Complex Phase Factors (6A)

- Complex Phase Factors in DFT
- N=8 DFT / IDFT
- N=8 DFT / IDFT Matrix
- N=8 DFT / IDFT Complex Factors
- N=8 DFT / IDFT Real Factors
- N=8 DFT / IDFT Imaginary Factors

Copyright (c) 2009, 2010 Young W. Lim.

Permission is granted to copy, distribute and/or modify this document under the terms of the GNU Free Documentation License, Version 1.2 or any later version published by the Free Software Foundation; with no Invariant Sections, no Front-Cover Texts, and no Back-Cover Texts. A copy of the license is included in the section entitled "GNU Free Documentation License".

Please send corrections (or suggestions) to youngwlim@hotmail.com.

This document was produced by using OpenOffice and Octave.

Young Won Lim
04/19/2010

Discrete Fourier Transform

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j(2\pi/N)kn} \quad \Leftrightarrow \quad x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k]\, e^{+j(2\pi/N)kn}$$

$$W_N \triangleq e^{-j(2\pi/N)}$$

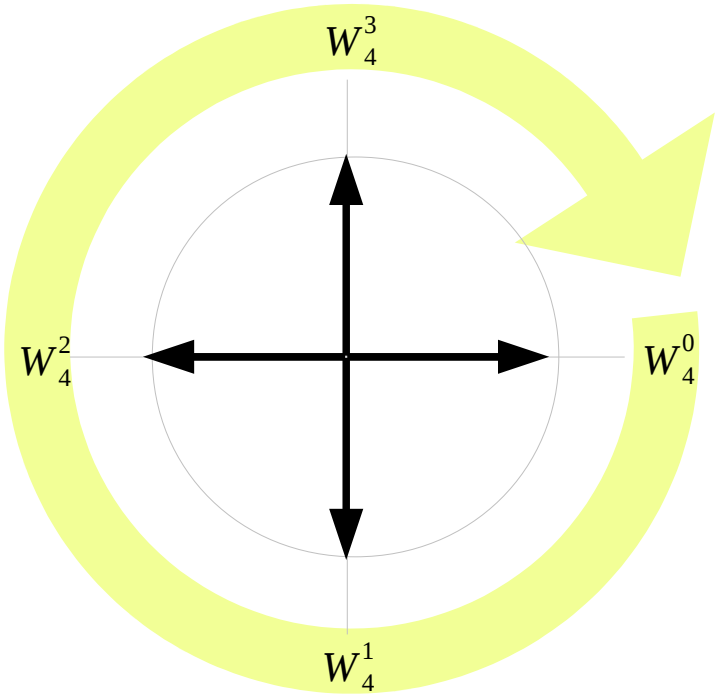
$$W_N^{nk} \triangleq e^{-j(2\pi/N)nk}$$

$$X[k] = \sum_{n=0}^{N-1} x[n]\, W_N^{kn} \quad \Leftrightarrow \quad x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k]\, W_N^{-kn}$$

Complex Phase Factor (1)

