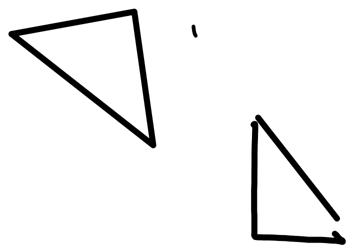
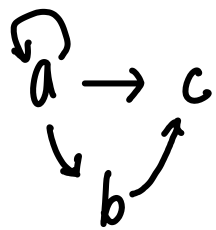
# Relations (3A)

equivalent R — $\begin{pmatrix} \text{Reflexive} \\ \text{Symmetric} \\ \text{Transitive} \end{pmatrix}$

$\geq$  Order  $\geq \; > \; < \; <=$

$\begin{pmatrix} \text{Reflexive} \\ \text{Anti-Symmetric} \\ \text{Transitive} \end{pmatrix}$





$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\underline{(i, j)}$ & $\underline{(j, k)} \rightarrow (i, k)$

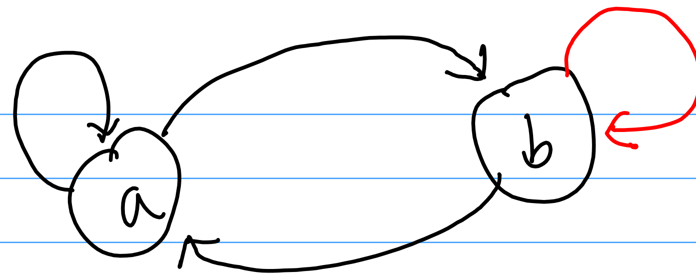
$6n^2 + 3n$

$O(n^2)$ $\quad O(n^3)$ $\quad \cancel{O(\log n)}$

$\Theta(n^2)$ $\quad \cancel{\Theta(n^3)}$ $\quad \cancel{\Theta(\log n)}$

$\Omega(n^2)$ $\quad \cancel{\Omega(n^3)}$ $\quad \Omega(\log n)$

$6n^2 + 3n \approx 6n^2$




$a \to b \to a$

$a \longrightarrow a$

$b \to a \to b$

$b \longrightarrow b$

$$\begin{array}{c|cc} & a & b \\ \hline a & 1 & 1 \\ b & 1 & 0 \end{array}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

Please send corrections (or suggestions) to youngwlim@hotmail.com.

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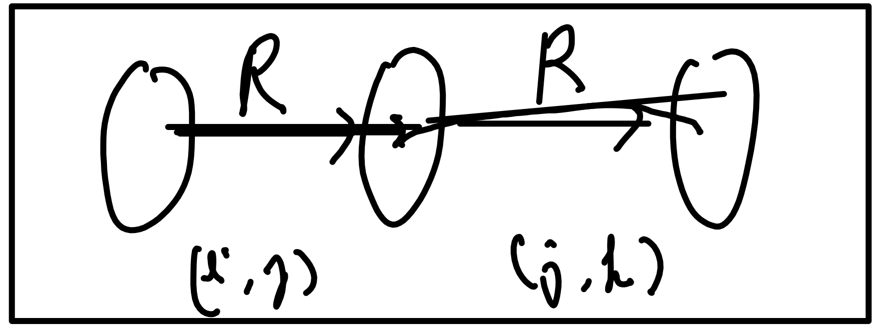
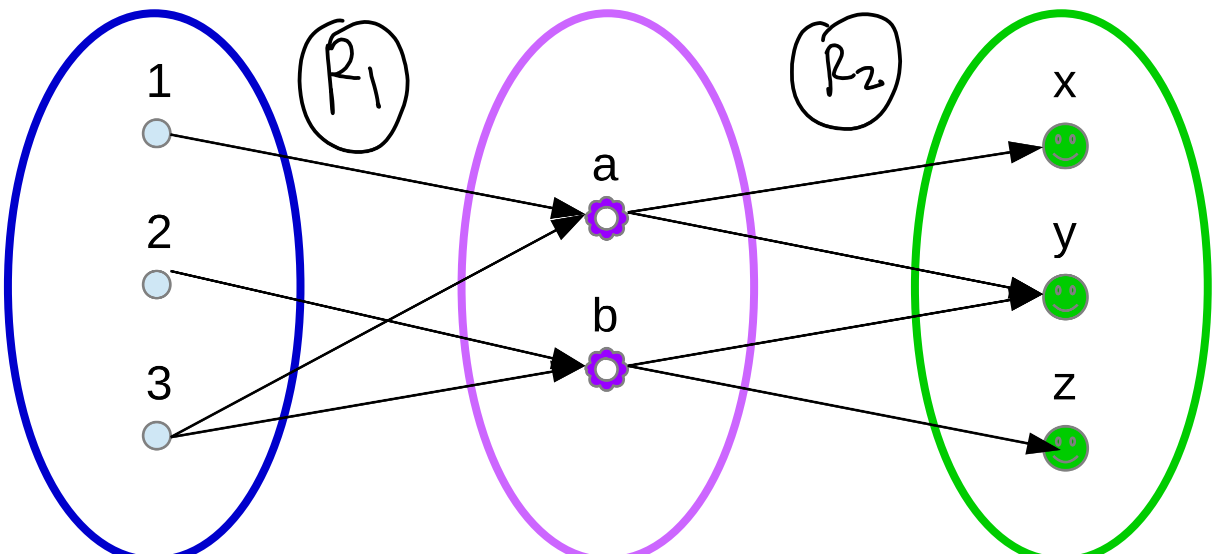
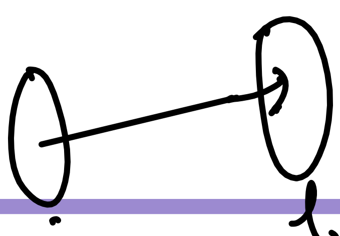
# Relation Examples

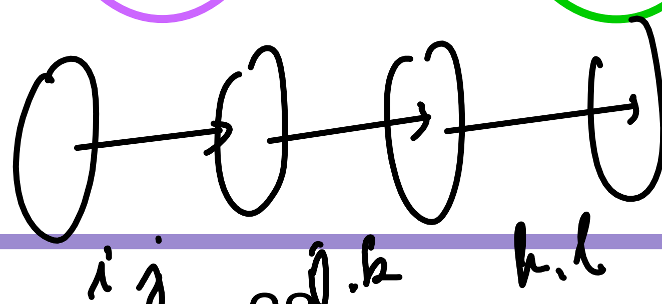
$R_1 \in \{(1,a),\ (2,b),\ (3,a),\ (3,b)\}$

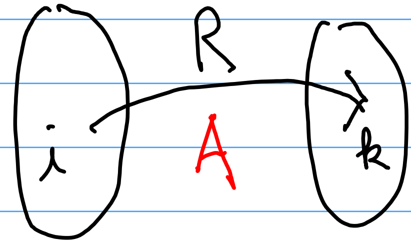
$R_2 \in \{(a,x),\ (a,y),\ (b,y),\ (b,z)\}$



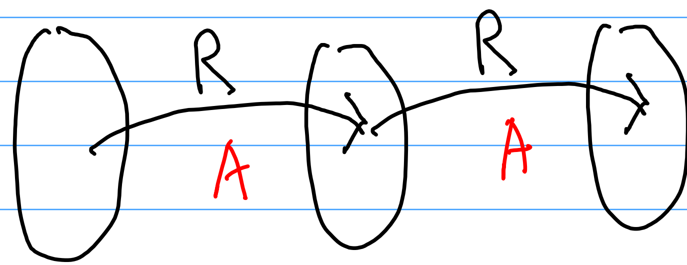






$R$




$R \circ R$

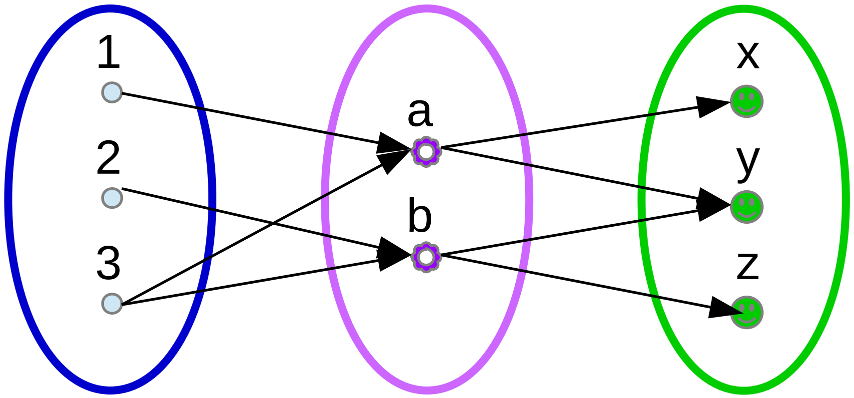
$A \cdot A$

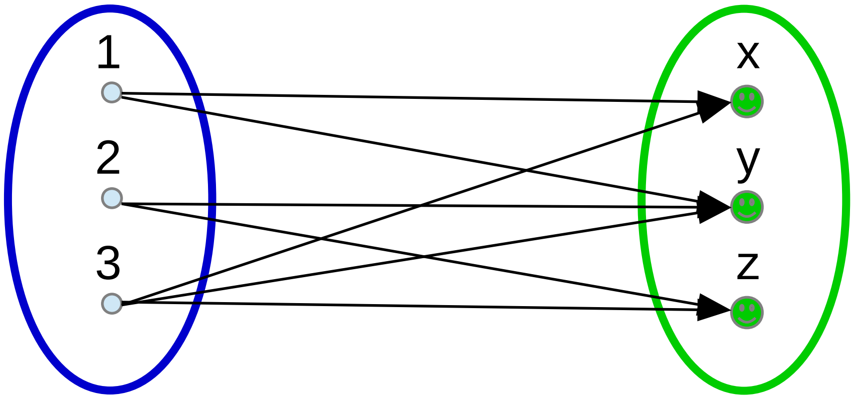
$i \longrightarrow j \longrightarrow k =$

# Composite Relation Examples

$R_1 \in \{(1,a), (2,b), (3,a), (3,b)\}$

$R_2 \in \{(a,x), (a,y), (b,y), (b,z)\}$



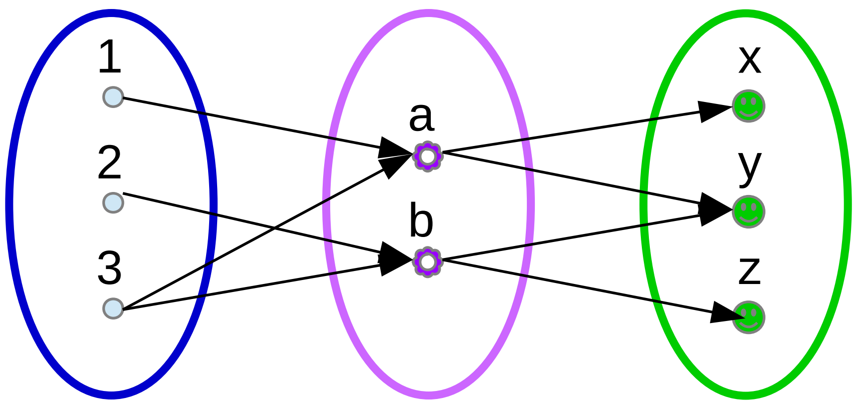

$R_2 \circ R_1 \in \{(1,x), (1,y), (2,y), (2,z), (3,x), (3,y), (3,z)\}$

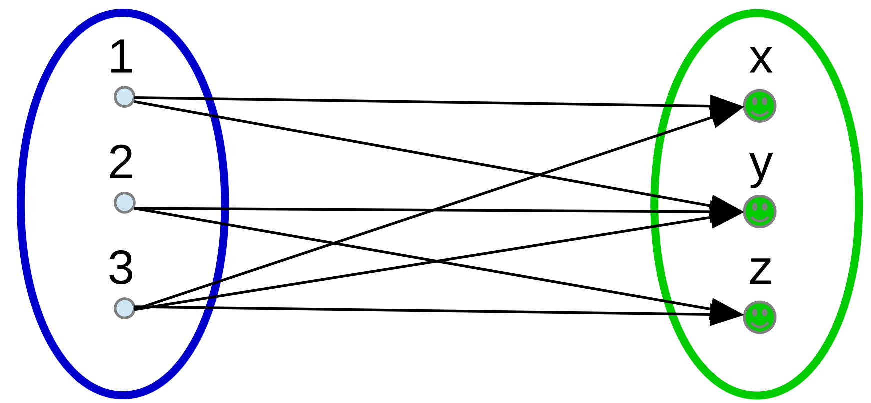
# Composite Relation Examples

$R_1 \in \{(1,a),\ (2,b),\ (3,a),\ (3,b)\}$

$R_2 \in \{(a,x),\ (a,y),\ (b,y),\ (b,z)\}$

$R_2 \circ R_1 \in$
$\{(1,x),\ (1,y),\ (2,y),\ (2,z),\ (3,x),\ (3,y),\ (3,z)\}$




$$A_1 = \begin{array}{c|c} & \begin{matrix} a & b \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \end{array} \qquad A_2 = \begin{array}{c|c} & \begin{matrix} x & y & z \end{matrix} \\ \begin{matrix} a \\ b \end{matrix} & \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \end{array}$$

$$A_1 A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{array}{c|c} & \begin{matrix} x & y & z \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix} \end{array}$$

# Matrix of a Relation

$R_1 \in \{(1,a),\ (2,b),\ (3,a),\ (3,b)\}$

$$A_1 = \begin{array}{c c} & \begin{array}{cc} a & b \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \end{array}$$

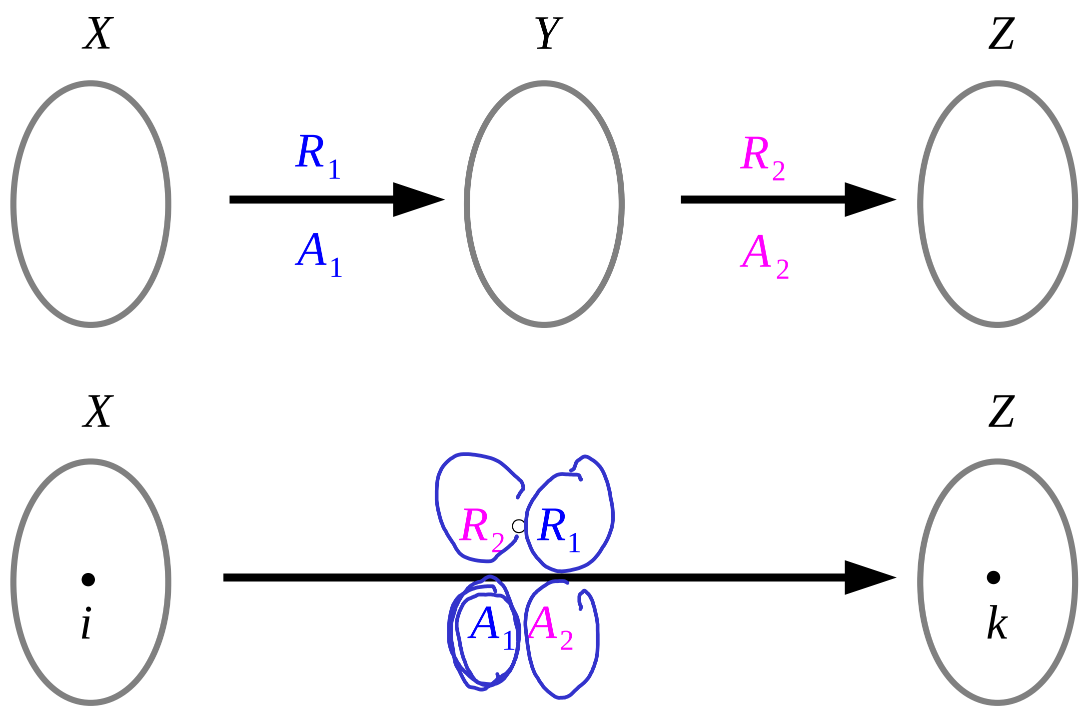
$R_2 \in \{(a,x),\ (a,y),\ (b,y),\ (b,z)\}$

$$A_2 = \begin{array}{c c} & \begin{array}{ccc} x & y & z \end{array} \\ \begin{array}{c} a \\ b \end{array} & \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \end{array}$$

$R_2 \circ R_1 \in \{(1,x),\ (1,y),\ (2,y),\ (2,z),\ (3,x),\ (3,y),\ (3,z)\}$

$$A_1 A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{array}{c c} & \begin{array}{ccc} x & y & z \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix} \end{array}$$

# Composite Relation Properties




$$\boxed{(i,k) \in R_2 \circ R_1} \quad \Longleftrightarrow \quad \boxed{a_{ik} \text{ of } A_1 A_2 \neq 0}$$

# Composite Relation Examples

$R_1 \in \{(1,a),\ (2,b),\ (3,a),\ (3,b)\}$

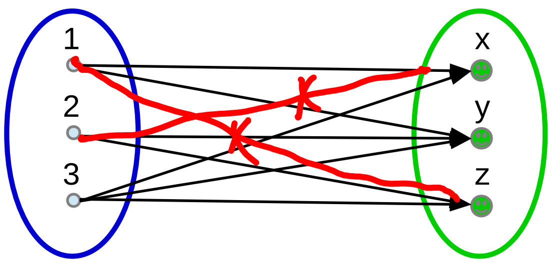
$R_2 \in \{(a,x),\ (a,y),\ (b,y),\ (b,z)\}$

$R_2 \circ R_1 \in \{(1,x),\ (1,y),\ (2,y),\ (2,z),\ (3,x),\ (3,y),\ (3,z)\}$

$$A_1 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cc} a & b \end{array} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \end{array} \qquad A_2 = \begin{array}{c} \\ a \\ b \end{array} \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \end{array}$$

$$A_1 A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix} \end{array}$$

$(i,k) \in R_2 \circ R_1$ $\Longleftrightarrow$ $a_{ik}$ *of* $A_1 A_2 \neq 0$



$R_1 = \{(1,a), (2,b), (3,a), (3,b)\}$

$R_2 = \{(a,x), (a,y), (b,y), (b,z)\}$

$$A_1 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\!\!\begin{array}{c} \begin{array}{cc} a & b \end{array} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \end{array} \qquad A_2 = \begin{array}{c} \\ a \\ b \end{array}\!\!\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \end{array}$$

$R_2 \circ R_1$

$\|$

$\{(1,x), (1,y), (2,y), (2,z), (3,x), (3,y), (3,z)\}$

$$A_1 \cdot A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

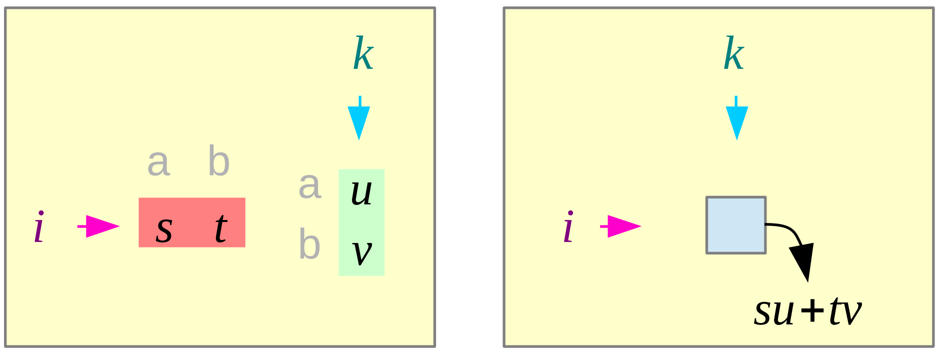
$$= \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix} \end{array}$$

# Composite Relation Property Examples

$$A_1 A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & \boxed{1} & 1 \\ 1 & 2 & 1 \end{bmatrix}$$

$$A_1 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cc} a & b \end{array} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \end{array}$$

$$A_2 = \begin{array}{c} \\ a \\ b \end{array} \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \end{array}$$




$i \in \{1, 2, 3\}$

$k \in \{x, y, z\}$

$s \in \{0, 1\}$

$t \in \{0, 1\}$

$u \in \{0, 1\}$

$v \in \{0, 1\}$

$su + tv \neq 0$ *nonzero* $(i, k)^{th}$ *element of* $A_1 A_2$

$(su \neq 0) \vee (tv \neq 0)$

$(s = 1 \wedge u = 1) \vee (t = 1 \wedge v = 1)$

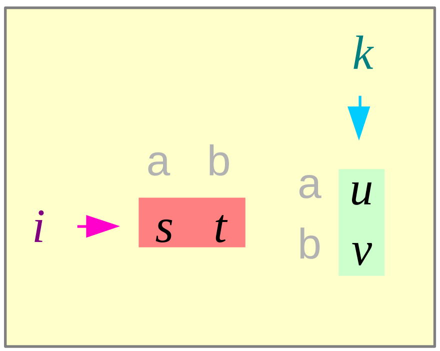
$\Longleftrightarrow (i, k) \in R_2 \circ R_1$

