at time n, $\phi = n F_{cw} / 2^M$

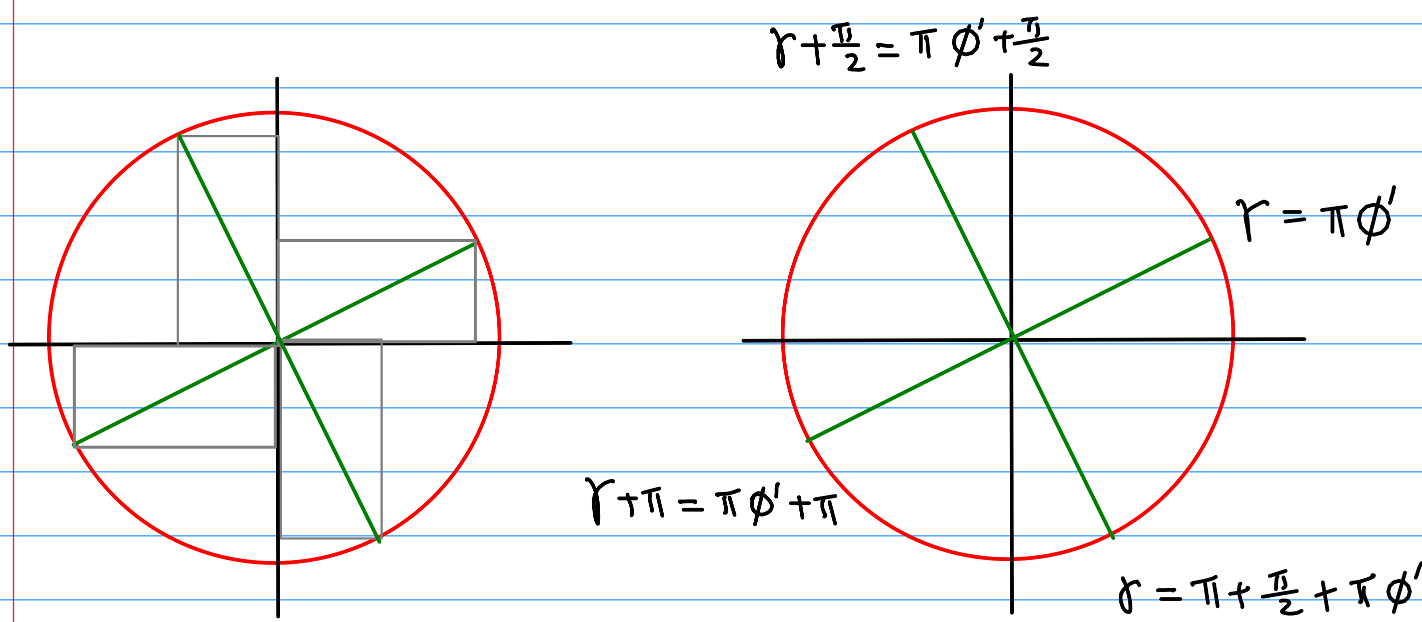
$$\cos\phi = \cos(n F_{cw} / 2^M)$$
$$\sin\phi = \sin(n F_{cw} / 2^M)$$

Radian Converter

normalized angle ϕ



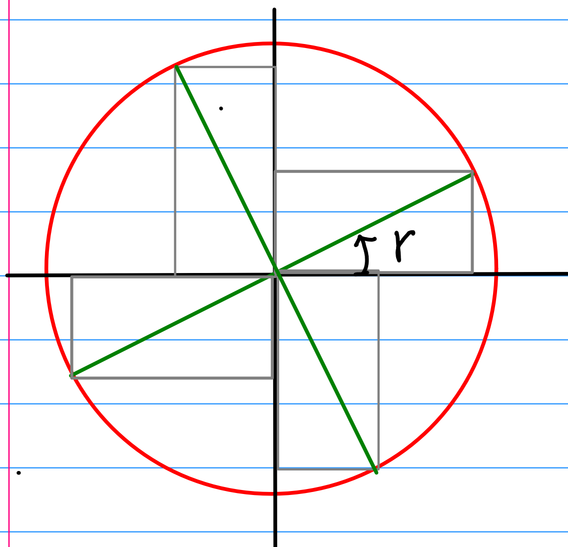




$$\phi \to \phi' \to \begin{array}{ll} \pi\phi' + 0\cdot\frac{\pi}{2} & 00 \\ \pi\phi' + 1\cdot\frac{\pi}{2} & 01 \\ \pi\phi' + 2\cdot\frac{\pi}{2} & 10 \\ \pi\phi' + 3\cdot\frac{\pi}{2} & 11 \end{array}$$