$$W_4^k = e^{-j(\frac{2\pi}{4})k}$$

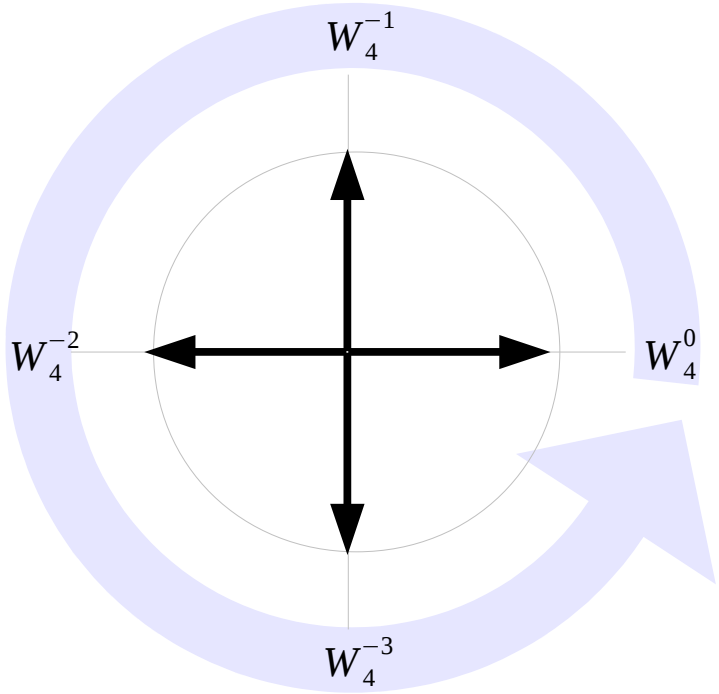
$$W_4^{-k} = e^{+j(\frac{2\pi}{4})k}$$



$$W_4^1 = W_4^{-3}$$

$$W_4^2 = W_4^{-2}$$

$$W_4^3 = W_4^{-1}$$

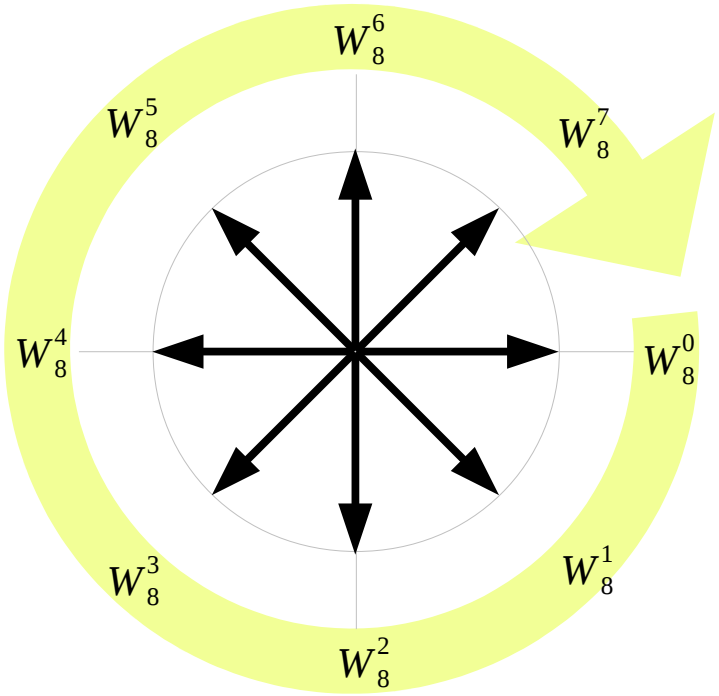


$$W_N^{k \pm N} = W_N^k$$

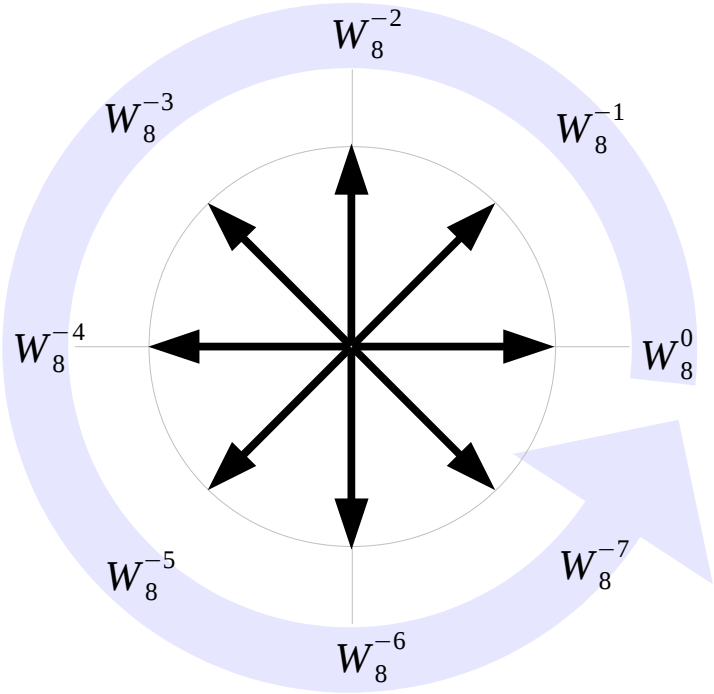
Complex Phase Factor (2)

$$W_8^k = e^{-j(\frac{2\pi}{8})k}$$

$$W_8^{-k} = e^{+j(\frac{2\pi}{8})k}$$



$$W_8^1 = W_8^{-7}$$
$$W_8^2 = W_8^{-6}$$
$$W_8^3 = W_8^{-5}$$
$$W_8^4 = W_8^{-4}$$
$$W_8^5 = W_8^{-3}$$
$$W_8^6 = W_8^{-2}$$
$$W_8^7 = W_8^{-1}$$