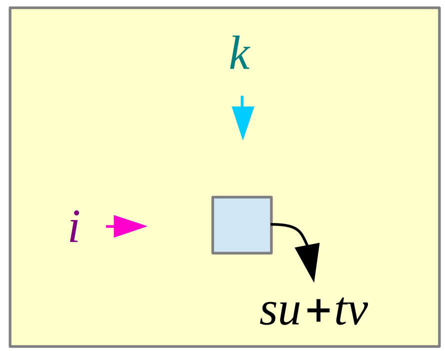
# Sufficient Part

$$A_1 A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & \boxed{1} & 1 \\ 1 & 2 & 1 \end{bmatrix}$$

$$A_1 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cc} a & b \end{array} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \end{array}$$

$$A_2 = \begin{array}{c} \\ a \\ b \end{array} \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \end{array}$$




$i \in \{1, 2, 3\}$

$k \in \{x, y, z\}$

$s \in \{0, 1\}$

$t \in \{0, 1\}$

$u \in \{0, 1\}$

$v \in \{0, 1\}$

$(a_{ik} \neq 0)$ $\quad su + tv \neq 0 \Rightarrow$

$su = 1$ $\quad \boxed{\begin{matrix}(s = 1)\\(u = 1)\end{matrix}}$

$tv = 1$ $\quad \boxed{\begin{matrix}(t = 1)\\(v = 1)\end{matrix}}$

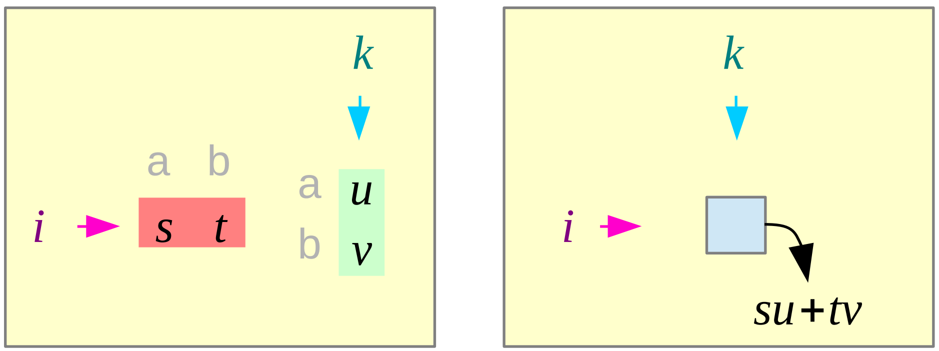
$$\frac{(i, a) \in R_1 \quad (a, k) \in R_2}{(i, k) \in R_2 \circ R_1} \qquad \frac{(i, b) \in R_1 \quad (b, k) \in R_2}{(i, k) \in R_2 \circ R_1}$$

# Necessary Part

$$A_1 A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & \boxed{1} & 1 \\ 1 & 2 & 1 \end{bmatrix}$$

$$A_1 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cc} a & b \end{array} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \end{array}$$

$$A_2 = \begin{array}{c} \\ a \\ b \end{array} \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \end{array}$$




$i \in \{1, 2, 3\}$

$k \in \{x, y, z\}$

$s \in \{0, 1\}$

$t \in \{0, 1\}$

$u \in \{0, 1\}$

$v \in \{0, 1\}$

$(i, k) \in R_2 \circ R_1 \Rightarrow$

| | |
|---|---|
| $(i, a) \in R_1$ | $(i, b) \in R_1$ |
| $(a, k) \in R_2$ | $(b, k) \in R_2$ |
| $(s = 1)$ $(u = 1)$ | $(t = 1)$ $(v = 1)$ |
| $su = 1$ | $tv = 1$ |

$su + tv \neq 0 \quad (a_{ik} \neq 0)$



Young Won Lim
3/27/18

# Transitivity Test Examples

$R \in \{(a,a),(b,b),(c,c),(d,d),(b,c),(c,b)\}$

the same relation

$R \circ R \in \{(a,a),(b,b),(c,c),(d,d),(b,c),(c,b)\}$

$$A = \begin{array}{c|cccc} & a & b & c & d \\ \hline a & & & & \\ b & & \left[\begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}\right] & & \\ c & & & & \\ d & & & & \end{array}$$

$$\begin{array}{cc} & \begin{matrix} a & b & c & d \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \end{matrix} & A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

$$AA = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{array}{cc} & \begin{matrix} a & b & c & d \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \end{matrix} & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$



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3/27/18

$a \to a$
$b \to b$
$b \to c$
$c \to b$
$c \to c$
$d \to d$

(%i2)

(%o2)
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(%i3)

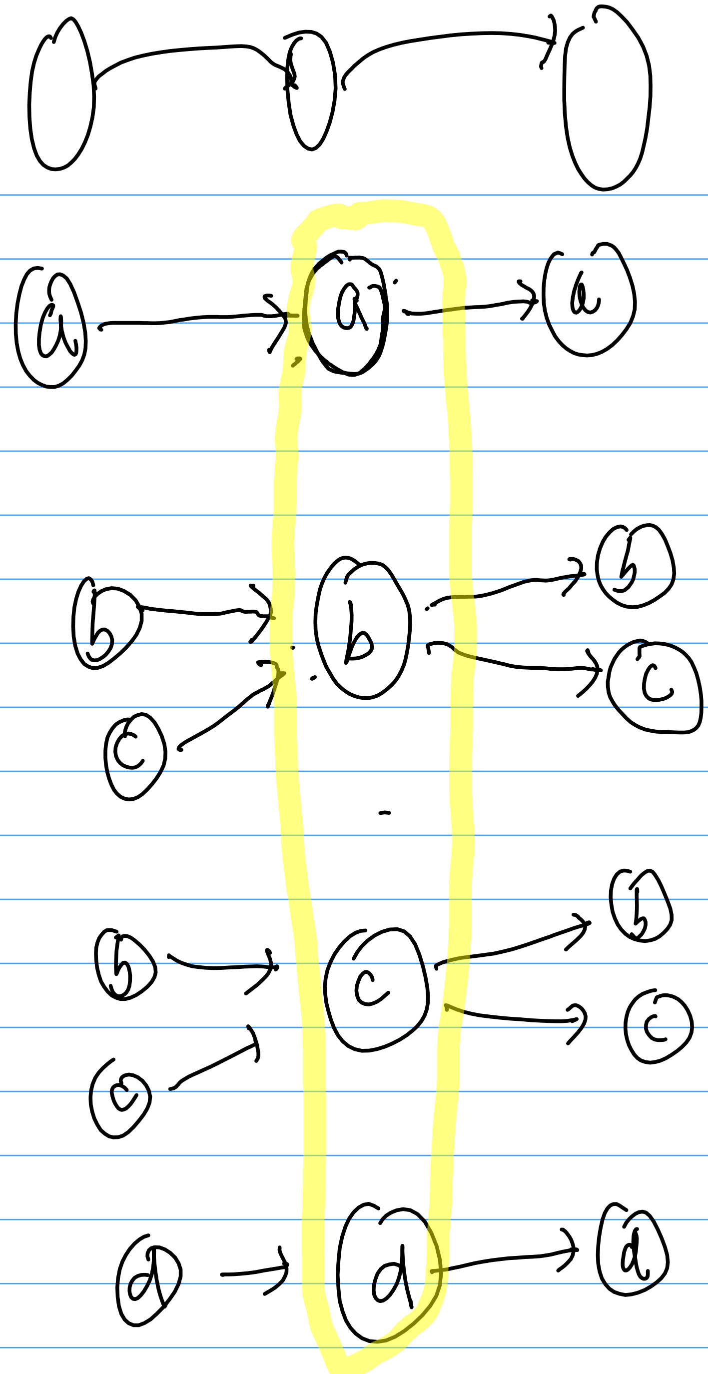
(%o3)
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(%i4)

(%o4)
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 4 & 0 \\ 0 & 4 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(%i5)

(%o5)
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 8 & 8 & 0 \\ 0 & 8 & 8 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$




$a \to a$

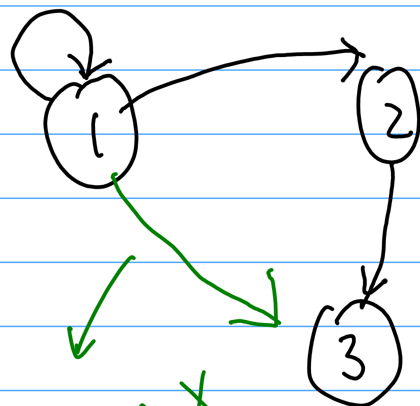
$b \to b$
$b \to c$
$c \to b$
$c \to c$

$b \to b$
$b \to c$
$c \to b$
$c \to c$

$d \to d$

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$R = \{ (1,1), (1,2), (2,3) \}$$




Transitive X

$1 \longrightarrow 1 \longrightarrow 1$ $\quad$ $1 \to 1$

$1 \longrightarrow 2 \longrightarrow 3$ $\quad$ $1 \to 3$

$2 \longrightarrow 3$ $\quad$ X $\quad$ X

$1 \to 3$ X $\qquad 1 \to 2 \to 3$

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

new

$$AA = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\underline{1 \to 2} \qquad \underline{2 \to 3}$

$1 \longrightarrow 3$

$$AAA = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A + AA + AAA = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

# Transitivity Test

$$A = \begin{array}{cc} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} * & * & * & * \\ * & \boxed{*} & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} \end{array}$$

$$A^2 = \begin{array}{cc} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} * & * & * & * \\ a & b & c & d \\ * & * & * & * \\ * & * & * & * \end{bmatrix} \end{array} \begin{array}{cc} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} * & e & * & * \\ * & f & * & * \\ * & g & * & * \\ * & h & * & * \end{bmatrix} \end{array} = \begin{array}{cc} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} * & * & * & * \\ * & \boxed{*} & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} \end{array}$$

$$nonzero\ (i,j)^{th}\ element\ of\ A^2 \quad \Rightarrow \quad nonzero\ (i,j)^{th}\ element\ of\ A$$

| $ae = 1$ | $bf = 1$ | $cg = 1$ | $dh = 1$ |
|---|---|---|---|
| $(2,1) \in R$ | $(2,2) \in R$ | $(2,3) \in R$ | $(2,4) \in R$ |
| $(1,2) \in R$ | $(2,2) \in R$ | $(3,2) \in R$ | $(4,2) \in R$ |

$$\Rightarrow (2,2) \in R$$

# Binary Relations and Digraphs

$$A = \{0,1,2,3,4,5,6\}$$

$$R \subset A \times A$$

$$R = \{(a,b) \mid a \equiv b \ (mod\ 3)\}$$

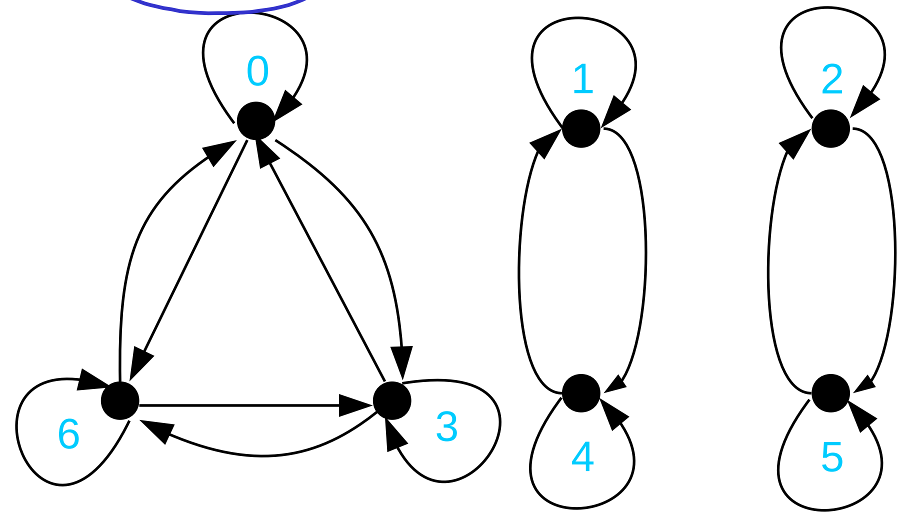
나머지 0 {3, 6, 9, ....}

1 {1, 4, 7, ....}

2 {2, 5, 8, ....}

{0, 3, 6} {1, 4} {2, 5}

$$R = \begin{array}{c|c} & \begin{matrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix} & \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$




http://www.math.fsu.edu/~pkirby/mad2104/SlideShow/s7_1.pdf

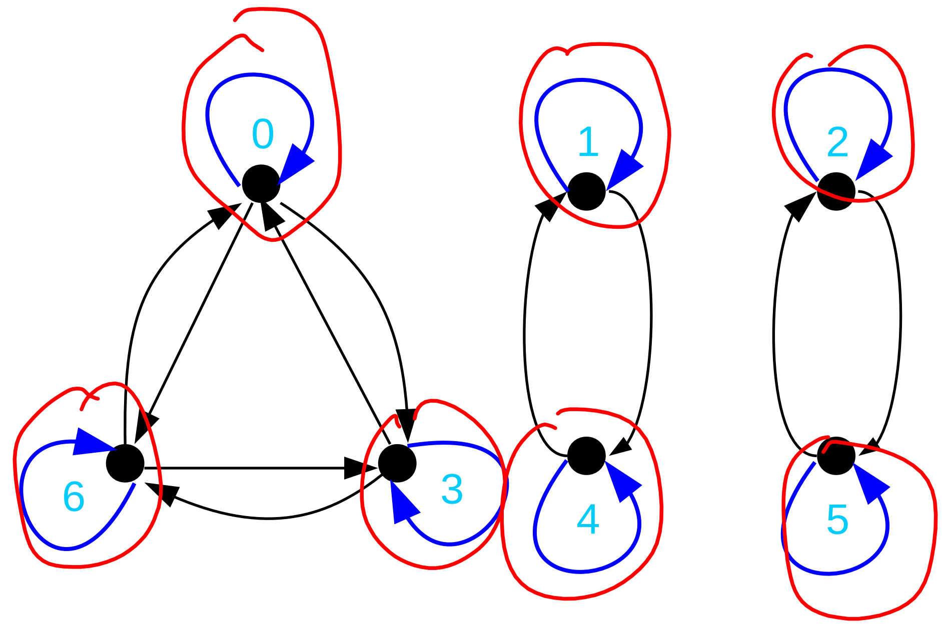
# Reflexive Relation

$$A = \{0,1,2,3,4,5,6\}$$

$$R \subset A \times A$$

$$R = \{(a,b) \mid a \equiv b \pmod{3}\}$$

$$R = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{ccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$



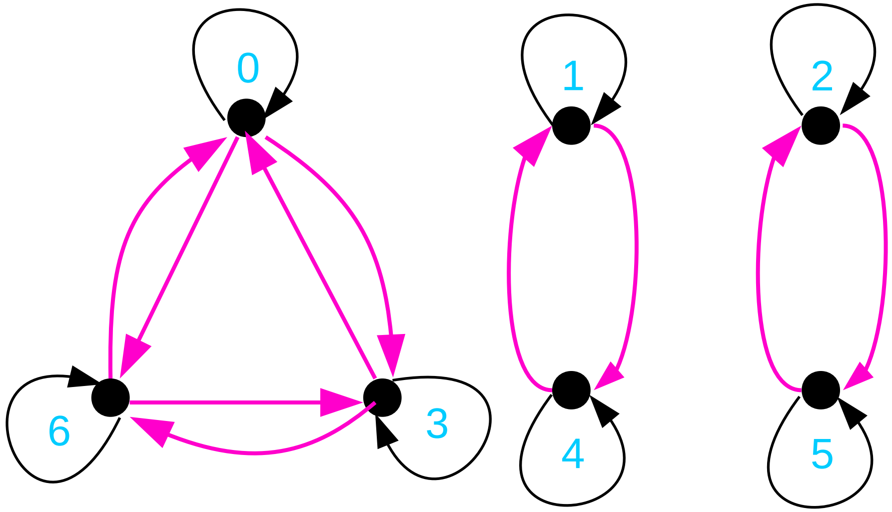
# Symmetric Relation

$$A = \{0,1,2,3,4,5,6\}$$

$$R \subset A \times A$$

$$R = \{(a,b) \mid a \equiv b \pmod 3\}$$

$$R = \begin{array}{c c} & \begin{array}{ccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} & \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$



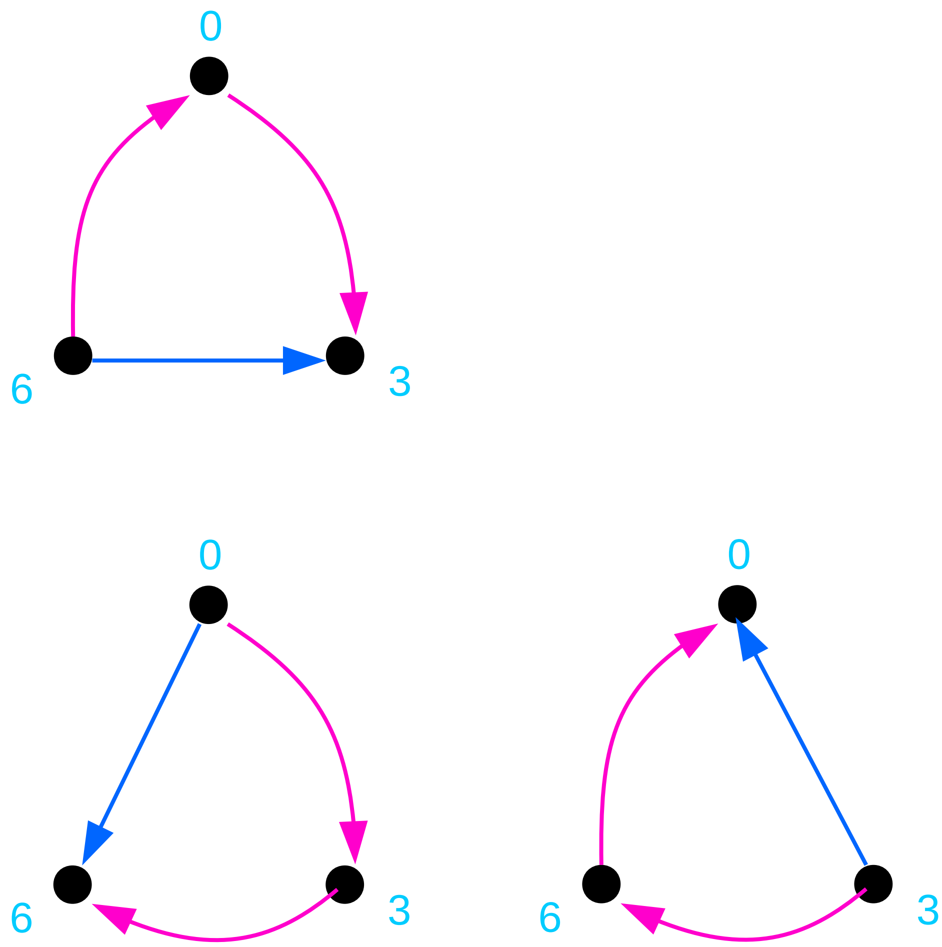
# Transitive Relation

$$A = \{0, 1, 2, 3, 4, 5, 6\}$$

$$R \subset A \times A$$

$$R = \{(a, b) \mid a \equiv b \ (mod\ 3)\}$$

$$RR = \begin{array}{c|c} & \begin{array}{ccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} & \begin{bmatrix} 3 & 0 & 0 & 3 & 0 & 0 & 3 \\ 0 & 2 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 2 & 0 \\ 3 & 0 & 0 & 3 & 0 & 0 & 3 \\ 0 & 2 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 2 & 0 \\ 3 & 0 & 0 & 3 & 0 & 0 & 3 \end{bmatrix} \end{array}$$



# Transitive Relation

```
(%i2) R:matrix(
        [1,0,0,1,0,0,1],
        [0,1,0,0,1,0,0],
        [0,0,1,0,0,1,0],
        [1,0,0,1,0,0,1],
        [0,1,0,0,1,0,0],
        [0,0,1,0,0,1,0],
        [1,0,0,1,0,0,1]
      );
```

(%o2) $$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

```
(%i4) R2: R.R;
```

(%o4) $$\begin{bmatrix} 3 & 0 & 0 & 3 & 0 & 0 & 3 \\ 0 & 2 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 2 & 0 \\ 3 & 0 & 0 & 3 & 0 & 0 & 3 \\ 0 & 2 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 2 & 0 \\ 3 & 0 & 0 & 3 & 0 & 0 & 3 \end{bmatrix}$$

```
(%i7) R3: R.R.R;
```

(%o7) $$\begin{bmatrix} 9 & 0 & 0 & 9 & 0 & 0 & 9 \\ 0 & 4 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 4 & 0 \\ 9 & 0 & 0 & 9 & 0 & 0 & 9 \\ 0 & 4 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 4 & 0 \\ 9 & 0 & 0 & 9 & 0 & 0 & 9 \end{bmatrix}$$

(%i11)

(%o11)
$$\begin{bmatrix} 27 & 0 & 0 & 27 & 0 & 0 & 27 \\ 0 & 8 & 0 & 0 & 8 & 0 & 0 \\ 0 & 0 & 8 & 0 & 0 & 8 & 0 \\ 27 & 0 & 0 & 27 & 0 & 0 & 27 \\ 0 & 8 & 0 & 0 & 8 & 0 & 0 \\ 0 & 0 & 8 & 0 & 0 & 8 & 0 \\ 27 & 0 & 0 & 27 & 0 & 0 & 27 \end{bmatrix}$$