1st Quad

Quadrant Symmetry



$\boxed{0}\boxed{1}$	$\boxed{0}\boxed{0}$
$r + \frac{\pi}{2} = \pi\phi' + \frac{\pi}{2}$	$r = \pi\phi'$
$r + \pi = \pi\phi' + \pi$	$r + \frac{3\pi}{2} = \pi\phi' + \frac{3\pi}{2}$
$\boxed{1}\boxed{0}$	$\boxed{1}\boxed{1}$

0 1	+1
0 0	0
1 1	-1
1 0	-2

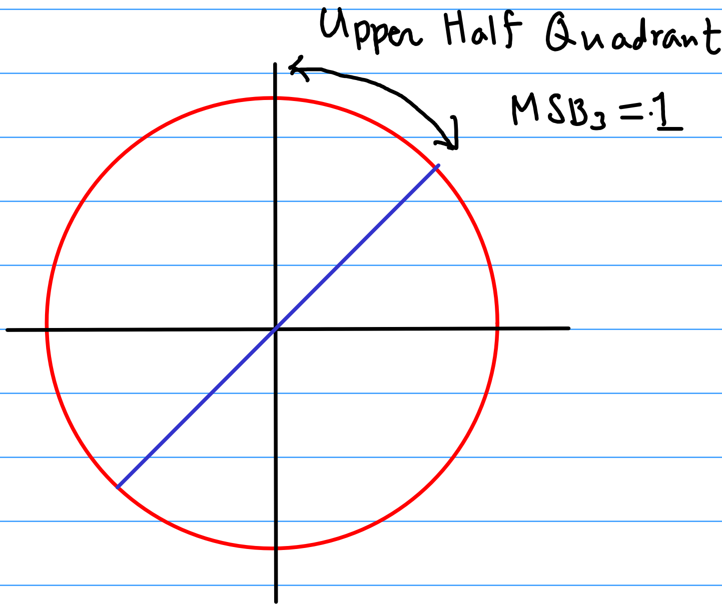
$$\theta = \pi\phi \Rightarrow \theta' = \pi\phi'$$

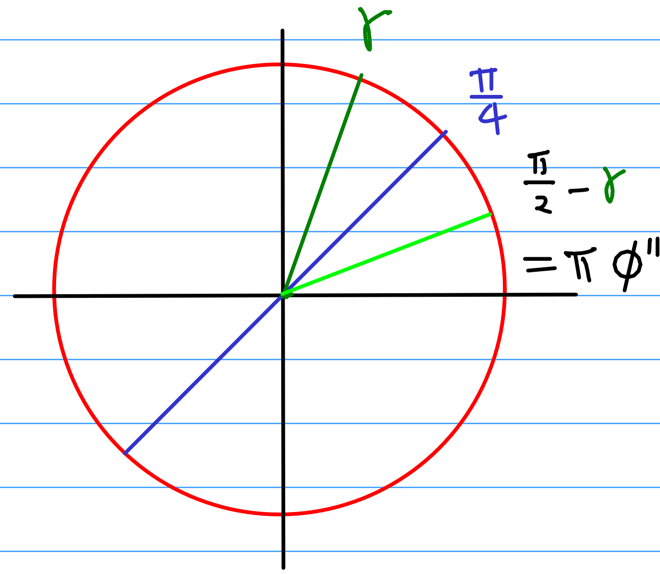
$$\theta \in [-\pi, +\pi] \Rightarrow \theta = [0, \tfrac{\pi}{2}]$$

$$\phi \in [-1, +1] \Rightarrow \phi' = [0, 0.5]$$

ϕ'									
	0	0	MSB3					…	

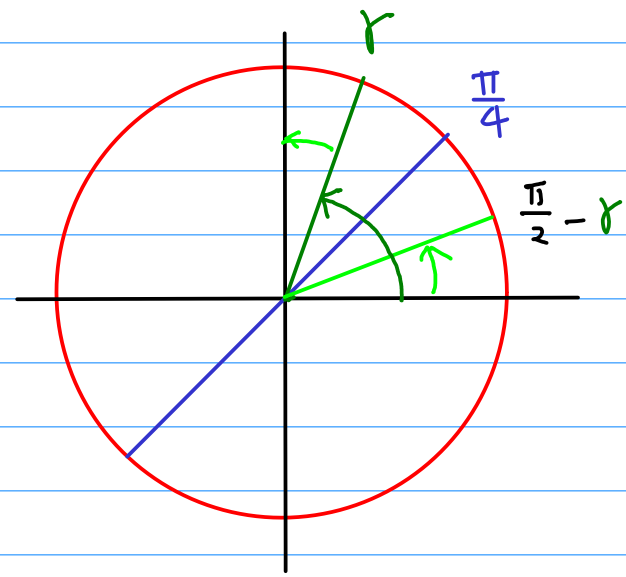
1st Quadrant





$\gamma > \frac{\pi}{4}$: Upper Half ($MSB_3 = 1$)