$$W_N^{k \pm N} = W_N^k$$

Complex Phase Factor (3)

$$W_N^{k} = e^{-j(\frac{2\pi}{N})k} \qquad W_N^{-k} = e^{+j(\frac{2\pi}{N})k}$$

$$W_N^{k-N} = W_N^{k} \qquad W_N^{k+N} = W_N^{k}$$

$$W_N^{k-N} = e^{-j(\frac{2\pi}{N})(k-N)}$$

$$\frac{W_N^{k-N}}{W_N^{k}} = \frac{e^{-j(\frac{2\pi}{N})(k-N)}}{e^{-j(\frac{2\pi}{N})k}} = e^{j2\pi} = 1$$

$$W_N^{k+N} = e^{-j(\frac{2\pi}{N})(k+N)}$$

$$\frac{W_N^{k+N}}{W_N^{k}} = \frac{e^{-j(\frac{2\pi}{N})(k+N)}}{e^{-j(\frac{2\pi}{N})k}} = e^{-j2\pi} = 1$$

$$W_N^{kN} = 1 \qquad W_N^{-kN} = 1$$

$$W_N^{kN} = e^{-j(\frac{2\pi}{N})kN} = e^{-j2\pi k} = 1$$

$$W_N^{kN} = e^{+j(\frac{2\pi}{N})kN} = e^{+j2\pi k} = 1$$

DFT Symmetry

$$X^*[k] = X[N-k]$$

$$X^*[k]$$

$$= \sum_{n=0}^{N-1} x[n]\, W_N^{-kn}$$

$$= \sum_{n=0}^{N-1} x[n]\, W_N^{nN}\, W_N^{-kn}$$

$$= \sum_{n=0}^{N-1} x[n]\, W_N^{n(N-k)}$$

$$= X[N-k]$$

$$X[k] = \sum_{n=0}^{N-1} x[n]\, W_N^{kn}$$

$$W_N^{nN} = 1$$

$$W_N^{k-N} = W_N^k$$

N=8 DFT

$$X[k] = \sum_{n=0}^{7} W_8^{kn}\, x[n] \qquad W_8^{kn} = e^{-j(\frac{2\pi}{8})kn}$$

$$\begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ X[3] \\ X[4] \\ X[5] \\ X[6] \\ X[7] \end{bmatrix} = \begin{bmatrix} W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 \\ W_8^0 & W_8^1 & W_8^2 & W_8^3 & W_8^4 & W_8^5 & W_8^6 & W_8^7 \\ W_8^0 & W_8^2 & W_8^4 & W_8^6 & W_8^8 & W_8^{10} & W_8^{12} & W_8^{14} \\ W_8^0 & W_8^3 & W_8^6 & W_8^9 & W_8^{12} & W_8^{15} & W_8^{18} & W_8^{21} \\ W_8^0 & W_8^4 & W_8^8 & W_8^{12} & W_8^{16} & W_8^{20} & W_8^{24} & W_8^{28} \\ W_8^0 & W_8^5 & W_8^{10} & W_8^{15} & W_8^{20} & W_8^{25} & W_8^{30} & W_8^{35} \\ W_8^0 & W_8^6 & W_8^{12} & W_8^{18} & W_8^{24} & W_8^{30} & W_8^{36} & W_8^{42} \\ W_8^0 & W_8^7 & W_8^{14} & W_8^{21} & W_8^{28} & W_8^{35} & W_8^{42} & W_8^{49} \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \\ x[4] \\ x[5] \\ x[6] \\ x[7] \end{bmatrix}$$

N=8 DFT Matrix (1)

$$-n \cdot k \mod 8$$

$N = 8$

$$W_N^{nk} = e^{-j(2\pi/N)nk}$$

$$\frac{W_N^{k \pm N}}{W_N^k} = \frac{e^{-j(\frac{2\pi}{N})(k \pm N)}}{e^{-j(\frac{2\pi}{N})k}} = e^{\mp j 2\pi} = 1$$