(%i12)

(%o12)
$$\begin{bmatrix} 81 & 0 & 0 & 81 & 0 & 0 & 81 \\ 0 & 16 & 0 & 0 & 16 & 0 & 0 \\ 0 & 0 & 16 & 0 & 0 & 16 & 0 \\ 81 & 0 & 0 & 81 & 0 & 0 & 81 \\ 0 & 16 & 0 & 0 & 16 & 0 & 0 \\ 0 & 0 & 16 & 0 & 0 & 16 & 0 \\ 81 & 0 & 0 & 81 & 0 & 0 & 81 \end{bmatrix}$$

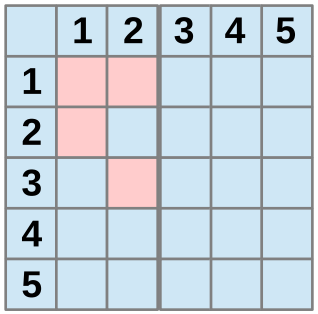
(%i13)

(%o13)
$$\begin{bmatrix} 243 & 0 & 0 & 243 & 0 & 0 & 243 \\ 0 & 32 & 0 & 0 & 32 & 0 & 0 \\ 0 & 0 & 32 & 0 & 0 & 32 & 0 \\ 243 & 0 & 0 & 243 & 0 & 0 & 243 \\ 0 & 32 & 0 & 0 & 32 & 0 & 0 \\ 0 & 0 & 32 & 0 & 0 & 32 & 0 \\ 243 & 0 & 0 & 243 & 0 & 0 & 243 \end{bmatrix}$$

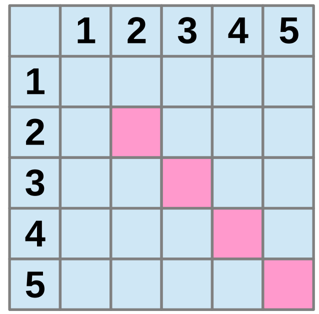
(%i14)

(%o14)
$$\begin{bmatrix} 729 & 0 & 0 & 729 & 0 & 0 & 729 \\ 0 & 64 & 0 & 0 & 64 & 0 & 0 \\ 0 & 0 & 64 & 0 & 0 & 64 & 0 \\ 729 & 0 & 0 & 729 & 0 & 0 & 729 \\ 0 & 64 & 0 & 0 & 64 & 0 & 0 \\ 0 & 0 & 64 & 0 & 0 & 64 & 0 \\ 729 & 0 & 0 & 729 & 0 & 0 & 729 \end{bmatrix}$$

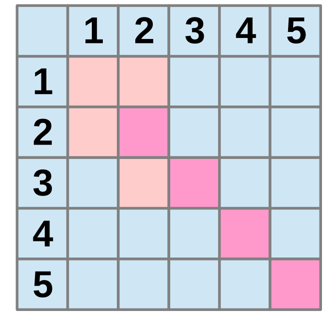
# Reflexive and Symmetric Closure



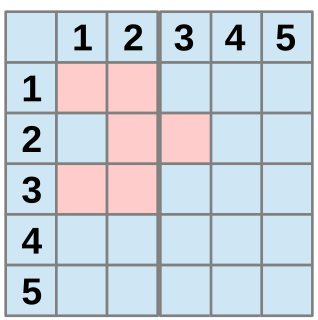
Not Reflexive **R**



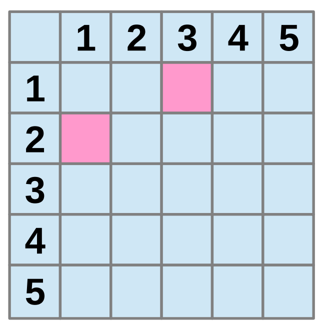
the minimal addition



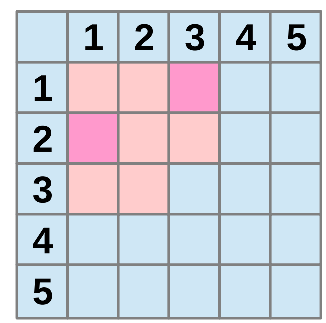
Reflexive Closure of **R**



Not Symmetric **R**



the minimal addition



Symmetric Closure of **R**

# Transitive Closure Examples

```
(%i9) A: matrix(
        [1,0,1],
        [0,1,0],
        [1,1,0]
      );
```

$$(\%o9)\quad \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

```
(%i11) A2: A.A;
```

$$(\%o11)\quad \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

```
(%i12) A3: A.A.A;
```

$$(\%o12)\quad \begin{bmatrix} 3 & 2 & 2 \\ 0 & 1 & 0 \\ 2 & 2 & 1 \end{bmatrix}$$

```
(%i13) A+A2+A3;
```

$$(\%o13)\quad \begin{bmatrix} 6 & 3 & 4 \\ 0 & 3 & 0 \\ 4 & 4 & 2 \end{bmatrix}$$

```
(%i18) A4: A.A.A.A;
```

$$(\%o18)\quad \begin{bmatrix} 5 & 4 & 3 \\ 0 & 1 & 0 \\ 3 & 3 & 2 \end{bmatrix}$$

```
(%i20) A5: A.A.A.A.A;
```

$$(\%o20)\quad \begin{bmatrix} 8 & 7 & 5 \\ 0 & 1 & 0 \\ 5 & 5 & 3 \end{bmatrix}$$

```
(%i19) matrix(
         [1,1,1],
         [0,1,0],
         [1,1,1]
       );
```

$$(\%o19)\quad \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

transitive closure

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

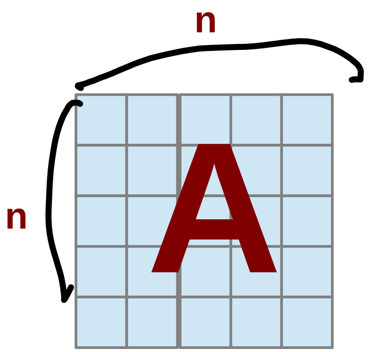
$$\Uparrow$$

$$\begin{bmatrix} 6 & 3 & 4 \\ 0 & 3 & 0 \\ 4 & 4 & 2 \end{bmatrix}$$

$$\Uparrow$$

$$A + A2 + A3 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 2 & 2 \\ 0 & 1 & 0 \\ 2 & 2 & 1 \end{bmatrix}$$

# Transitive Closure




R

$$R^* = \cup_{n=1}^{\infty} R^n$$

$$= R \cup R^2 \cup \cdots \cup R^n$$

$$A \vee A^2 \vee \cdots \vee A^n$$

$$\boxed{A + A^2 + \cdots + A^n}$$

then non-zero → 1

→ transitive closure

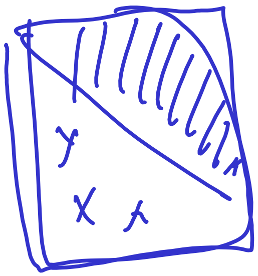
# Equivalence Relation

Equivalence Relation $\Leftrightarrow$ Reflexive Relation & Symmetric Relation & Transitive Relation

$$A = \{0,1,2,3,4,5,6,7,8\}$$

$$R \subset A \times A$$

$$R = \{(a,b) \mid a \equiv b \ (mod\ 3)\}$$




Ref
cnt.
Tr

anti symmetric

partial ordering

$(1,1) \in R \qquad a \equiv 1 \ (mod\ 3)$

$(1,4) \in R \qquad 1 \equiv 4 \ (mod\ 3)$

$(4,1) \in R \qquad 4 \equiv 1 \ (mod\ 3)$

$(1,4) \in R \qquad 1 \equiv 4 \ (mod\ 3)$

$(4,7) \in R \qquad 4 \equiv 7 \ (mod\ 3)$

$(1,7) \in R \qquad 1 \equiv 7 \ (mod\ 3)$

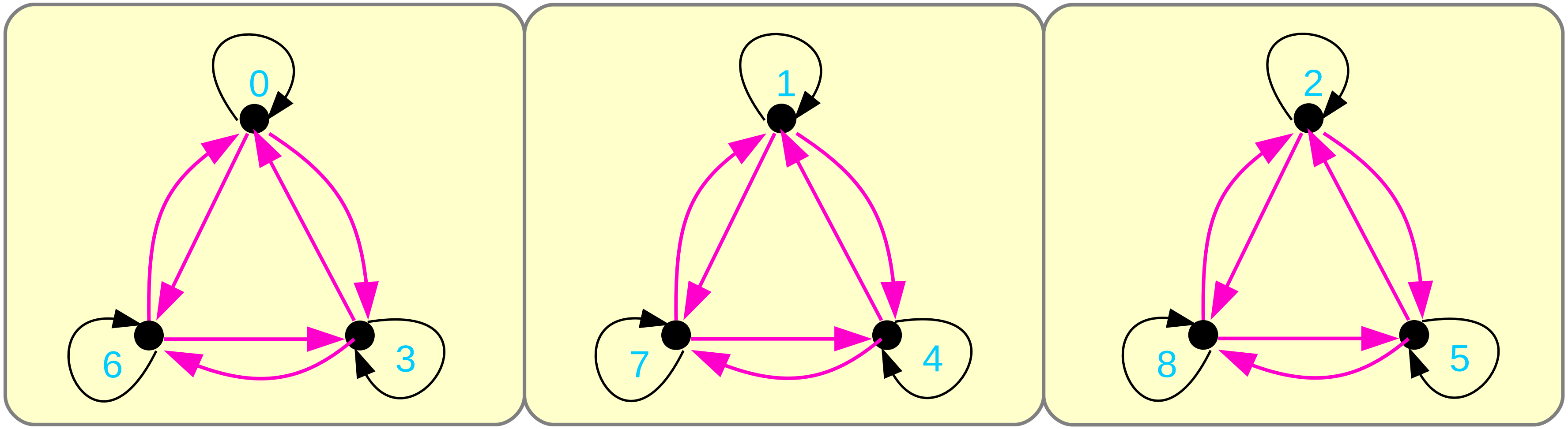
# Equivalence Relation Examples

$$A = \{0,1,2,3,4,5,6,7,8\}$$

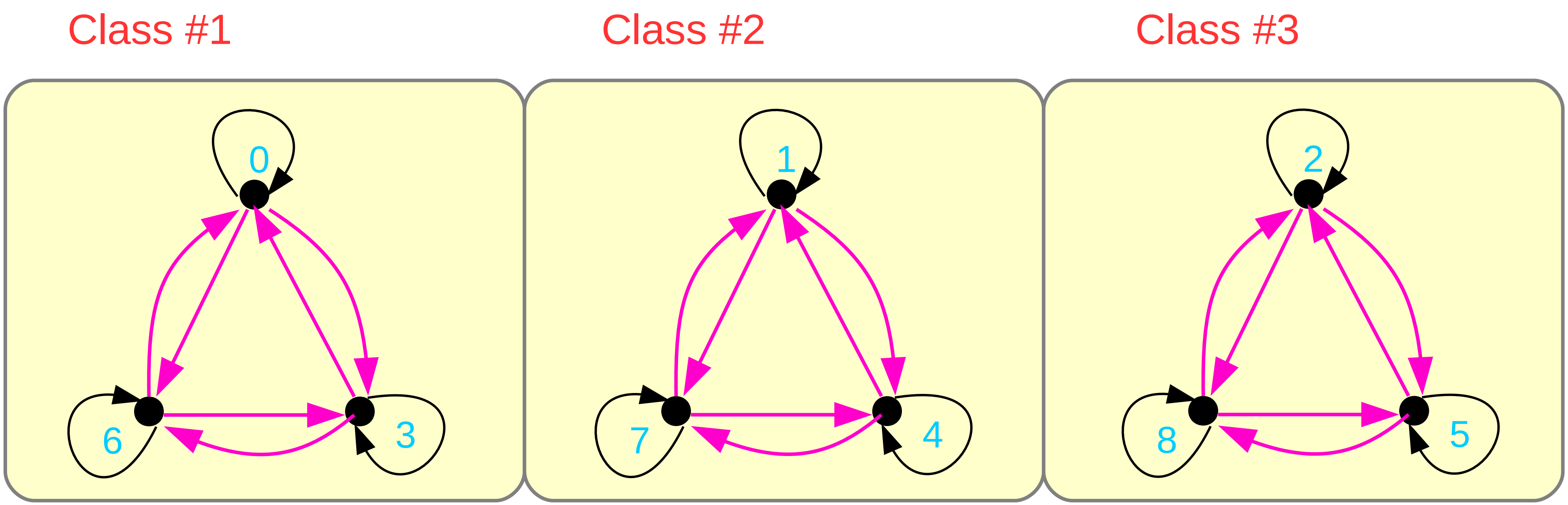
$$R \subset A \times A$$

$$R = \{(a,b) \mid a \equiv b \ (mod\ 3)\}$$

$$RR = \begin{array}{c|ccccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \hline 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 3 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 4 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 5 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 6 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 7 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 8 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{array}$$



# Equivalence Classes




$$\begin{array}{c c} & \begin{array}{ccc} 0 & 3 & 6 \end{array} \\ \begin{array}{c} 0 \\ 3 \\ 6 \end{array} & \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \end{array} \qquad \begin{array}{c c} & \begin{array}{ccc} 1 & 4 & 7 \end{array} \\ \begin{array}{c} 1 \\ 4 \\ 7 \end{array} & \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \end{array} \qquad \begin{array}{c c} & \begin{array}{ccc} 2 & 5 & 8 \end{array} \\ \begin{array}{c} 2 \\ 5 \\ 8 \end{array} & \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \end{array}$$

partition $P_1 = \{0, 3, 6\}$ partition $P_2 = \{1, 4, 7\}$ partition $P_3 = \{2, 5, 8\}$

# Equivalence Class

$$A \;=\; \boldsymbol{Z}^{+} \;=\; \{0, 1, 2, 3, 4, 5, 6, \cdots\}$$

$$R \;\subset\; A \times A$$

$$R \;=\; \{\, (a, b) \mid a \equiv b \;(mod\ 3)\}$$

| | | |
|---|---|---|
| $\{0,\ 3,\ 6,\ 9,\ \cdots\}$ | $[0]$ | $[33]$ |
| $\{1,\ 4,\ 7,\ 10,\ \cdots\}$ | $[1]$ | $[331]$ |
| $\{2,\ 5,\ 8,\ 11,\ \cdots\}$ | $[2]$ | $[3332]$ |

https://www.cse.iitb.ac.in/~nutan/courses/cs207-12/notes/lec7.pdf

**References**

[1] http://en.wikipedia.org/
[2]

# The Growth of Functions (2A)

Please send corrections (or suggestions) to youngwlim@hotmail.com.

This document was produced by using LibreOffice and Octave.

# Functions and Ranges

$$\left\{\begin{array}{l} \boxed{x^2+2x+1} \\ \quad = \\ x^2 \\ 2x \\ 1 \end{array}\right.$$