$\gamma < \frac{\pi}{4}$: Lower Half ($MSB_3 = 0$)

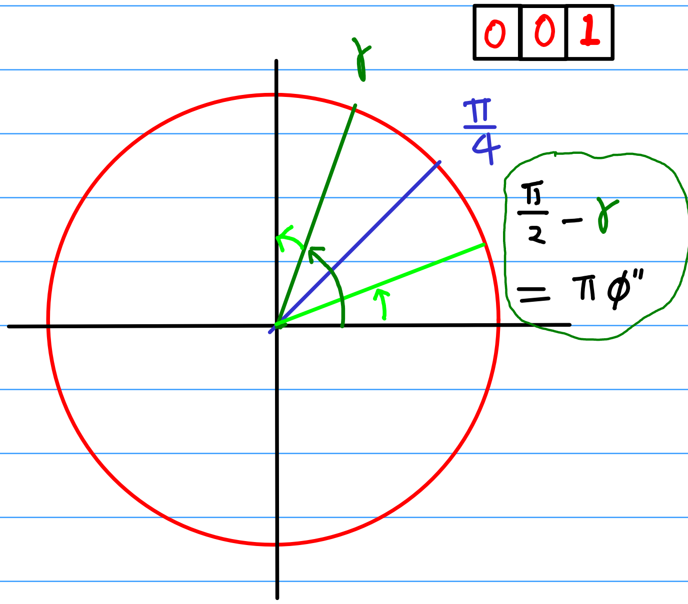


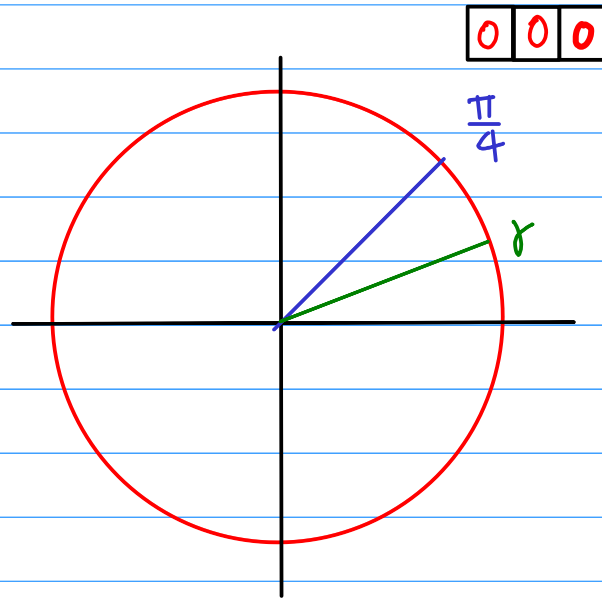
$$\cos \gamma = \sin\left(\frac{\pi}{2} - \gamma\right)$$

$$\sin \gamma = \cos\left(\frac{\pi}{2} - \gamma\right)$$

$\gamma > \frac{\pi}{4}$ $\qquad$ $\frac{\pi}{2} - \gamma < \frac{\pi}{4}$.

π/4 mirror





$\frac{\pi}{4} < \gamma < \frac{\pi}{2}$

$\phi'' = 0.5 - \phi'$

$0 < \gamma < \frac{\pi}{4}$

$\phi'' = \phi'$

$\theta = \pi \phi \rightarrow \theta' = \pi \phi' \rightarrow \theta'' = \pi \phi''$

$\theta \in [-\pi, +\pi] \rightarrow \theta' = [0, \pi/2] \rightarrow \theta'' = [0, \pi/4]$

$\phi \in [-1, +1] \rightarrow \phi' = [0, 0.5] \rightarrow \phi'' = [0, 0.25]$

ϕ'	0	0	MSB3				$\cdots$	

$MSB3 = 1 \qquad \phi' > \frac{\pi}{4}$

$$\phi'' = \frac{\pi}{2} - \phi'$$

	0.5	1	0	0				$\cdots$	
$\ominus$	ϕ'	0	0	MSB3				$\cdots$	

$$\begin{cases} MSB3 = 0 & \phi'' = \phi' \\ MSB_3 = 1 & \phi'' = 0.5 - \phi' \end{cases}$$

$\theta = \pi \phi''$ (Hardwired Multiplier)