example:
$-49 \mod 8 \equiv -1 \mod 8$

k \ n	0	1	2	3	4	5	6	7
0	0 0	0 0	0 0	0 0	0 0	0 0	0 0	0 0
1	0 0	-1 -1	-2 -2	-3 -3	-4 -4	-5 -5	-6 -6	-7 -7
2	0 0	-2 -2	-4 -4	-6 -6	-8 0	-10 -2	-12 -4	-14 -6
3	0 0	-3 -3	-6 -6	-9 -1	-12 -4	-15 -7	-18 -2	-21 -5
4	0 0	-4 -4	-8 0	-12 -4	-16 0	-20 -4	-24 0	-28 -4
5	0 0	-5 -5	-10 -2	-15 -7	-20 -4	-25 -1	-30 -6	-35 -3
6	0 0	-6 -6	-12 -4	-18 -2	-24 0	-30 -6	-36 -4	-42 -2
7	0 0	-7 -7	-14 -6	-21 -5	-28 -4	-35 -3	-42 -2	-49 -1

N=8 DFT Matrix (2)

$$-n \cdot k \mod 8$$

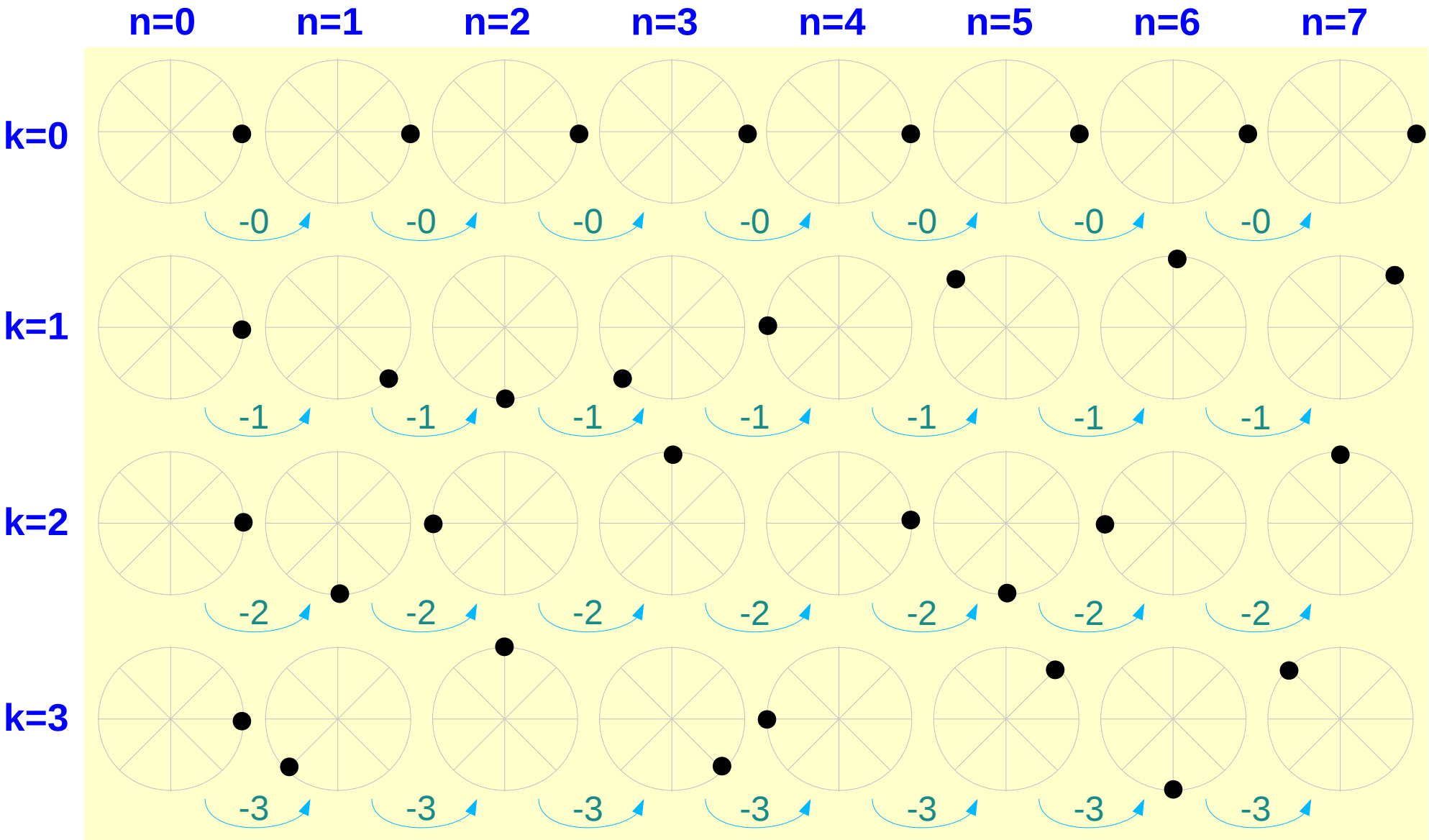
$N = 8$

$$W_N^{nk} = e^{-j(2\pi/N)nk}$$

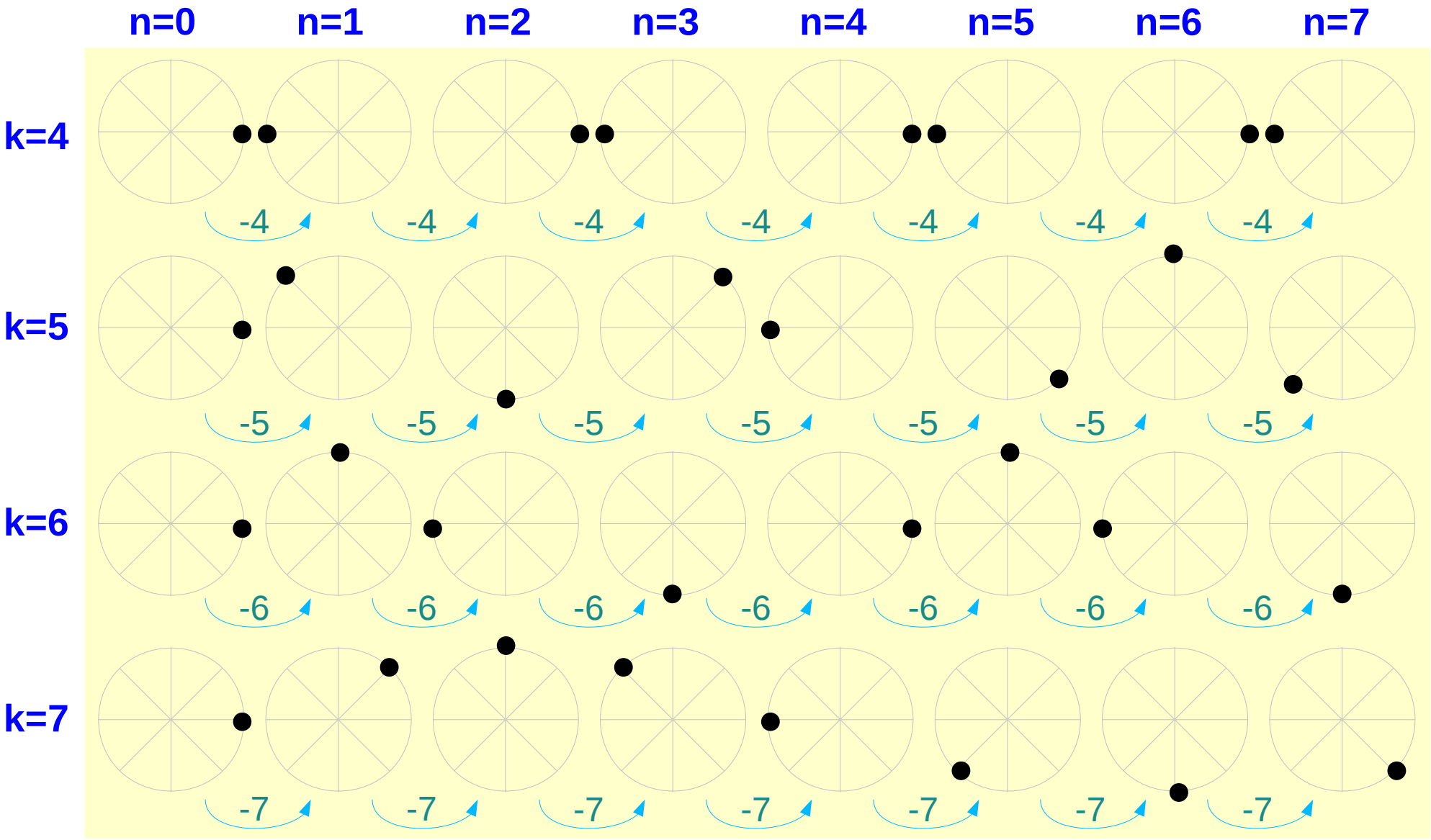
$$\frac{W_N^{k \pm N}}{W_N^k} = \frac{e^{-j(\frac{2\pi}{N})(k \pm N)}}{e^{-j(\frac{2\pi}{N})k}} = e^{\mp j 2\pi} = 1$$