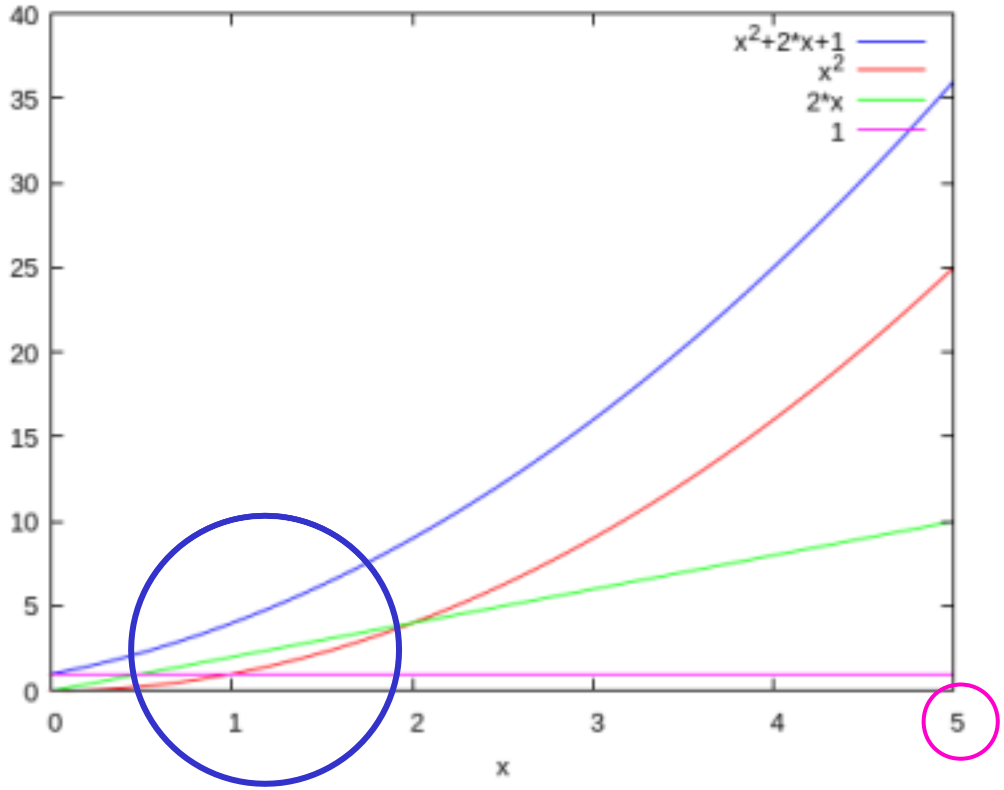
$\Theta(x^2)$

$$A_1 = [0, 5]$$

$$A_2 = [0, 100]$$

$$A_3 = [0, 500]$$

All are distinguishable




$x^2+2x+1$

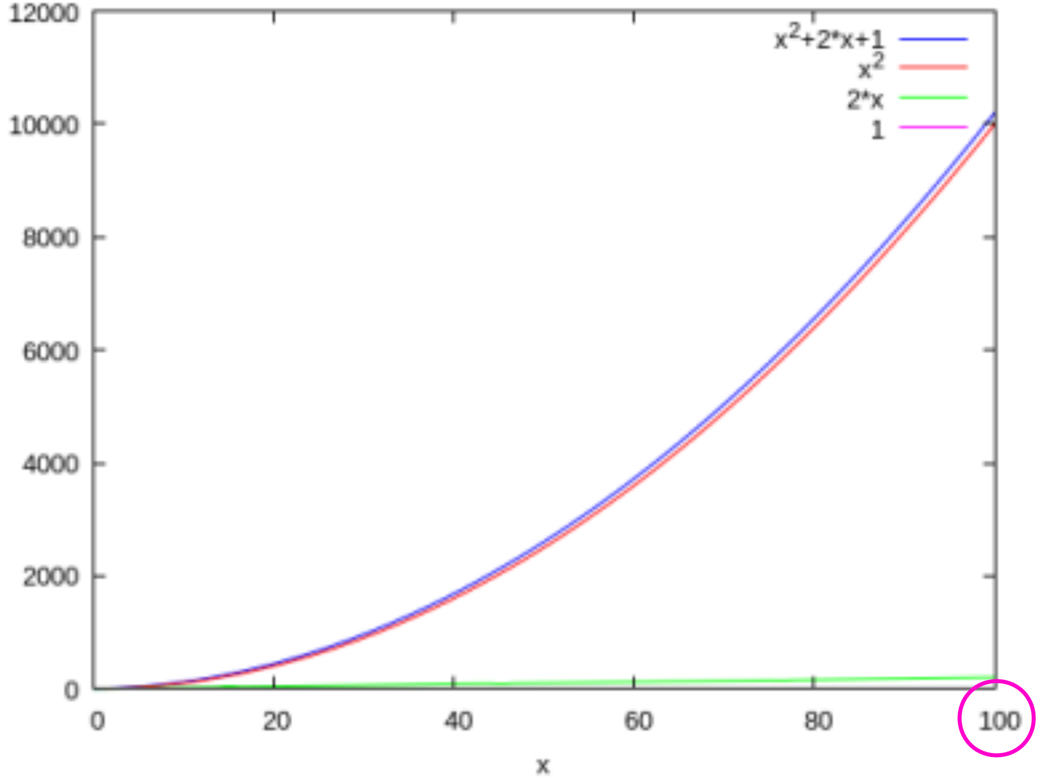
$x^2$

$2x$

$1$

Zoom Out

*for* $x > -0.5$ $\qquad x^2 < x^2+2x+1$

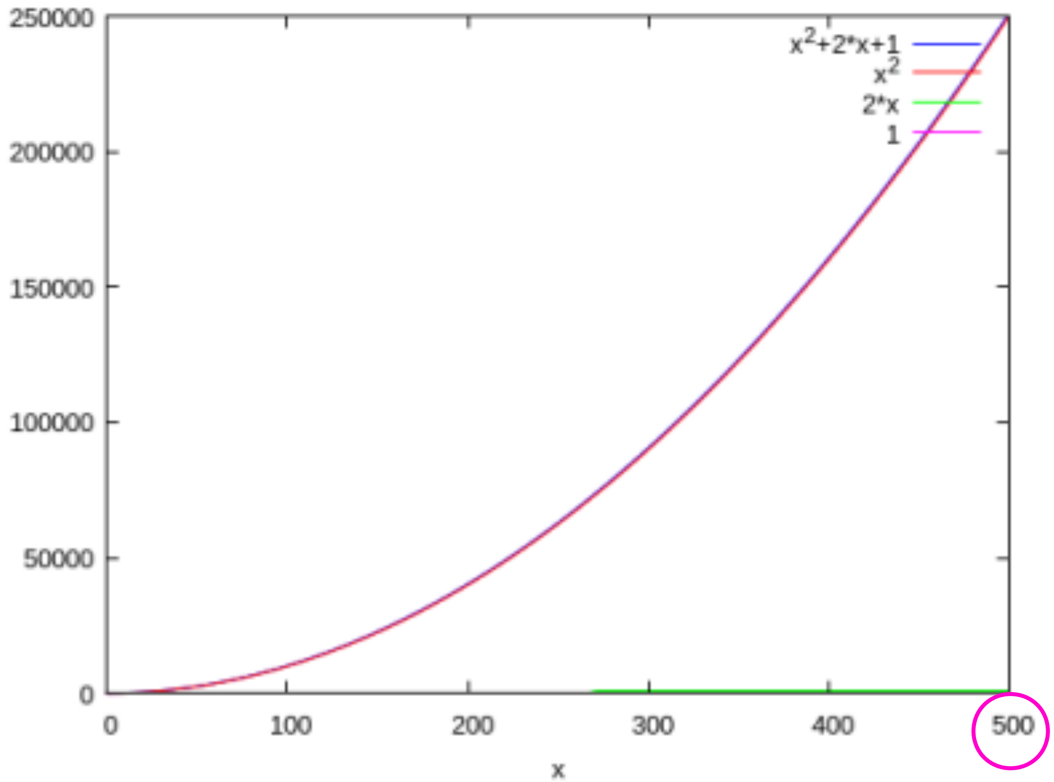



similar

$\begin{cases} x^2+2x+1 \\ x^2 \end{cases}$ $\quad 10000+201$ $\quad 10000$

$2x$ ⇨ Zoom Out More

$for \quad x > -0.5 \qquad x^2 < x^2+2x+1$




Indistinguishable

$$\begin{cases} x^2 + 2x + 1 \\ x^2 \end{cases}$$

$250000 + 1001$

$250000$

$for \quad x > -0.5 \qquad\qquad x^2 < x^2 + 2x + 1$

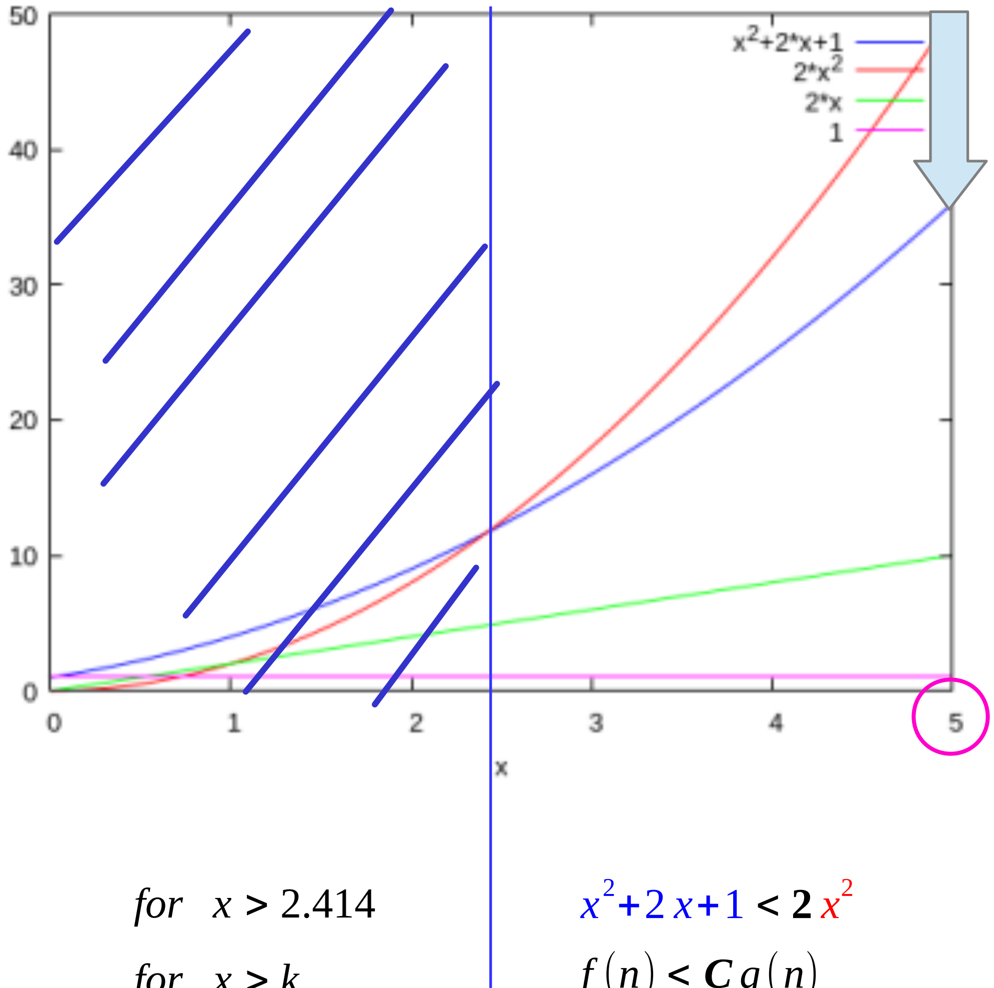
# Functions and Ranges

$$\begin{cases} \boxed{2 \cdot x^2} \\ x^2 + 2x + 1 \\ 2x \\ 1 \end{cases}$$

$$B_1 = [0, 5]$$
$$B_2 = [0, 100]$$
$$B_3 = [0, 500]$$

# Small Range, $2x^2$

distinguishable




$2x^2$ 50

$x^2+2x+1$ $25+11$

$2x$

$1$

Zoom Out

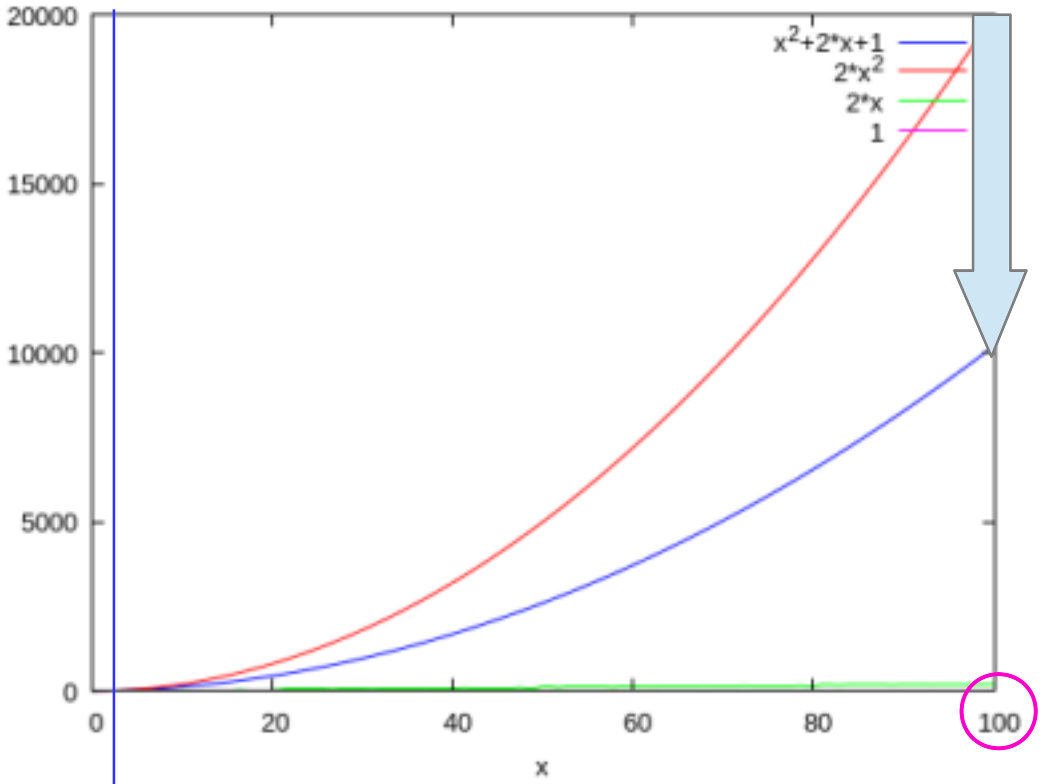
*for* $x > 2.414$ $x^2+2x+1 < 2x^2$

*for* $x > k$ $f(n) < Cg(n)$

# Medium Range, $2x^2$



distinguishable




$2x^2$ 20000

$x^2+2x+1$ 10000+201

$2x$

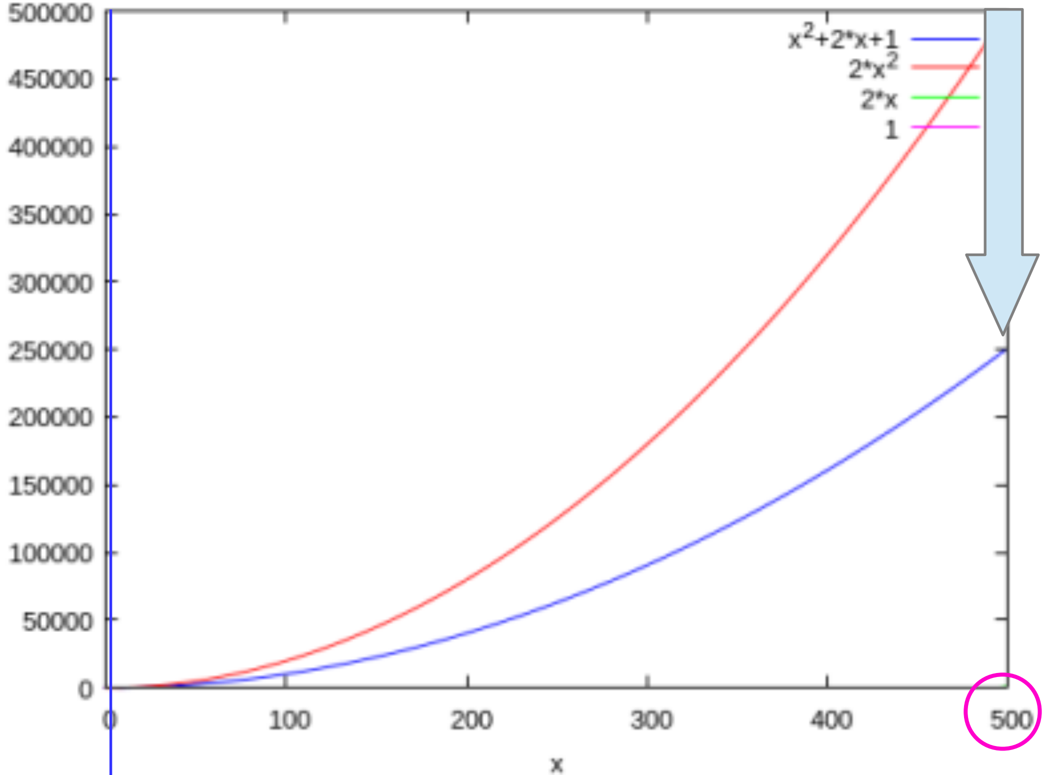
$1$

Zoom Out

*for* $x > 2.414$ $x^2+2x+1 < 2x^2$

*for* $x > k$ $f(n) < C\,g(n)$

# Large Range, $2x^2$




distinguishable

$2x^2$ 500000

$x^2+2x+1$ 250000 + 1001

$2x$

$1$

Zoom Out

$for \quad x > 2.414 \qquad x^2+2x+1 < 2x^2$

$for \quad x > k \qquad f(n) < C g(n)$

# Functions and Ranges

$$\left\{\begin{array}{l} \boxed{10 \cdot x^2} \\ \\ x^2 + 2x + 1 \\ \\ 2x \\ \\ 1 \end{array}\right.$$

$$C_1 = [0, 5]$$

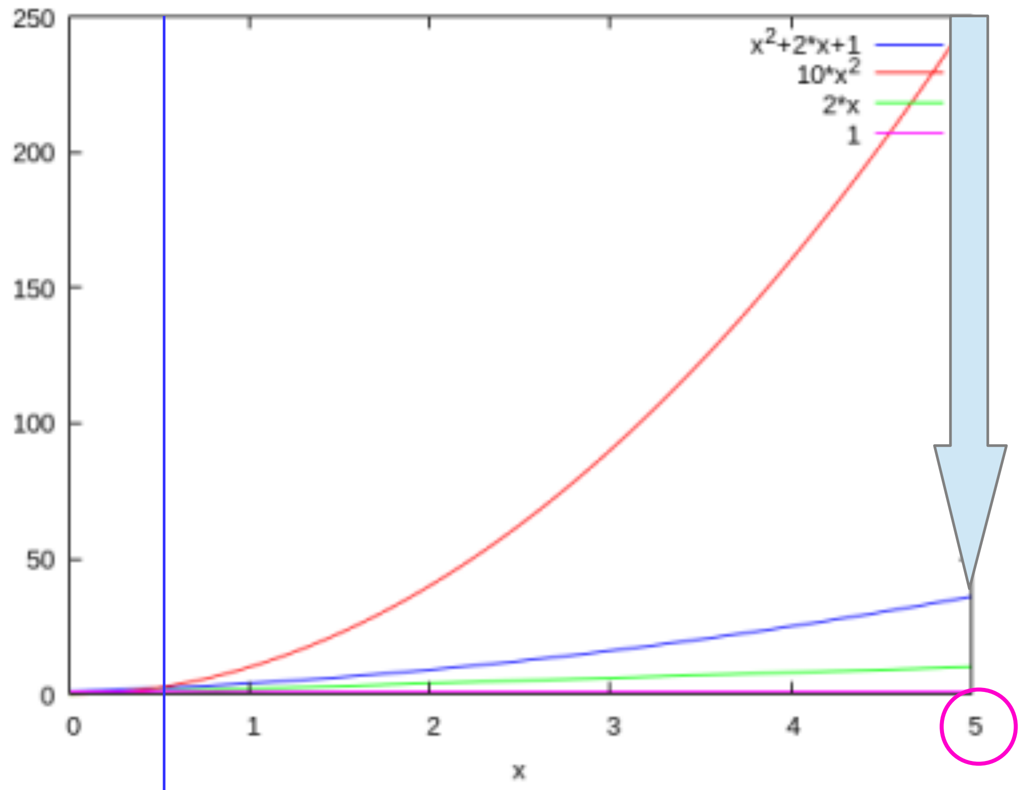
$$C_2 = [0, 100]$$

$$C_3 = [0, 500]$$

# Small Range, $10x^2$



distinguishable




$\mathbf{10}\,x^2$ 250

$x^2+2x+1$ $25+11$

$2x$

$1$

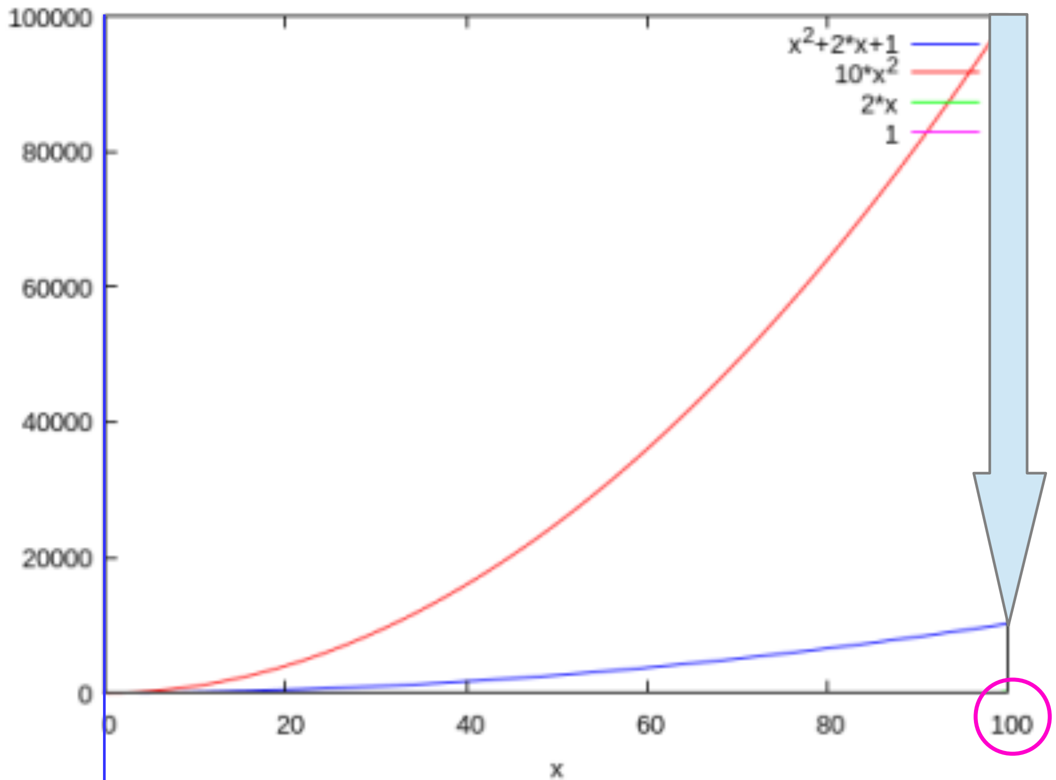
Zoom Out

$for \quad x > 0.462$ $\quad x^2+2x+1 < \mathbf{10}\,x^2$

$for \quad x > k$ $\quad f(n) < \boldsymbol{C}\,g(n)$

# Medium Range, $10x^2$

distinguishable




$10\,x^2$ 100000

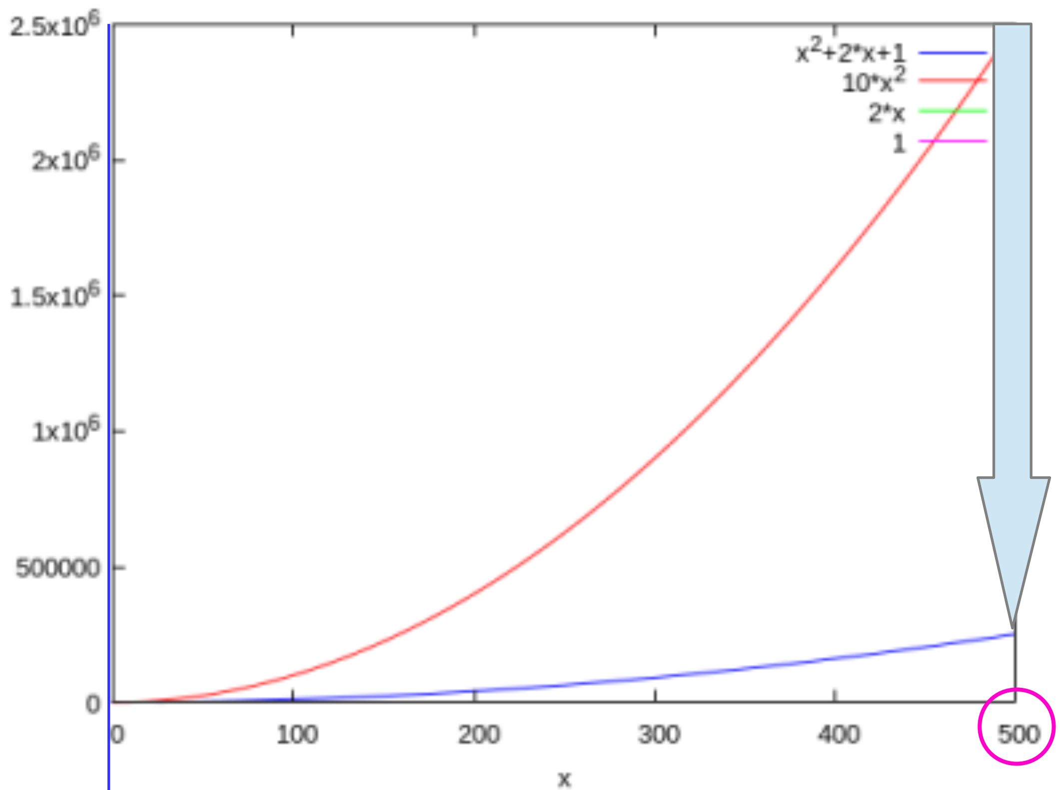
$x^2+2\,x+1$ $10000+201$

Zoom Out

for $x > 0.462$ $x^2+2\,x+1 < \mathbf{10}\,x^2$

for $x > k$ $f(n) < \boldsymbol{C}\,g(n)$

distinguishable




$\mathbf{10}\,x^2$ 2500000

$x^2+2x+1$ $250000+1001$

$for \quad x > 0.462 \qquad x^2+2x+1 < \mathbf{10}\,x^2$

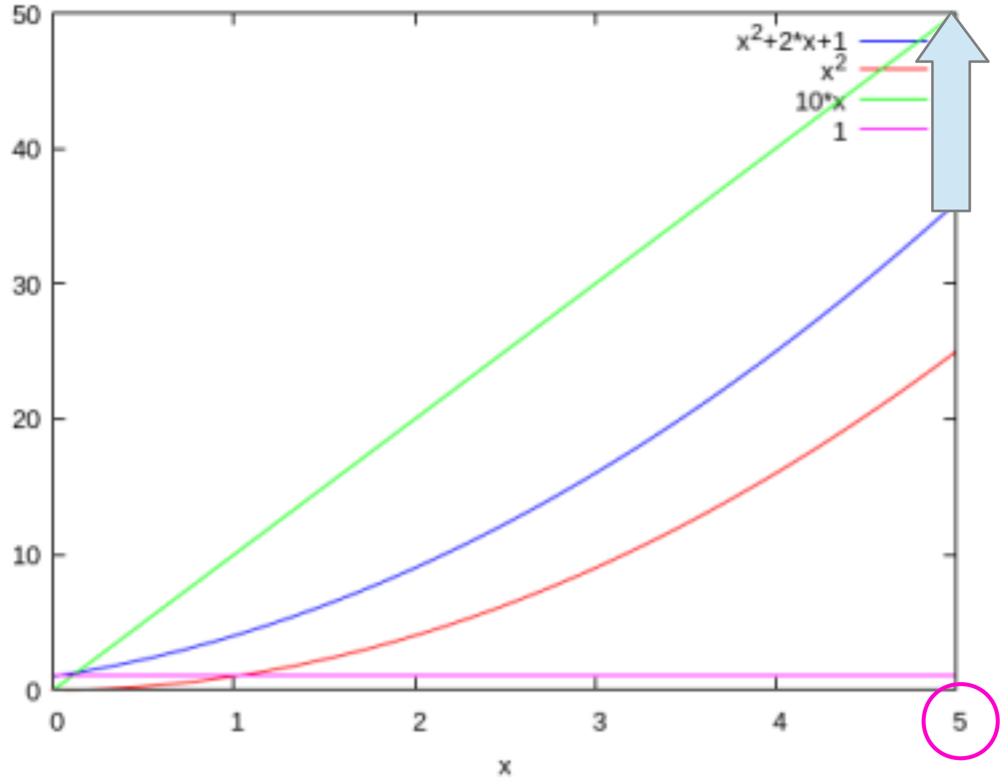
$for \quad x > k \qquad f(n) < \boldsymbol{C}\,g(n)$