$0 < \theta < \frac{\pi}{4}$

$$\phi \longrightarrow \phi' \longrightarrow \phi''$$

1st Quad Lower Half

$$\theta = \pi\phi \longrightarrow \theta' = \pi\phi' \longrightarrow \theta'' = \pi\phi''$$

$$\theta \in [-\pi, +\pi] \longrightarrow \theta' = [0, \pi/2] \longrightarrow \theta'' = [0, \pi/4]$$

$$\phi \in [-1, +1] \longrightarrow \phi' = [0, 0.5] \longrightarrow \phi'' = [0, 0.25]$$

$\theta \in [-\pi, +\pi]$ $\boxed{\phi = \theta / \pi}$

$\downarrow$

$$\phi \in [-1, +1] \longrightarrow \phi' = [0, 0.5] \longrightarrow \phi'' = [0, 0.25]$$

$\downarrow$

Compute

$\downarrow$

$$\phi \in [-1, +1] \longleftarrow \phi' = [0, 0.5] \longleftarrow \phi'' = [0, 0.25]$$

$\downarrow$

$\theta \in [-\pi, +\pi]$ $\boxed{\theta = \pi\phi}$

* radian converter

Sine / Cosine Generator

given angle θ (in radian) $0 \le \theta \le \pi/4$ < 1

$\pi/4 = 0.785398163$

compute $\cos\theta$, $\sin\theta$

$$\begin{bmatrix} X_\theta \\ Y_\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} X_0 \\ Y_0 \end{bmatrix}$$

$$= \cos\theta \begin{bmatrix} 1 & -\tan\theta \\ \tan\theta & 1 \end{bmatrix} \begin{bmatrix} X_0 \\ Y_0 \end{bmatrix}$$

$$= \cos\theta \begin{bmatrix} 1 & -\tan\theta \\ \tan\theta & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Suppose: θ as a sequence of sub-rotation

$\{\theta_i\}$ the subrotation angles are <u>known</u> a priori

then $\theta = \sigma_0 \theta_0 + \sigma_1 \theta_1 + \cdots + \sigma_N \theta_N$

$\sigma_k = \{-1, 0, +1\}$

$$\theta = \sigma_0 \theta_0 + \sigma_1 \theta_1 + \cdots + \sigma_N \theta_N$$

$$\sigma_k = \{-1, 0, +1\}$$

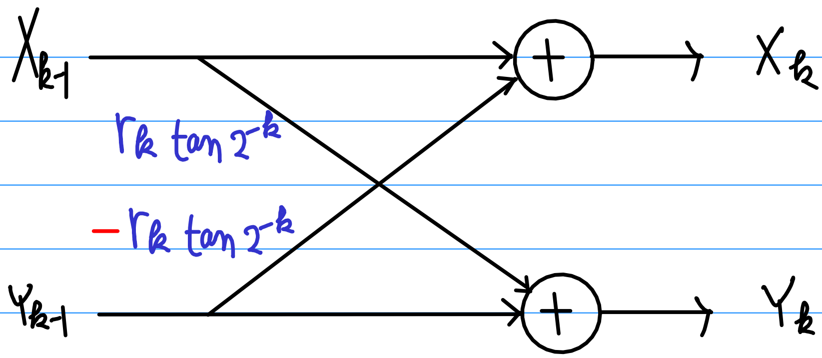
$$\begin{bmatrix} X_\theta \\ Y_\theta \end{bmatrix} = K \begin{bmatrix} 1 & -\tan \sigma_N \theta_N \\ \tan \sigma_N \theta_N & 1 \end{bmatrix} \cdots \begin{bmatrix} 1 & -\tan \sigma_0 \theta_0 \\ \tan \sigma_0 \theta_0 & 1 \end{bmatrix} \begin{bmatrix} X_0 \\ Y_0 \end{bmatrix}$$

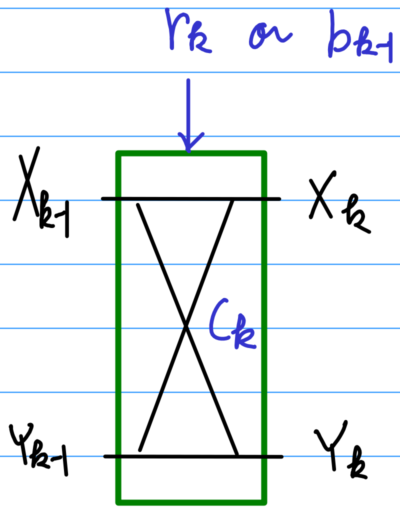
$$K = \cos \sigma_0 \theta_0 \cdot \cos \sigma_1 \theta_1 \cdots \cos \sigma_N \theta_N$$

Subrotation

$$X_{k+1} = X_k - (r_k \tan\theta_k) Y_k$$

$$Y_{k+1} = Y_k + (r_k \tan\theta_k) X_k$$





Butterfly