	n=0	n=1	n=2	n=3	n=4	n=5	n=6	n=7
k=0	0	0	0	0	0	0	0	0
	-0	-0	-0	-0	-0	-0	-0	
k=1	0	-1	-2	-3	-4	-5	-6	-7
	-1	-1	-1	-1	-1	-1	-1	
k=2	0	-2	-4	-6	0	-2	-4	-6
	-2	-2	-2	-2	-2	-2	-2	
k=3	0	-3	-6	-1	-4	-7	-2	-5
	-3	-3	-3	-3	-3	-3	-3	
k=4	0	-4	0	-4	0	-4	0	-4
	-4	-4	-4	-4	-4	-4	-4	
k=5	0	-5	-2	-7	-4	-1	-6	-3
	-5	-5	-5	-5	-5	-5	-5	
k=6	0	-6	-4	-2	0	-6	-4	-2
	-6	-6	-6	-6	-6	-6	-6	
k=7	0	-7	-6	-5	-4	-3	-2	-1
	-7	-7	-7	-7	-7	-7	-7	

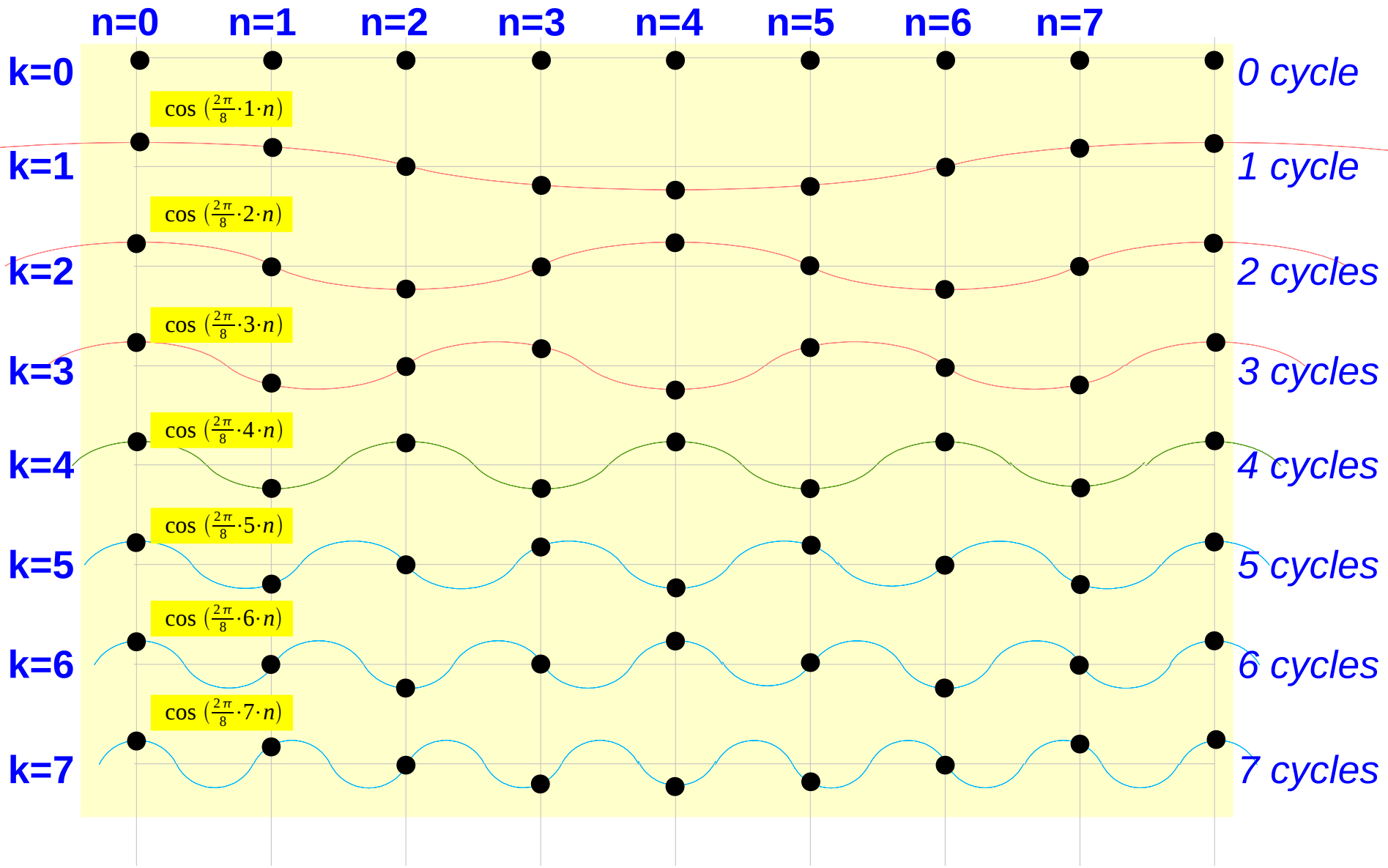
N=8 DFT Complex Factors (3)