$$\begin{cases} \boxed{10 \cdot x} \\ x^2 + 2x + 1 \\ x^2 \\ 1 \end{cases}$$

$$D_1 = [0, 5]$$

$$D_2 = [0, 100]$$

$$D_3 = [0, 500]$$




$\mathbf{10}\,x$

$x^2+2x+1$

distinguishable

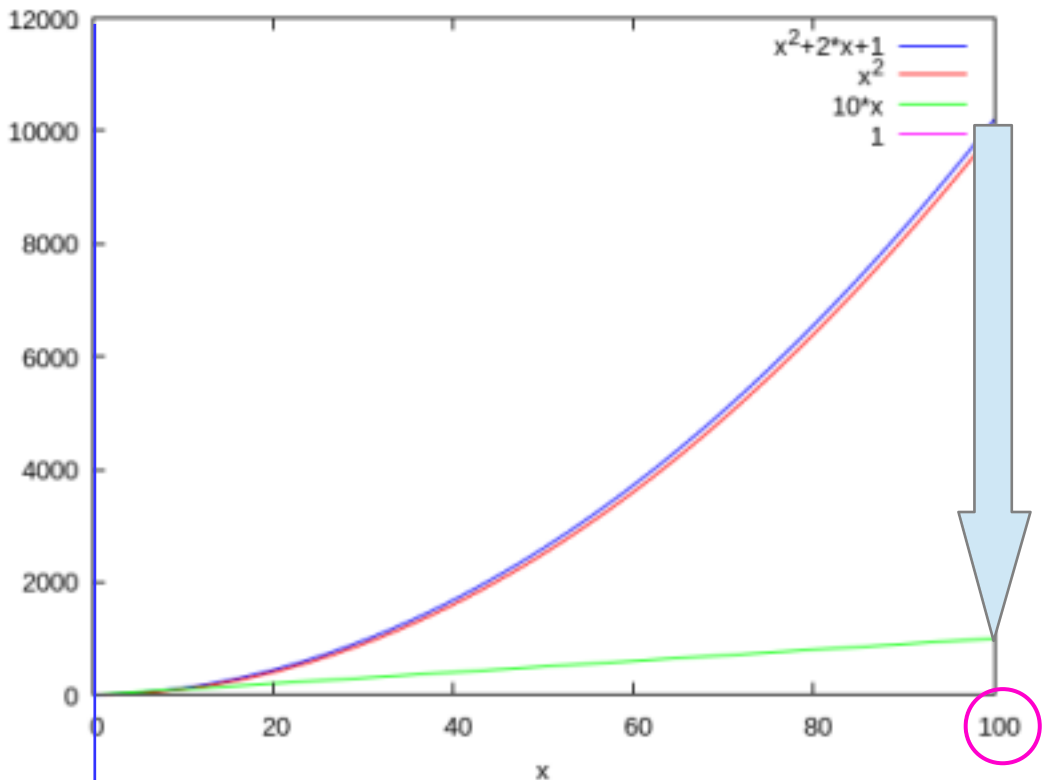
$x^2$

$1$

Zoom Out

*for* $0.127 < x < 7.873 \qquad x^2+2x+1 < \mathbf{10}\,x$

# Medium Range, *10x*




$x^2+2x+1$

$x^2$

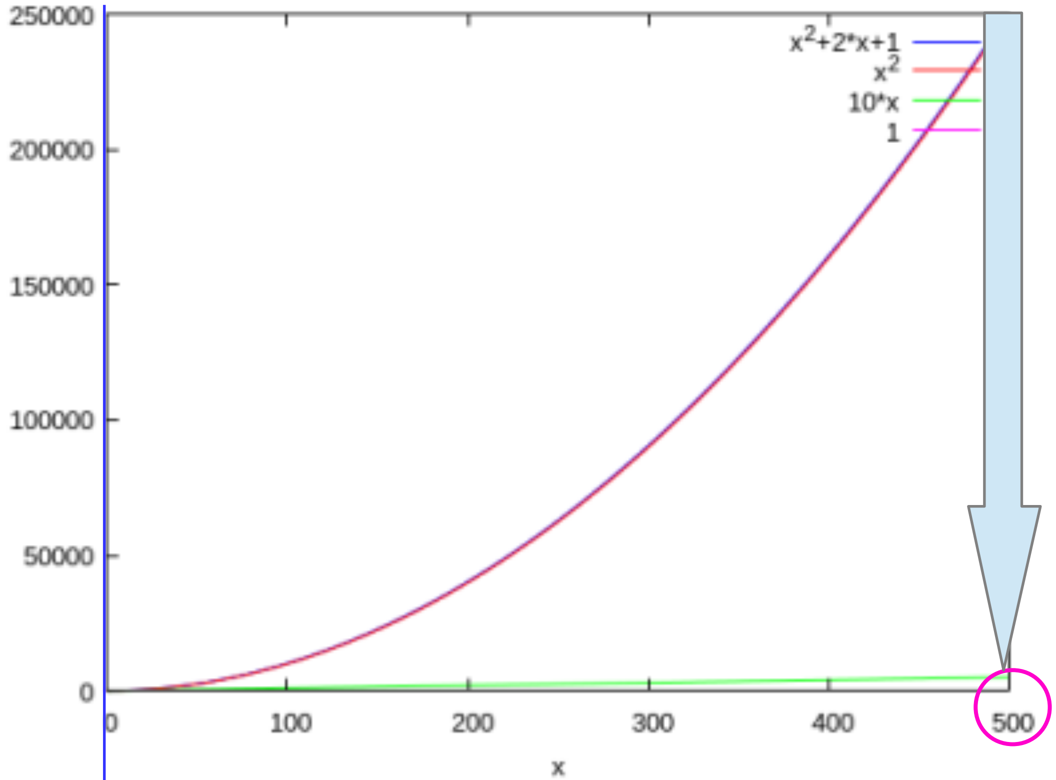
indistinguishable

$\mathbf{10}\,x$

Zoom Out

*for* $x > 7.873$ $\qquad \mathbf{10}\,x < x^2+2x+1$

*for* $x > k$ $\qquad \mathbf{C}\,g(n) < f(n)$




$x^2+2x+1$

$x^2$

indistinguishable

$\mathbf{10}\,x$

*for* $x > 7.873$ $\qquad$ $\mathbf{10}\,x < x^2+2x+1$

*for* $x > k$ $\qquad$ $\mathbf{C}\,g(n) < f(n)$

# Big-**O** Definition

Let f and g be functions (Z→R or R→R)
from the set of integers or
the set of real numbers
to the set of real numbers.

We say f(x) is **O(**g(x)**)** "f(x) is **big-oh** of g(x)"

If there are constants C and k such that

$$|f(x)| \le C|g(x)| \quad \text{whenever } x > k.$$

g(x) : **upper bound** of f(x)

# Big-Ω Definition

Let f and g be functions (Z→R or R→R)
from the set of integers or
the set of real numbers
to the set of real numbers.

We say f(x) is **Ω(**g(x)**)** "f(x) is **big-omega** of g(x)"

If there are constants C and k such that

$C|g(x)| \leq |f(x)|$ whenever $x > k$.

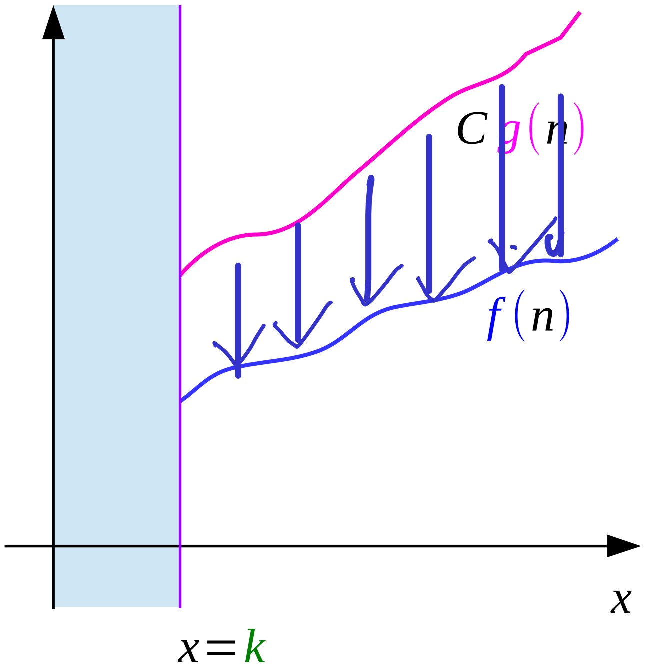
g(x) : **lower bound** of f(x)

# Big-**O** Definition

$$\textit{for} \quad k < x$$

$$f(x) \le C|g(x)|$$

$$f(x) \;\; \textit{is} \;\; \boldsymbol{O}(g(x))$$




g(x) : **upper bound** of f(x)

g(x) has a simpler form than f(x)

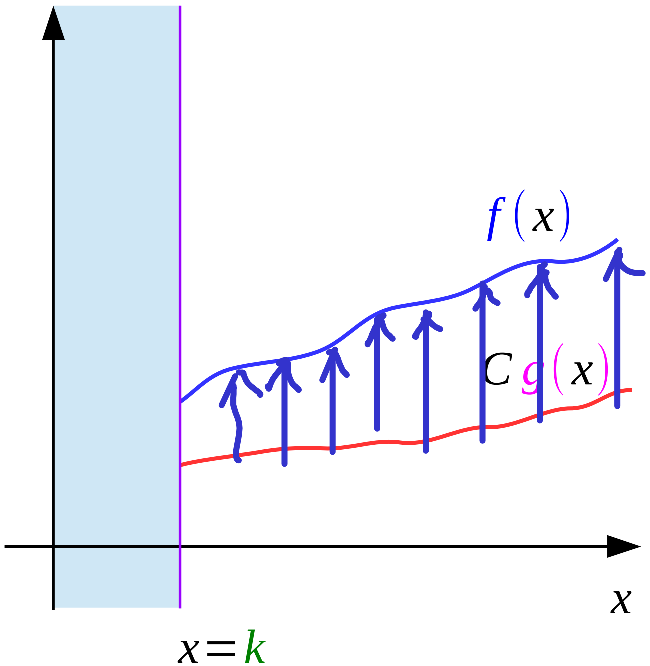
is usually a single term

# Big-Ω Definition

$$\text{for} \quad k < x$$

$$f(x) \geq C|g(x)|$$

$$f(x) \text{ is } \Omega(g(x))$$




g(x) : **lower bound** of f(x)

g(x) has a simpler form than f(x)
is usually a single term

# Big-Θ definition

$$for \quad k < x$$

$$f(x) \le C|g(x)| \iff f(x) \text{ is } O(g(x))$$

$$C|g(x)| \le f(x) \iff f(x) \text{ is } \Omega(g(x))$$

$$\underbrace{C_1|g(x)| \le f(x)}_{\Omega(g(x))} \underbrace{\le C_2|g(x)|}_{O(g(x))} \iff f(x) \text{ is } \Theta(g(x))$$

AND

# Big-Θ = Big-Ω ∩ Big-O

$for \quad k < x$

$0.1x^2 < x^2 + 2x + 1 < 10x^2$

$\Omega(x^2) \Rightarrow \Theta(x^2) \Leftarrow O(x^2)$

$$\underbrace{C_1|g(x)| \le \overbrace{f(x) \le C_2|g(x)|}^{O(g(x))}}_{\Omega(g(x))} \iff f(x) \text{ is } \Theta(g(x))$$

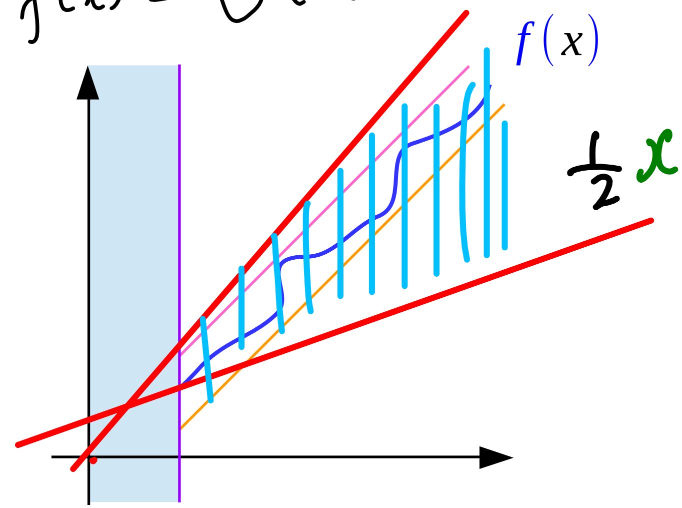
$$\Omega(g(x)) \wedge O(g(x)) \iff \Theta(g(x))$$

# $\Theta(x)$ and $\Theta(1)$

*for* $0 < k < x$

$\frac{1}{2}x < f(x) < 2x$

$f(x) = \Theta(x)$




$f(x) \le C\,x \iff f(x)$ *is* $O(x)$

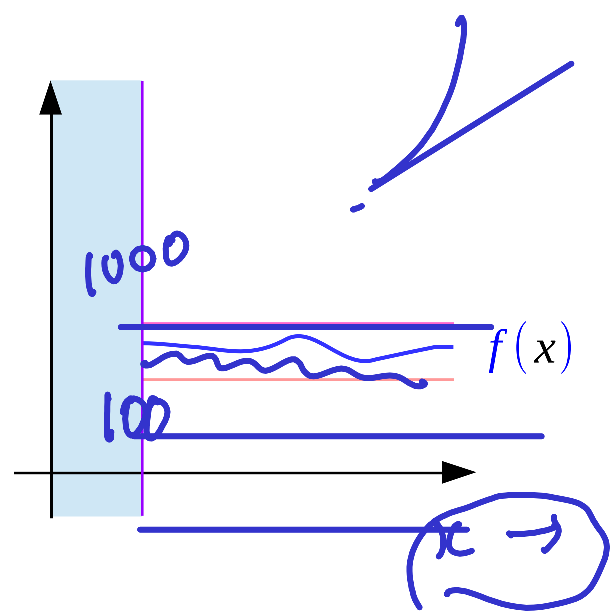
$C\,x \le f(x) \iff f(x)$ *is* $\Omega(x)$

$C_1 x \le f(x) \le C_2 x \iff f(x)$ *is* $\Theta(x)$

$f(x) \le C \cdot 1 \iff f(x)$ *is* $O(1)$

$C \cdot 1 \le f(x) \iff f(x)$ *is* $\Omega(1)$

$C_1 \cdot 1 \le f(x) \le C_2 \cdot 1 \iff f(x)$ *is* $\Theta(1)$



# Big-O, Big-Ω, Big-Θ Examples

$for \quad x > -0.5 \qquad 1x^2 < x^2+2x+1 \qquad x^2+2x+1 \;\; is \;\; \Omega(x^2)$

lower bound

$x^2+2x+1 \;\; is \;\; \Theta(x^2)$

upper bound

$for \quad x > 2.414 \qquad x^2+2x+1 < 2x^2 \qquad x^2+2x+1 \;\; is \;\; O(x^2)$

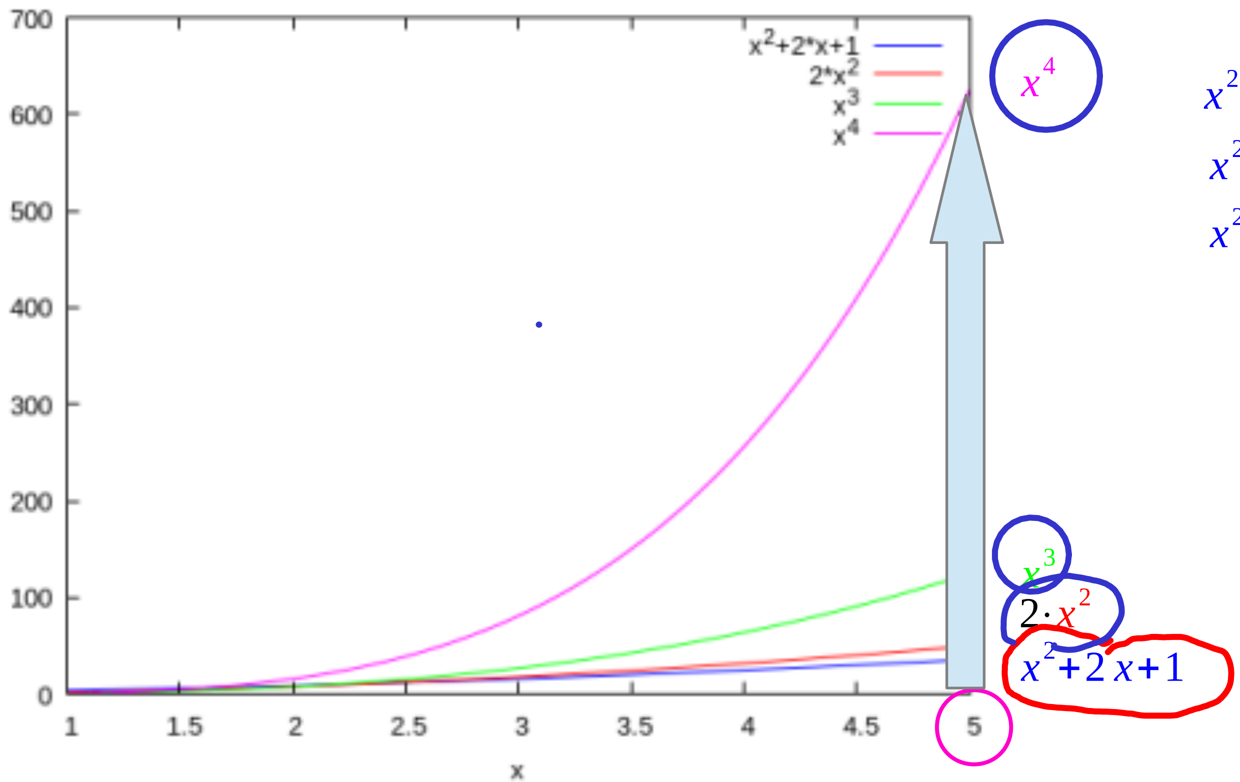
upper bound

$for \quad x > 0.462 \qquad x^2+2x+1 < 10x^2 \qquad x^2+2x+1 \;\; is \;\; O(x^2)$

$for \quad x > 7.873 \qquad 10x < x^2+2x+1 \qquad x^2+2x+1 \;\; is \;\; \Omega(x)$

lower bound

# Many Larger Upper Bounds




$x^2+2x+1$ *is* $\boldsymbol{O}(x^2)$

$x^2+2x+1$ *is* $\boldsymbol{O}(x^3)$

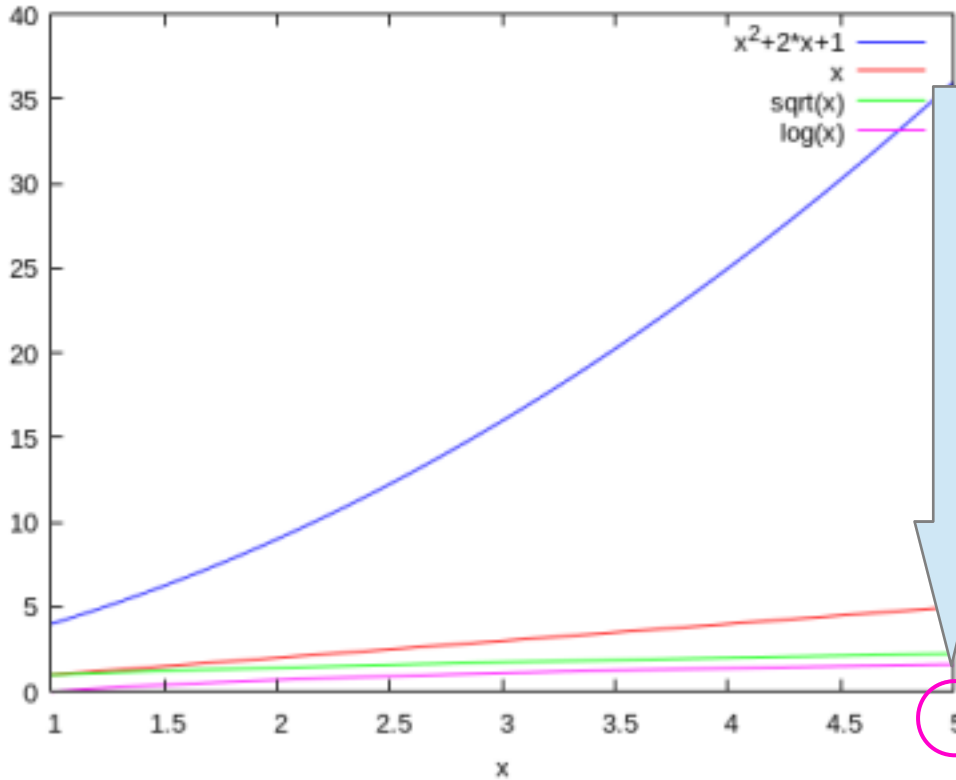
$x^2+2x+1$ *is* $\boldsymbol{O}(x^4)$

•
•
•

$x^2 < x^3 < x^4$

the <u>least</u> upper bound?

# Many Smaller Lower Bounds




$x^2+2x+1$

$x$

$\sqrt{x}$

$\log x$

$\underline{x^2+2x+1}$ *is* $\mathbf{\Omega}(x^2)$

$x^2+2x+1$ *is* $\mathbf{\Omega}(x)$

$x^2+2x+1$ *is* $\mathbf{\Omega}(\sqrt{x})$

$x^2+2x+1$ *is* $\mathbf{\Omega}(\log x)$

$\vdots$

$x^2 > x > \sqrt{x} > \lg x$

the <u>greatest</u> lower bound?

# Many Upper and Lower Bounds

| | | | |
|---|---|---|---|
| $x^2+2x+1$ *is* $O(x^2)$ | $\Leftrightarrow$ | $x^2+2x+1 \le Cx^2$ | upper bound |
| $x^2+2x+1$ *is* $O(x^3)$ | $\Leftrightarrow$ | $x^2+2x+1 \le Cx^3$ | upper bound |
| $x^2+2x+1$ *is* $O(x^4)$ | $\Leftrightarrow$ | $x^2+2x+1 \le Cx^4$ | upper bound |
| $\vdots$ | | $\vdots$ | |

the least

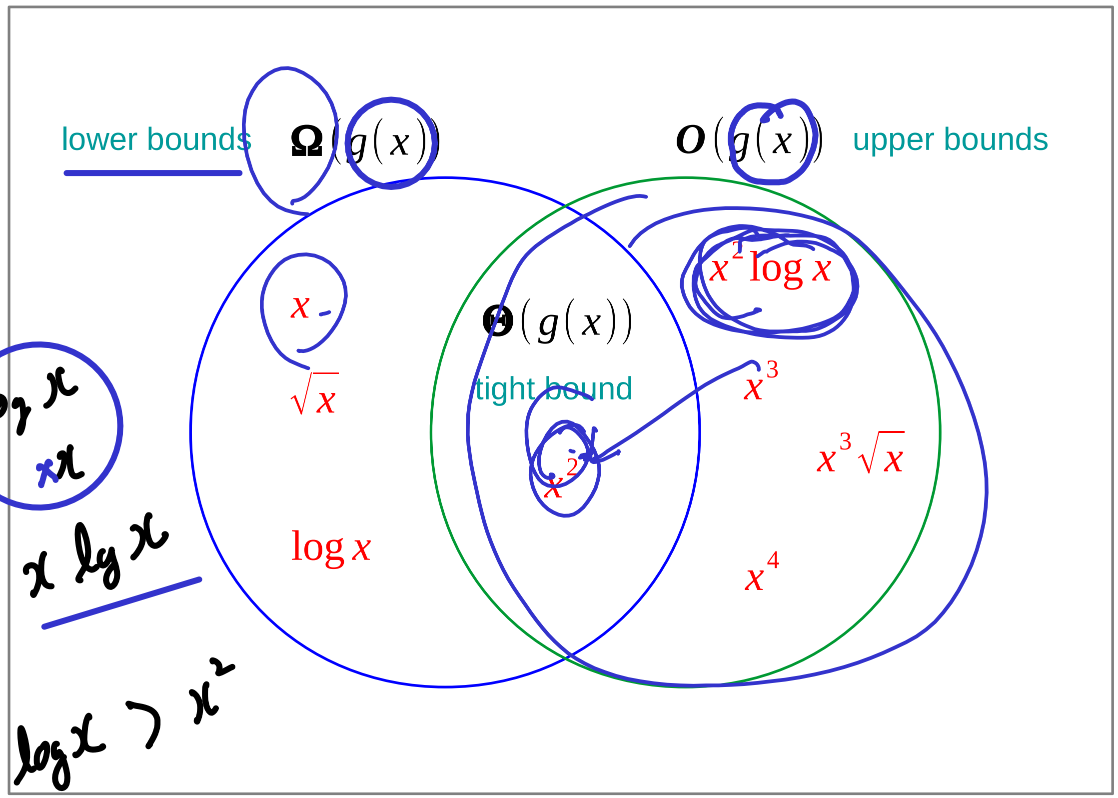
lower

| | | | |
|---|---|---|---|
| $x^2+2x+1$ *is* $O(x^2)$ | $\Leftrightarrow$ | $x^2+2x+1 \ge Cx^2$ | ~~upper bound~~ |
| $x^2+2x+1$ *is* $\Omega(x)$ | $\Leftrightarrow$ | $x^2+2x+1 \ge Cx$ | lower ~~lower bound~~ |
| $x^2+2x+1$ *is* $\Omega(\sqrt{x})$ | $\Leftrightarrow$ | $x^2+2x+1 \ge C\sqrt{x}$ | lower bound |
| $x^2+2x+1$ *is* $\Omega(\log x)$ | $\Leftrightarrow$ | $x^2+2x+1 \ge C\log x$ | lower bound |
| $\vdots$ | | $\vdots$ | |

the greatest

# Simultaneously being lower and upper bound

$\boxed{f(x) = x^2 + 2x + 1}$ $\cancel{\Theta(n)}$ $\Theta(n^2)$

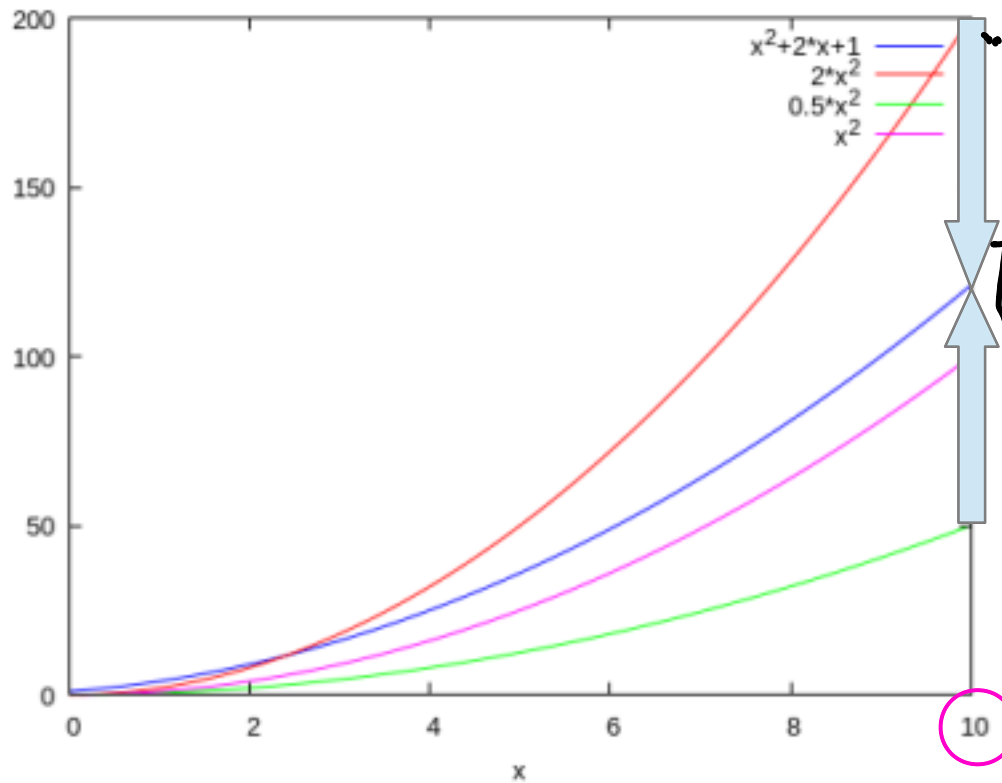



$x > \log x$

$x \cdot x$ $\quad x \log x$

$x^2 > x \log x$

$x^3 > x^2 \log x > x^2$




$2 \cdot x^2$

$x^2 + 2 * x + 1$

$1 \cdot x^2$

$0.5 \cdot x^2$

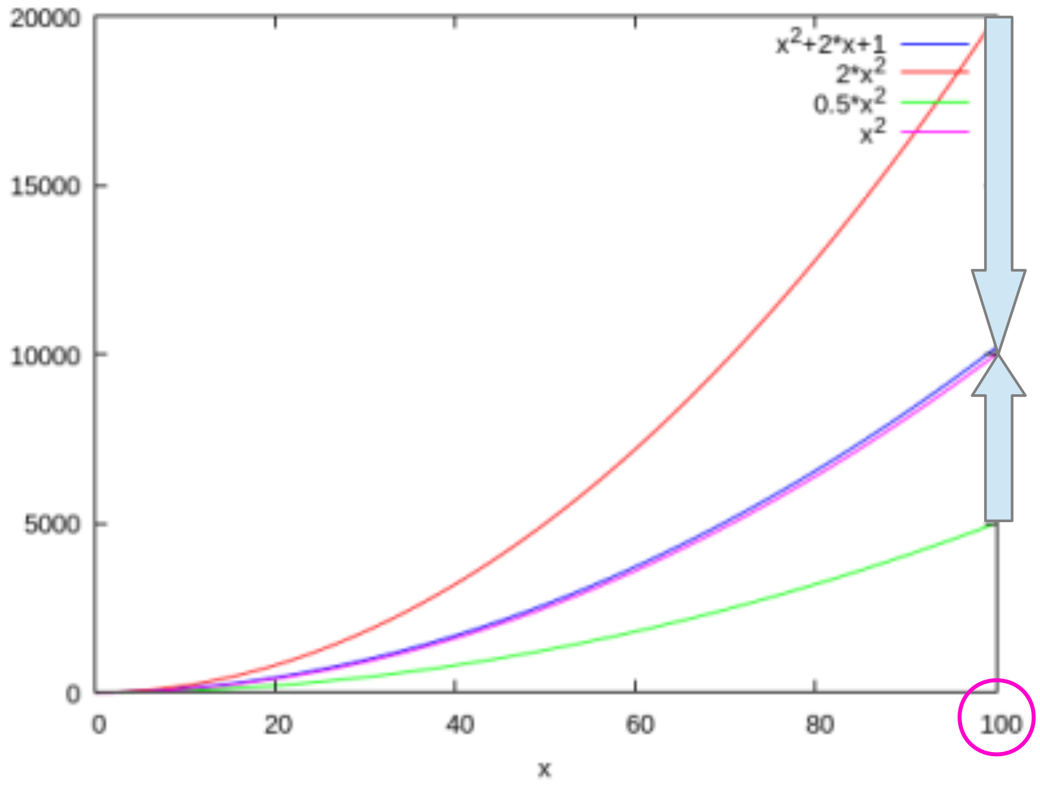
$x^2 + 2x + 1$ *is* $O(x^2)$

$x^2 + 2x + 1$ *is* $\Theta(x^2)$

$x^2 + 2x + 1$ *is* $\Omega(x^2)$

⇨ Zoom Out

# Big-Θ Examples (2)




$2 \cdot x^2$

$x^2 + 2 * x + 1$

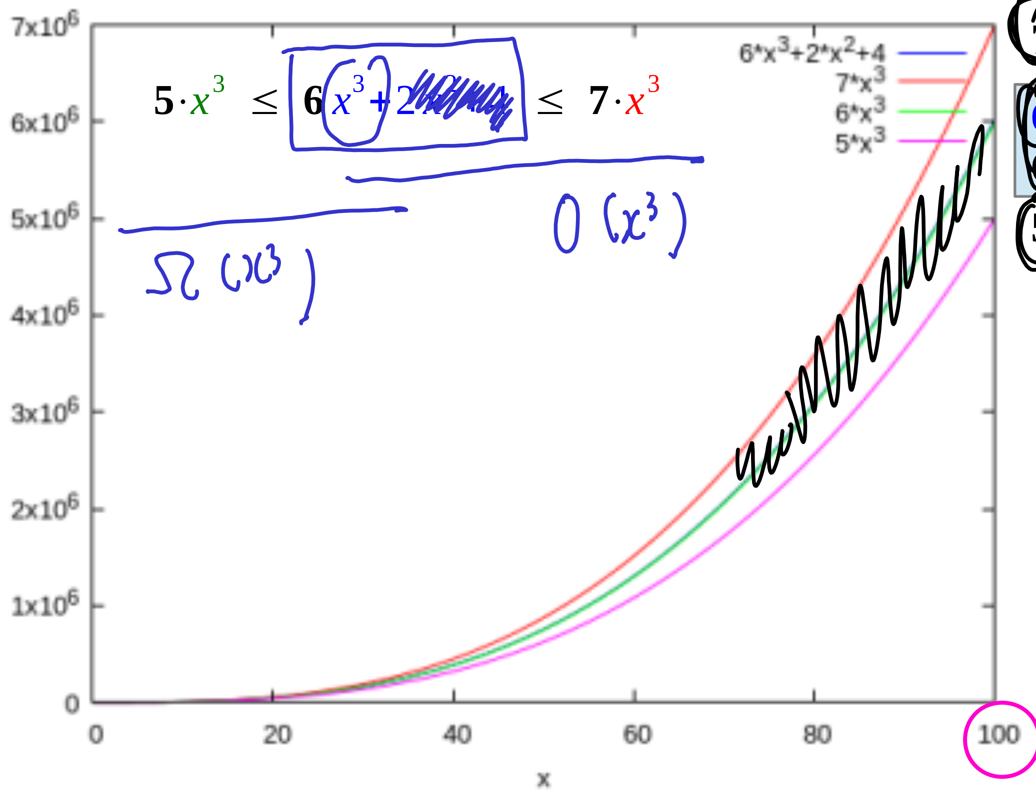
$1 \cdot x^2$

$0.5 \cdot x^2$

$x^2 + 2x + 1$ *is* $\Theta(x^2)$

# Big-Θ Examples (3)

$O(x^3)$ & $\Omega(x^3)$ = $\Theta(x^3)$




$7 \cdot x^3$

$6x^3 + 2x^2 + 4$

$6 \cdot x^3$

$5 \cdot x^3$

indistinguishable

$x^3$

$6x^3 + 2x^2 + 4$ *is* $\Theta(x^3)$

$6x^3 + 2x^2 + 4 \approx 6 \cdot x^3$

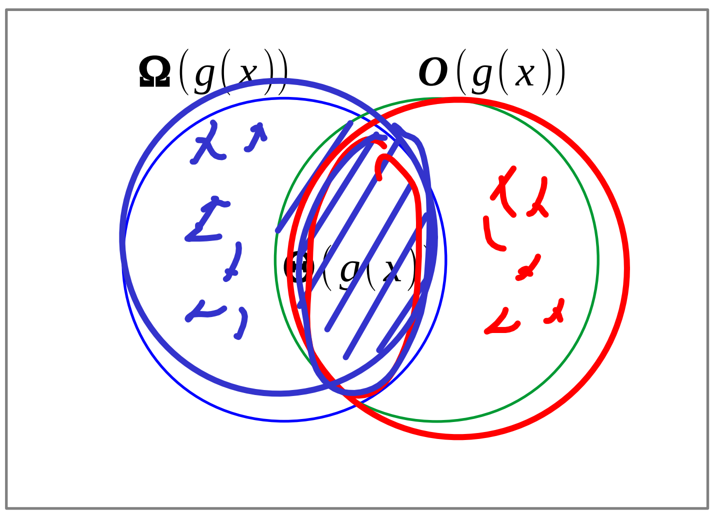
# Tight bound Implications

$$f(x) \text{ is } \Theta(g(x)) \Longrightarrow f(x) \text{ is } O(g(x))$$

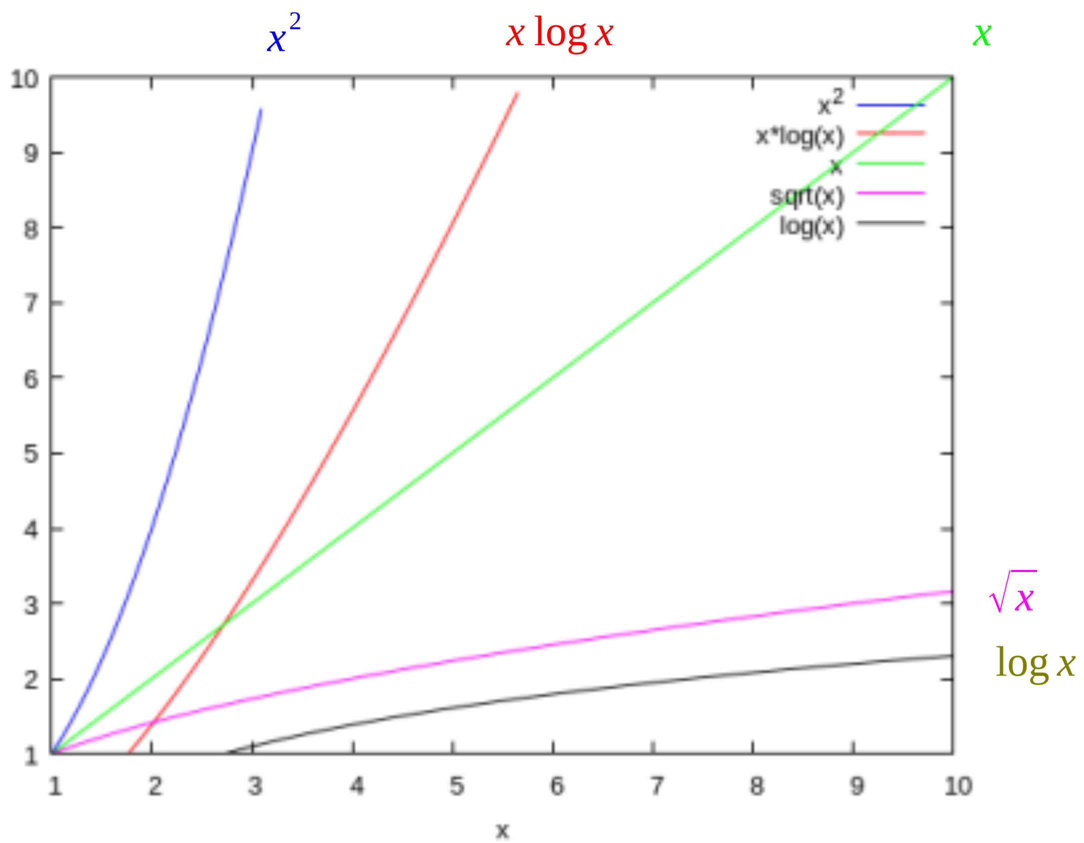
$$f(x) \text{ is } \Theta(g(x)) \Longrightarrow f(x) \text{ is } \Omega(g(x))$$

$$f(x) \text{ is } \Theta(g(x)) \;\not\Longleftarrow\; f(x) \text{ is } O(g(x))$$