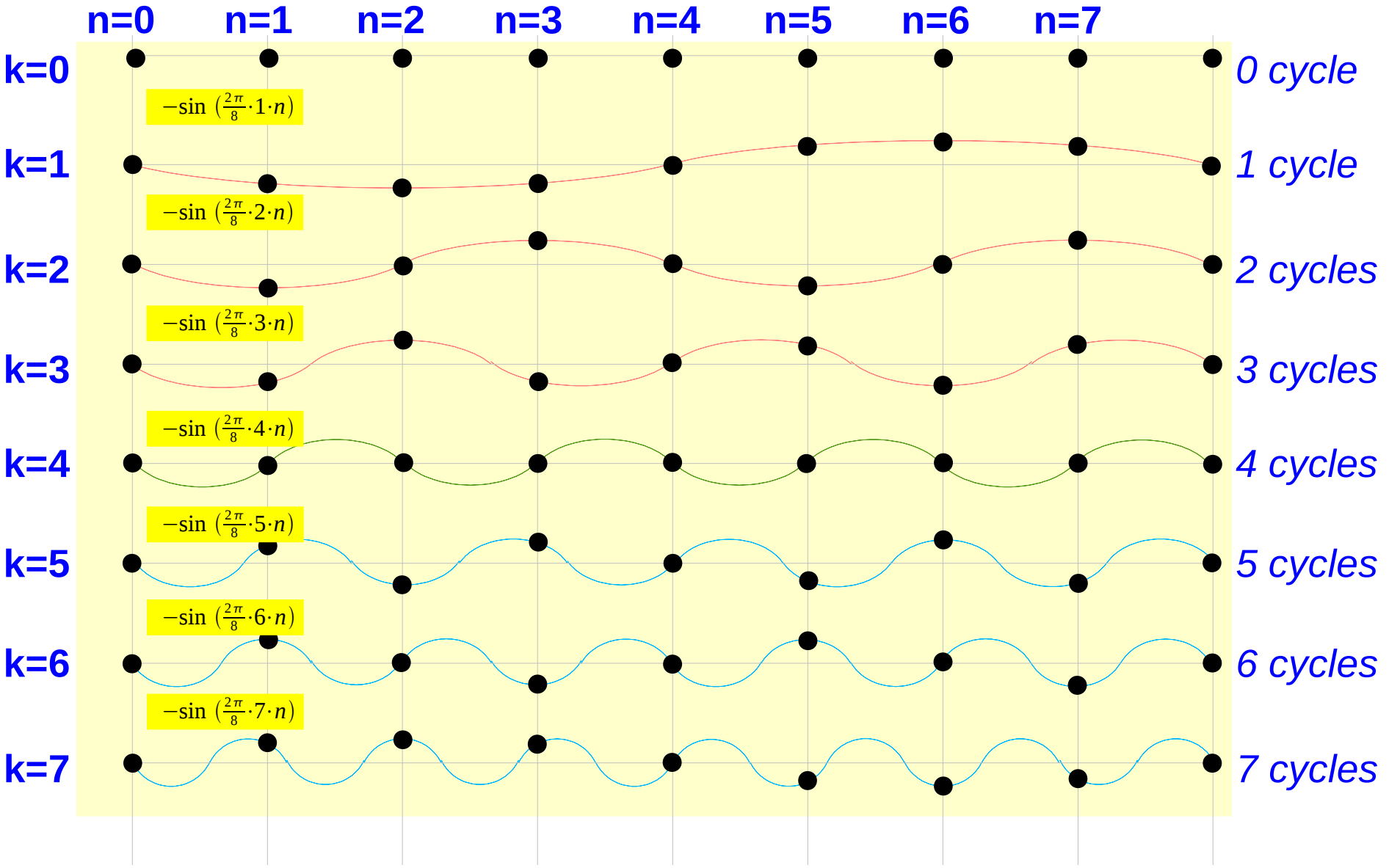
N=8 DFT Complex Factors (4)