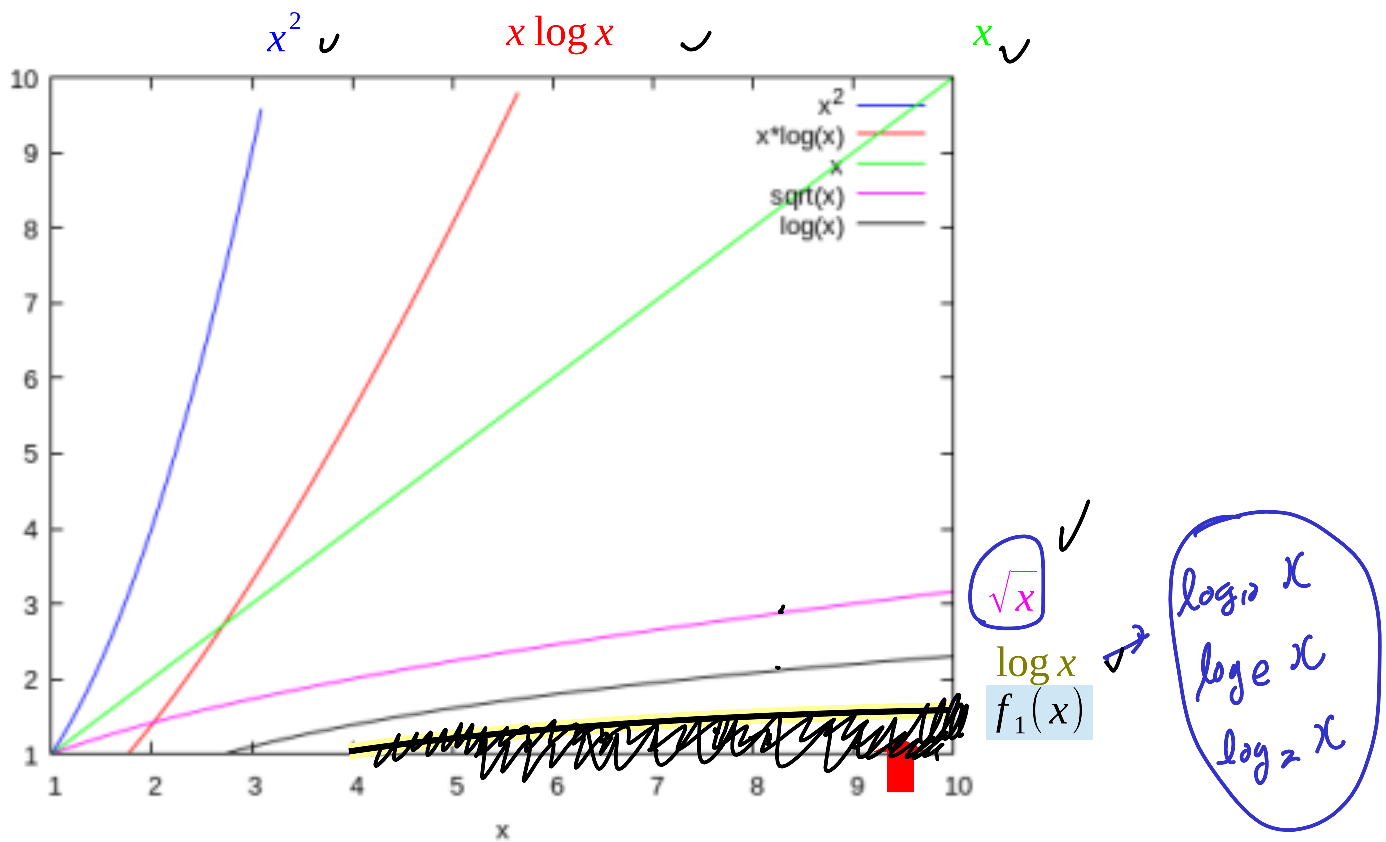
$$f(x) \text{ is } \Theta(g(x)) \;\not\Longleftarrow\; f(x) \text{ is } \Omega(g(x))$$



# Common Growth Functions

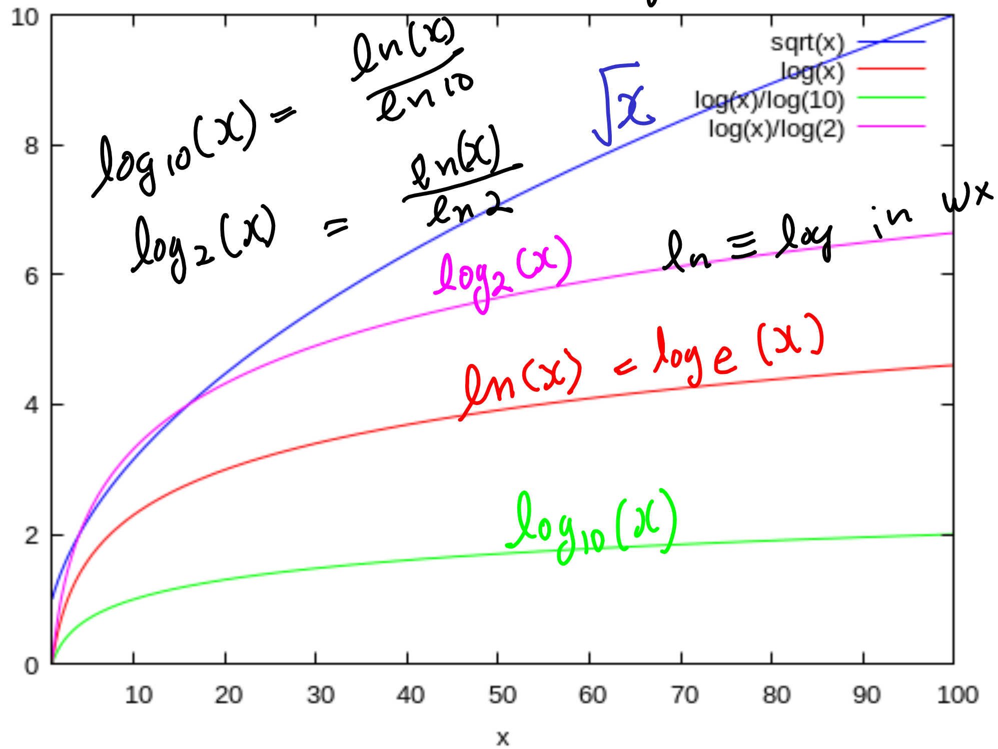


# Upper bounds




$f_1(x)$ *is* $\boldsymbol{O}(\log x)$ ➡ $\boldsymbol{O}(\sqrt{x})$ ➡ $\boldsymbol{O}(x)$ ➡ $\boldsymbol{O}(x \log x)$ ➡ $\boldsymbol{O}(x^2)$

wxMaxima $\log(x) \equiv \ln(x) \equiv \log_e(x)$




$\log_{10}(x) = \dfrac{\ln(x)}{\ln 10}$

$\log_2(x) = \dfrac{\ln(x)}{\ln 2}$

$\sqrt{x}$

$\log_2(x)$

$\ln \equiv \log$ in wxMaxima

$\ln(x) = \log_e(x)$

$\log_{10}(x)$

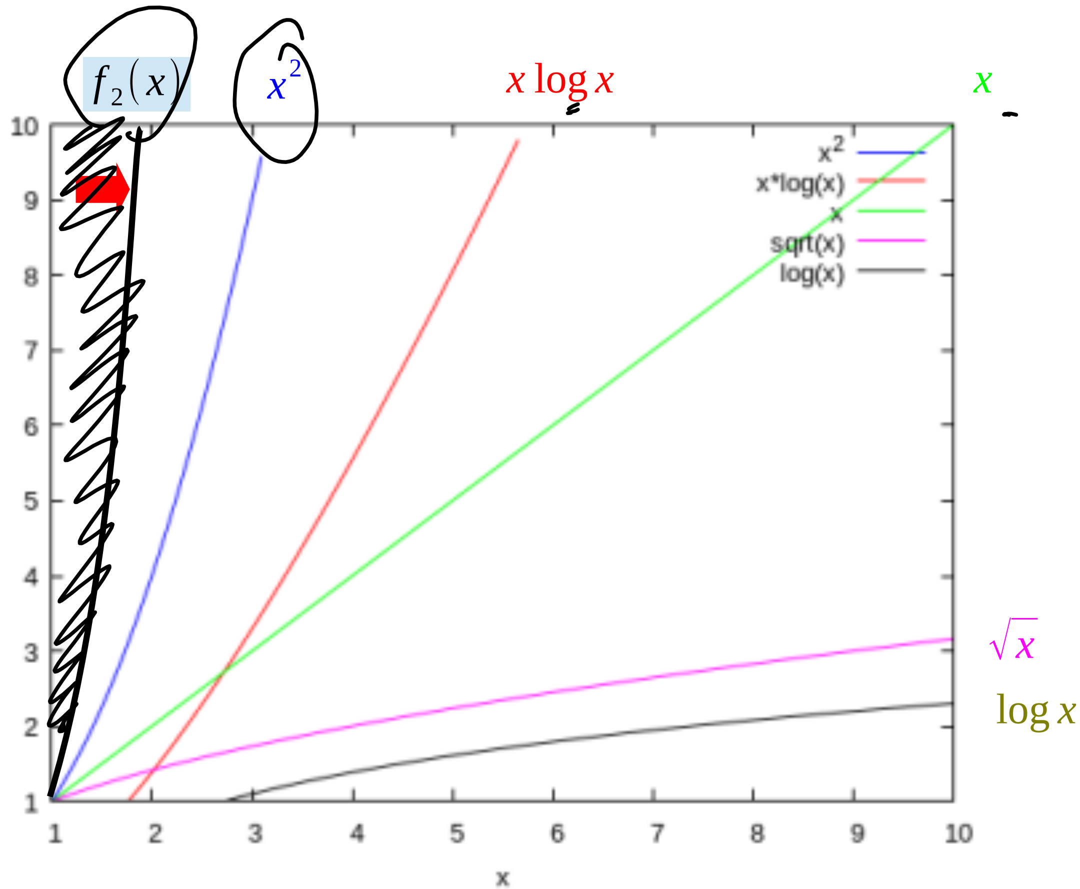
$\log_{10} x$

$\log_2 x$

$\log_e x = \ln x \Rightarrow \log x$

$e = 2{,}7818\ldots$

# Lower bounds




$f_2(x)$ *is* $\mathbf{\Omega}(x^2)$ ➡ $\mathbf{\Omega}(x \log x)$ ➡ $\mathbf{\Omega}(x)$ ➡ $\mathbf{\Omega}(\sqrt{x})$ ➡ $\mathbf{\Omega}(\log x)$

# Example 1

$f(n) = n^6 + 3n$
$f(n) = 2^n + 12$
$f(n) = 2^n + 3^n$
$f(n) = n^n + n$

$f(n) = O(n^6)$
$f(n) = O(2^n)$
$f(n) = O(3^n)$
$f(n) = O(n^n)$

$f(n) = \Omega(n)$
$f(n) = \Omega(1)$
$f(n) = \Omega(2^n)$
$f(n) = \Omega(n)$

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# Example 2

| | | | |
|---|---|---|---|
| $f(n)=n^6+3n$ | $f(n)=O(n^6)$ | $f(n)=\Omega(n^6)$ | $f(n)=\Theta(n^6)$ |
| $f(n)=2^n+12$ | $f(n)=O(2^n)$ | $f(n)=\Omega(2^n)$ | $f(n)=\Theta(2^n)$ |
| $f(n)=2^n+3^n$ | $f(n)=O(3^n)$ | $f(n)=\Omega(3^n)$ | $f(n)=\Theta(3^n)$ |
| $f(n)=n^n+n$ | $f(n)=O(n^n)$ | $f(n)=\Omega(n^n)$ | $f(n)=\Theta(n^n)$ |

$$0.5\, n^6 \le n^6+3n \le 2\, n^6$$

$$0.5\, 2^n \le 2^n+12 \le 2\, 2^n$$

$$0.5\, 3^n \le 2^n+3^n \le 2 \cdot 3^n$$

$$0.5\, n^2 \le n^2+n \le 2\, n^2$$

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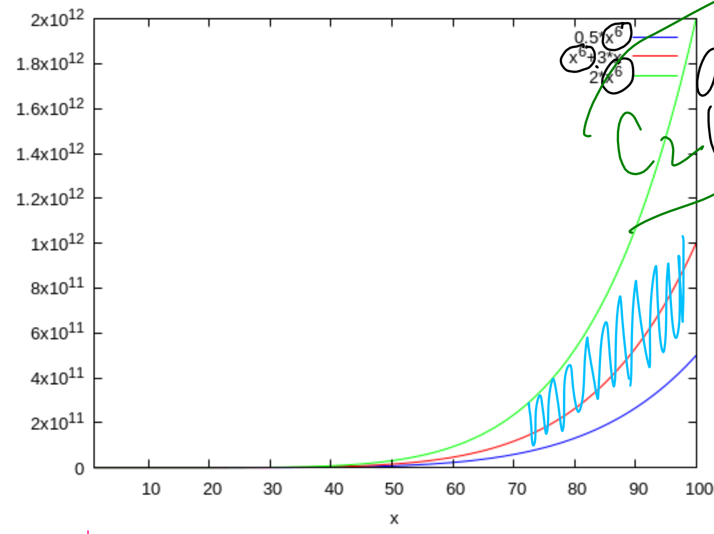
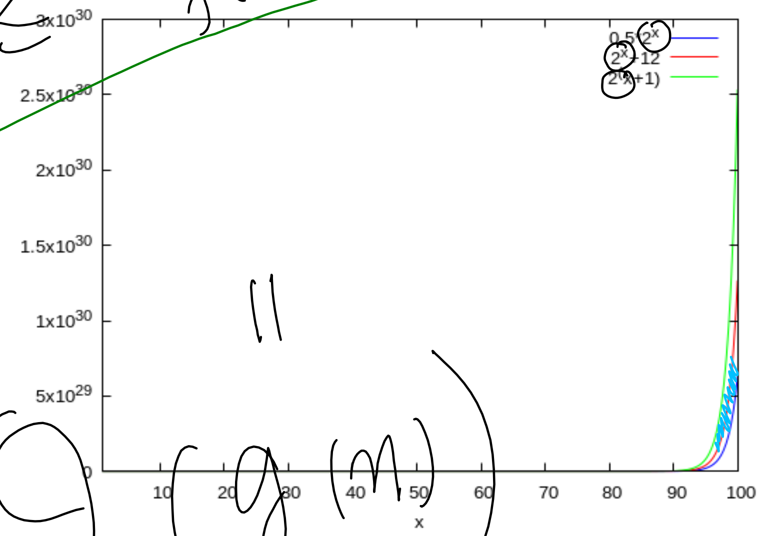
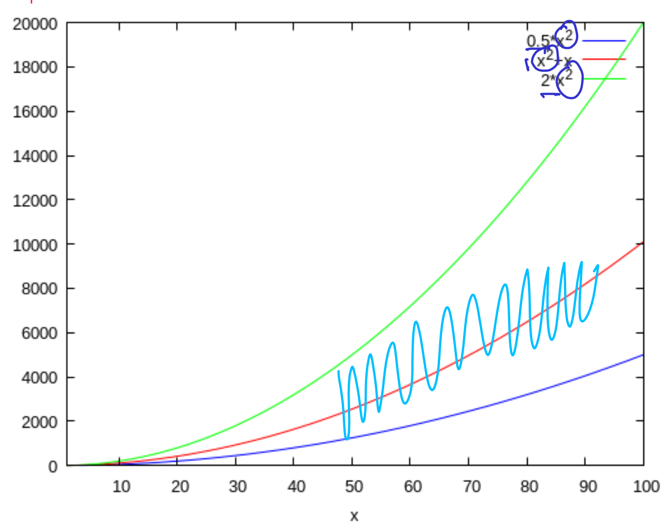
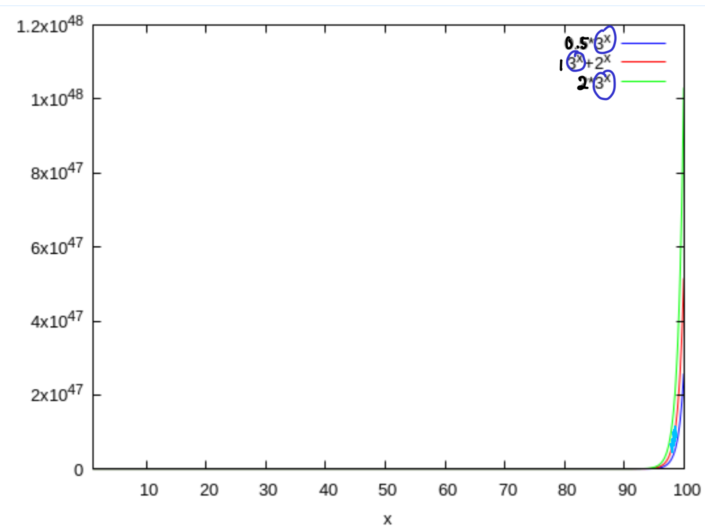
$c_2 g(n) \le f(n) \le c_1 g(n)$

$= \Theta(g(n))$



# Example 3

$\Theta(n) \longrightarrow O(n)$

$\Theta(n) \longrightarrow O(n^2)$

$\Theta(n^2) \longrightarrow O(n^3)$

$\Theta(n) \not\longrightarrow O(1)$

$O(1) \longrightarrow \Theta(1)$

$O(n) \not\longrightarrow \Theta(1)$

$\Theta(n) \to O(n) \to O(n^2)$

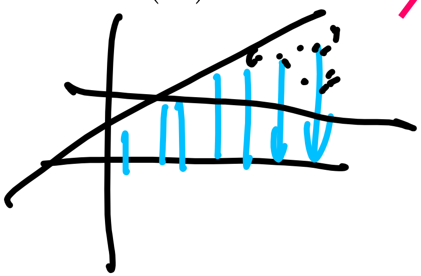
$\Theta(n^2) \to O(n^2) \to O(n^3)$

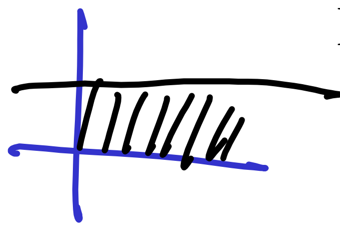
$\Theta(n) \to \Omega(n) \not\to O(1)$

$\Theta(1) \equiv \Omega(1) \equiv O(1)$

$\log(n) \; : \; O(\log n) \to O(n)$

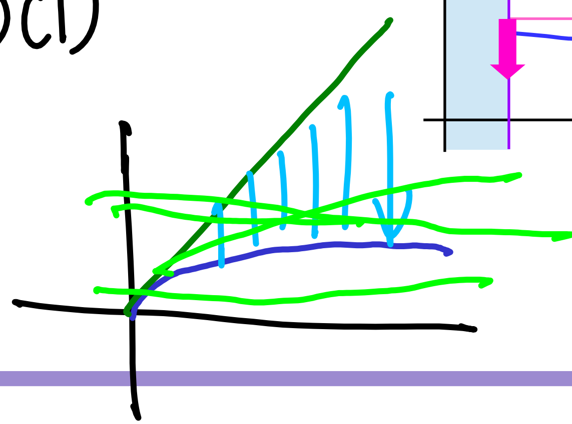
$\log(n) \; : \; \Omega(\log n) \not\to O(1)$

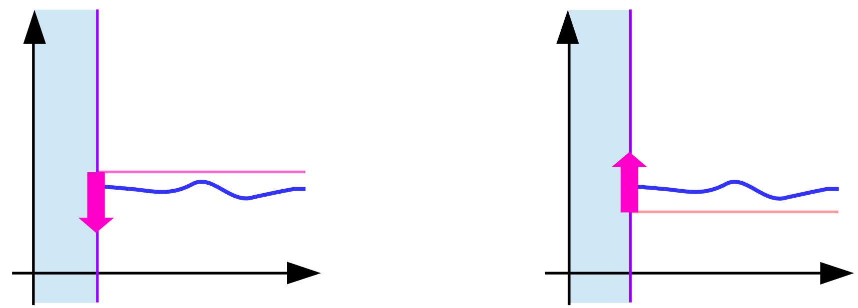




$\Theta(n) \not\to O(1)$

$\log(n)$





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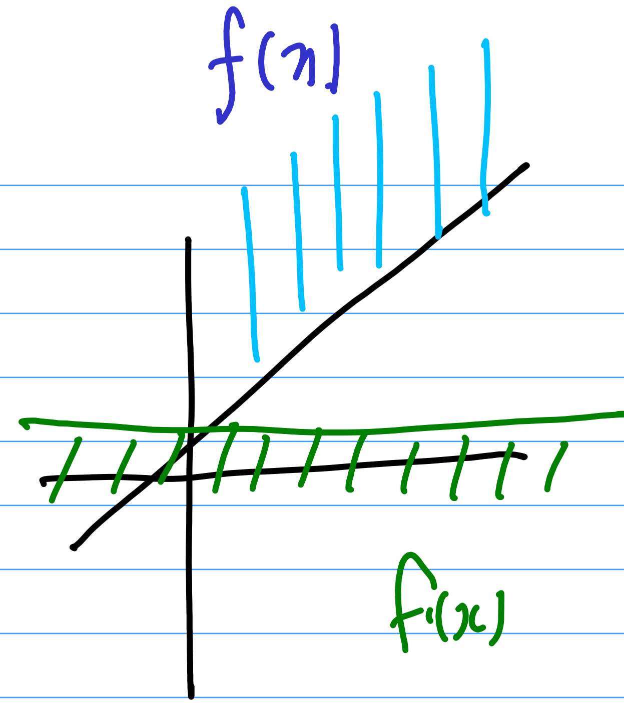
$f(n)$

$\Theta(n) \not\to O(1)$

$\Theta(n) \to O(n)$

$\to \Omega(n)$

$O(1)$



# Example 4

| | | | |
|---|---|---|---|
| $\Theta(n) \rightarrow O(n)$ | upper bound | tight | |
| $\Theta(n^2) \rightarrow O(n^3)$ | upper bound | | |
| $\Theta(1) \rightarrow O(n)$ | upper bound | | |
| $\Theta(n) \rightarrow O(1)$ | upper bound | wrong | |
| $\Theta(n) \rightarrow O(2n)$ | upper bound | tight | |

$O(n^3)$

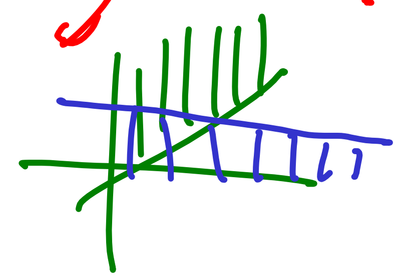
$O(n^2)$ $\Omega(n^4)$

$O(n)$ $\Omega(n)$

$\Theta(n) \rightarrow \Omega(n) \not\rightarrow O(1)$

$O(2n) = O(n)$

$O(n)$



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# Example 5

| | | | | | | |
|---|---|---|---|---|---|---|
| $\Theta(1)$ | $\rightarrow$ | $O(1)$ | tight upper bound | $\rightarrow$ | $O(n)$ | upper bound |
| $\Theta(\sqrt{n})$ | $\rightarrow$ | $O(\sqrt{n})$ | tight upper bound | $\rightarrow$ | $O(n)$ | upper bound |
| $\Theta(n)$ | $\rightarrow$ | $O(n)$ | tight upper bound | $\rightarrow$ | $O(n^2)$ | upper bound |
| $\Theta(n^2)$ | $\rightarrow$ | $O(n^2)$ | tight upper bound | $\rightarrow$ | $O(n^3)$ | upper bound |
| $\Theta(n^3)$ | $\rightarrow$ | $O(n^3)$ | tight upper bound | $\rightarrow$ | $O(n^4)$ | upper bound |

| | | | | | | |
|---|---|---|---|---|---|---|
| $\Theta(1)$ | $\rightarrow$ | $\Omega(1)$ | tight lower bound | | | |
| $\Theta(\sqrt{n})$ | $\rightarrow$ | $\Omega(\sqrt{n})$ | tight lower bound | $\rightarrow$ | $\Omega(1)$ | lower bound |
| $\Theta(n)$ | $\rightarrow$ | $\Omega(n)$ | tight lower bound | $\rightarrow$ | $\Omega(\sqrt{n})$ | lower bound |
| $\Theta(n^2)$ | $\rightarrow$ | $\Omega(n^2)$ | tight lower bound | $\rightarrow$ | $\Omega(n)$ | lower bound |
| $\Theta(n^3)$ | $\rightarrow$ | $\Omega(n^3)$ | tight lower bound | $\rightarrow$ | $\Omega(n^2)$ | lower bound |

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## References


[1] http://en.wikipedia.org/
[2]

# The Complexity of Algorithms (3A)

Please send corrections (or suggestions) to youngwlim@hotmail.com.

This document was produced by using LibreOffice and Octave.

# Complexity Analysis

- to compare algorithms at the idea level
  ignoring the low level details
- To measure how fast a program is
- To explain how an algorithm behaves
  as the input grows larger

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# Counting Instructions

| | |
|---|---|
| • Assigning a value to a variable | x= 100; |
| • Accessing a value of a particular array element | A[i] |
| • Comparing two values | (x > y) |
| • Incrementing a value | i++ |
| • Basic arithmetic operations | +, –, *, / |
| • Branching is not counted | if else |

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# Asymptotic Behavior

- avoiding tedious instruction counting
- eliminate all the minor details
- focusing how algorithms behaves when treated badly
- drop all the terms that grow slowly
- only keep the ones that grow fast as **n** becomes larger

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# Finding the Maximum

```
M = A[0];                    // M is set to the 1st element
for (i=0; i<n; ++i) {
    if (A[i] >= M) {         // if the (i+1)th element is greater than M,
        M = A[i];            // M is set to that element (new maximum value)
    }
}
```

```
int A[n];       // n element integer array A
int M;          // the current maximum value found so far
                // set to the 1st element, initially
```

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# Worst and Best Cases

int A[4];

| | |
|---|---|
| i=0 | A[0] |
| i=1 | A[1] |
| i=2 | A[2] |
| i=3 | A[3] |

Case 1:
Worst Case

| | | |
|---|---|---|
| A[0]=1 | ➡ | M=1 |
| A[1]=2 | ➡ | M=2 |
| A[2]=3 | ➡ | M=3 |
| A[3]=4 | ➡ | M=4 |

Case 2:
Best Case

| | | |
|---|---|---|
| A[0]=4 | ➡ | M=4 |
| A[1]=3 | | |
| A[2]=2 | | |
| A[3]=1 | | |

```
for (i=0; i<n; ++i) {
    if (A[i] >= M) {        // always n comparisons
        M = A[i];           // the updating of M depends on the data
    }                       // minimum 1 update, maximum n updates
```

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# Assignment

```
M = A[0];                       // 2 instructions
for (i=0; i<n; ++i) {
    if (A[i] >= M) {
        M = A[i];
    }
}
```

A[0] – **1** instruction

M = – **1** instruction

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# Loop instructions

```
for (i=0; i<n; ++i) {
        if (A[i] >= M) {
                M = A[i];
        }
```

Initialization * **1**

| | |
|---|---|
| i=0 | : **1** instruction |
| i<n | : **1** instruction |

Update * **n**

| | |
|---|---|
| ++i | : **1** instruction |
| i<n | : **1** instruction |

Loop body * **n**

| | | | |
|---|---|---|---|
| A[i] | : **1** instruction | **n** | always |
| >= M | : **1** instruction | | |
| A[i] | : **1** instruction | **1 ~ n** | depending on the comparison |
| M= | : **1** instruction | | |

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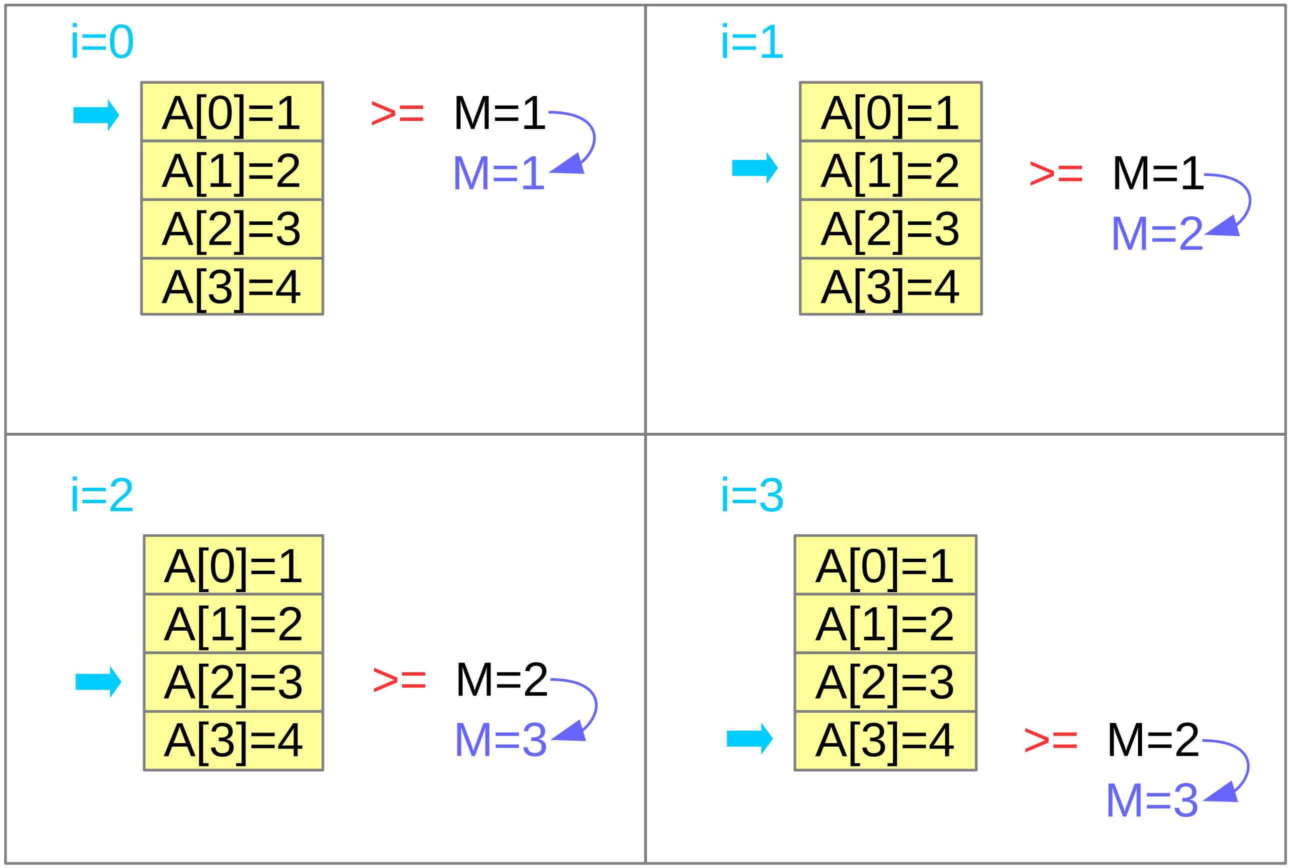
# Worst case examples




```
for (i=0; i<n; ++i) {
    if (A[i] >= M) {
        M = A[i];
    }
```

2**n** + 2**n** = 4**n**

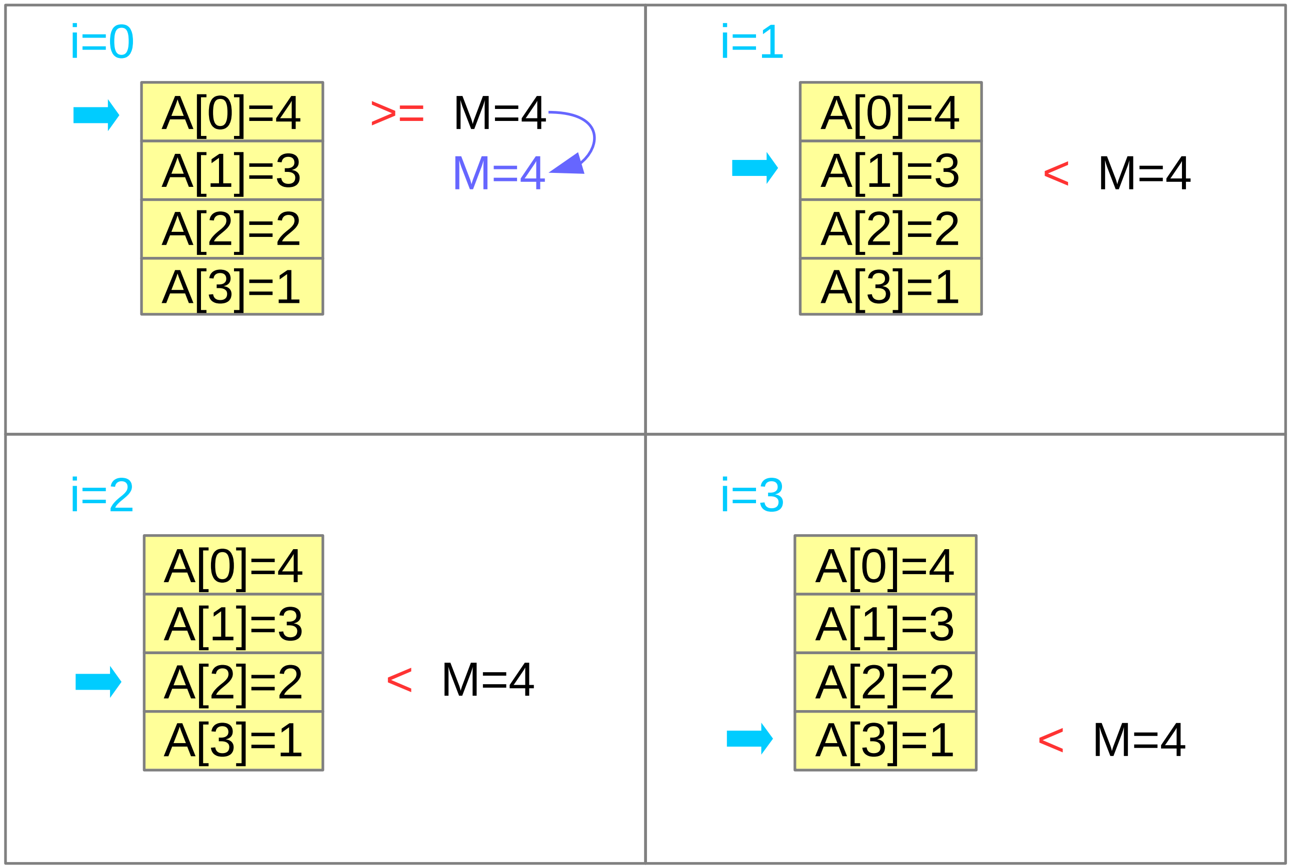
Instructions

**n** comparisons

**n** updates

https://discrete.gr/complexity/

# Best case examples




```
for (i=0; i<n; ++i) {
    if (A[i] >= M) {
        M = A[i];
    }
```

2**n** + 2

Instructions

**n** comparisons

**1** update

https://discrete.gr/complexity/

# Asymptotic behavior

```
M = A[0]; ------------------ 2        instructions
for (i=0; i<n; ++i) { --------- 2 + 2n   instructions (init + update)
    if (A[i] >= M) { --------- 2n       instructions
        M = A[i]; ---------- 2 ~ 2n   instructions
    }
}
```

$$f(n) = \begin{cases} 6n+4 & \text{instructions for the worst case} \\ 4n+6 & \text{instruction for the best case} \end{cases}$$

$f(n) = O(n)$

$f(n) = \Omega(n)$

$f(n) = \Theta(n)$

https://discrete.gr/complexity/

# O(**n**) codes

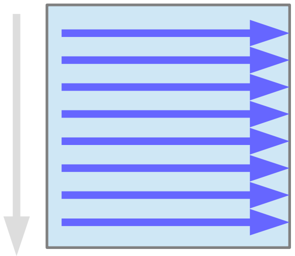
```
// Here c is a positive integer constant
for (i = 1; i <= n; i += c) {
    // some O(1) expressions
}

for (int i = n; i > 0; i -= c) {
    // some O(1) expressions
}
```

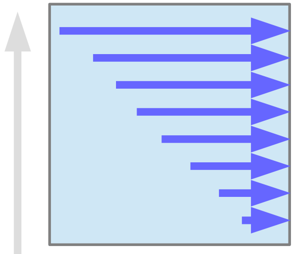
https://stackoverflow.com/questions/11032015/how-to-find-time-complexity-of-an-algorithm

# O($n^2$) codes

```
for (i = 1; i <=n; i += c) {
   for (j = 1; j <=n; j += c) {
      // some O(1) expressions
   }
}
```



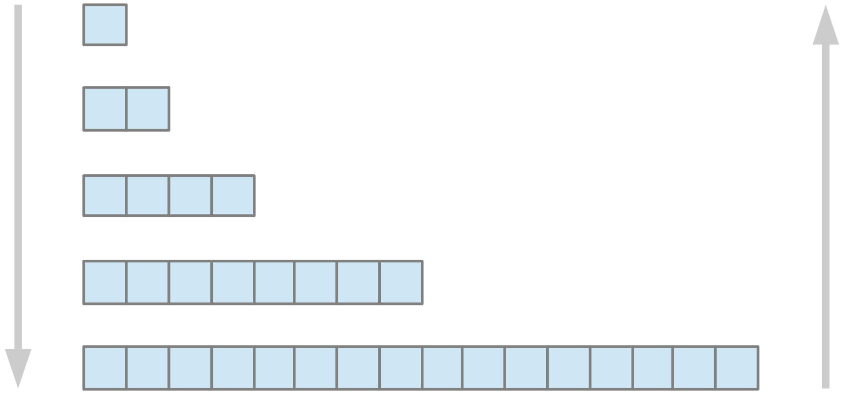
```
for (i = n; i > 0; i += c) {
   for ( j = i+1; j <=n; j += c) {
      // some O(1) expressions
}
```



https://stackoverflow.com/questions/11032015/how-to-find-time-complexity-of-an-algorithm

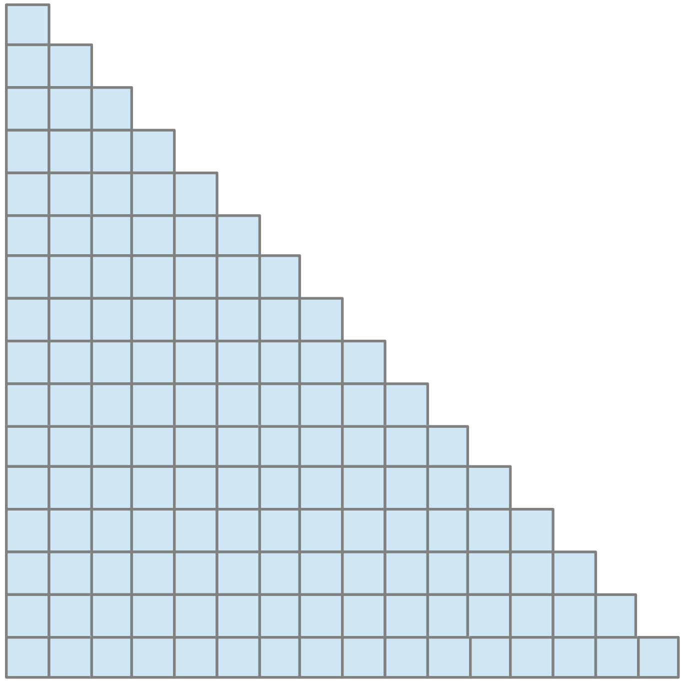
# O(log n) codes

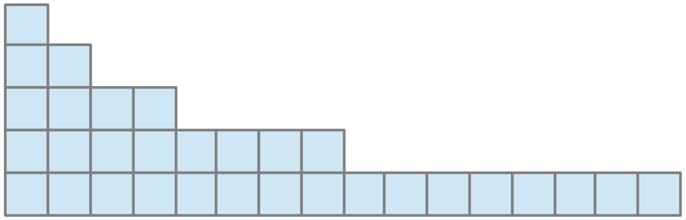
```
for (int i = 1; i <=n; i *= c) {
   // some O(1) expressions
}
for (int i = n; i > 0; i /= c) {
   // some O(1) expressions
}
```



https://stackoverflow.com/questions/11032015/how-to-find-time-complexity-of-an-algorithm

# O(**n**) vs. O(log **n**)





https://stackoverflow.com/questions/11032015/how-to-find-time-complexity-of-an-algorithm

# O(log **n**) codes

```
// Here c is a constant greater than 1
for (int i = 2; i <=n; i = pow(i, c)) {          // i = i^c
   // some O(1) expressions
}
//Here fun is sqrt or cuberoot or any other constant root
for (int i = n; i > 0; i = fun(i)) {             // i = i^(1/c)
   // some O(1) expressions
}
```

$i = i^2, i = i^3$

https://stackoverflow.com/questions/11032015/how-to-find-time-complexity-of-an-algorithm

# O(log log **n**) codes

```
// Here c is a constant greater than 1
for (int i = 2; i <=n; i = pow(i, c)) {        // i = i^c
  // some O(1) expressions
}
//Here fun is sqrt or cuberoot or any other constant root
for (int i = n; i > 0; i = fun(i)) {           // i = i^(1/c)
  // some O(1) expressions
}
```

$i = i^2 \quad (2, 2^2, 2^4, 2^8, 2^{16}, \cdots)$

$i = i^{\frac{1}{2}} \quad (n, n^{\frac{1}{2}}, n^{\frac{1}{4}}, n^{\frac{1}{8}}, n^{\frac{1}{16}}, \cdots)$

https://stackoverflow.com/questions/11032015/how-to-find-time-complexity-of-an-algorithm

# O(log log **n**) codes

```
// Here c is a constant greater than 1
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}
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$i = i^2 \quad (2, 2^2, 2^4, 2^8, 2^{16}, \cdots)$

$i = i^{\frac{1}{2}} \quad (n, n^{\frac{1}{2}}, n^{\frac{1}{4}}, n^{\frac{1}{8}}, n^{\frac{1}{16}}, \cdots)$

https://stackoverflow.com/questions/11032015/how-to-find-time-complexity-of-an-algorithm

# Some Algorithm Complexities and Examples (1)

**O(1) – Constant Time**

not affected by the input size **n**.

**O(n) – Linear Time**

Proportional to the input size **n**.

**O(log n) – Logarithmic Time**

recursive subdivisions of a problem

binary search algorithm

**O(n log n) – Linearithmic Time**

Recursive subdivisions of a problem and then merge them

merge sort algorithm.

https://stackoverflow.com/questions/11032015/how-to-find-time-complexity-of-an-algorithm

# Some Algorithm Complexities and Examples (2)

**O(n2) – Quadratic Time**

bubble sort algorithm

**O(n3) – Cubic Time**

straight forward matrix multiplication

**O(2^n) – Exponential Time**

Tower of Hanoi

**O(n!) – Factorial Time**

Travel Salesman Problem (TSP)

https://stackoverflow.com/questions/11032015/how-to-find-time-complexity-of-an-algorithm

**References**

[1] http://en.wikipedia.org/
[2]