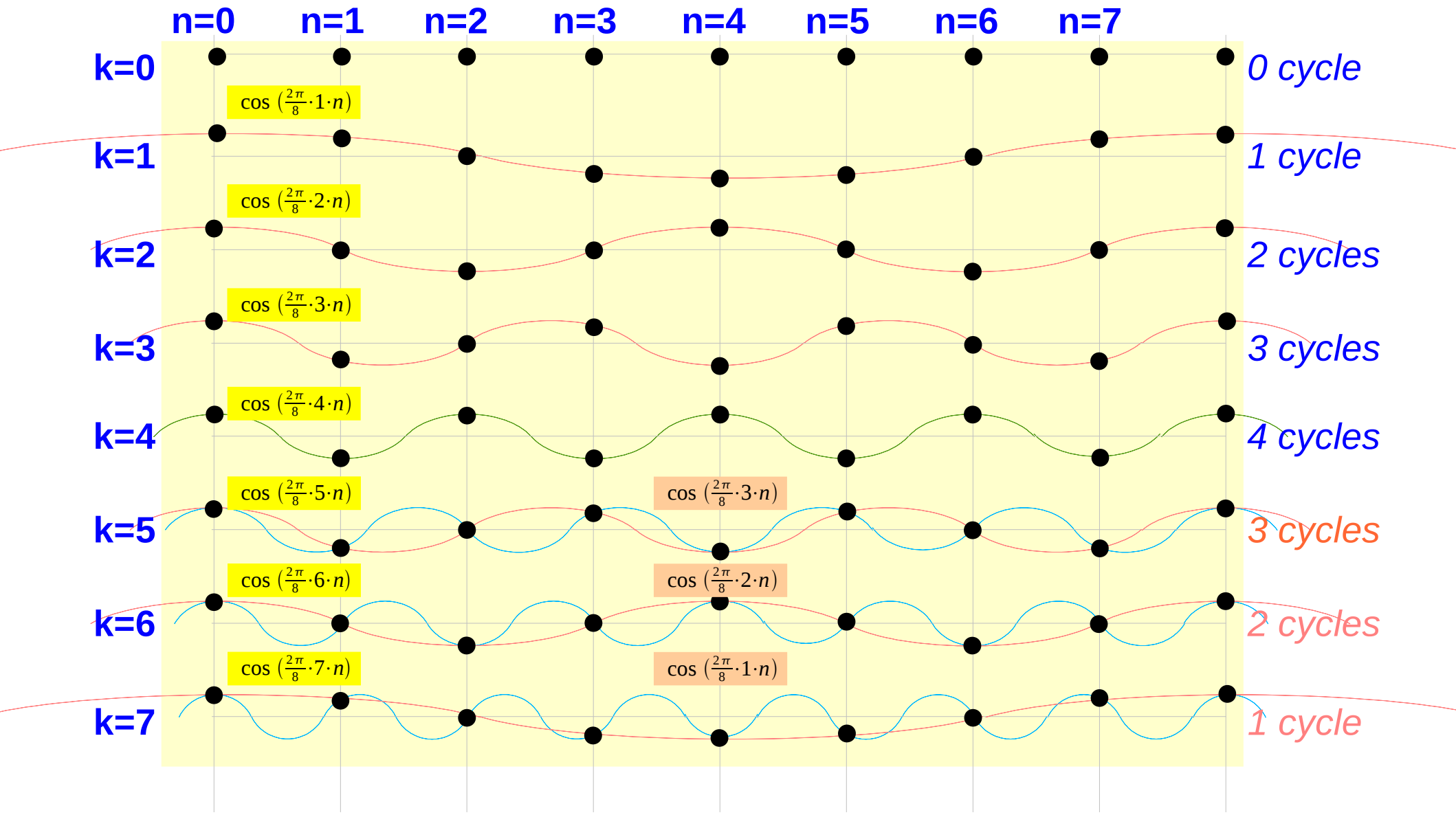
N=8 DFT Real Factors



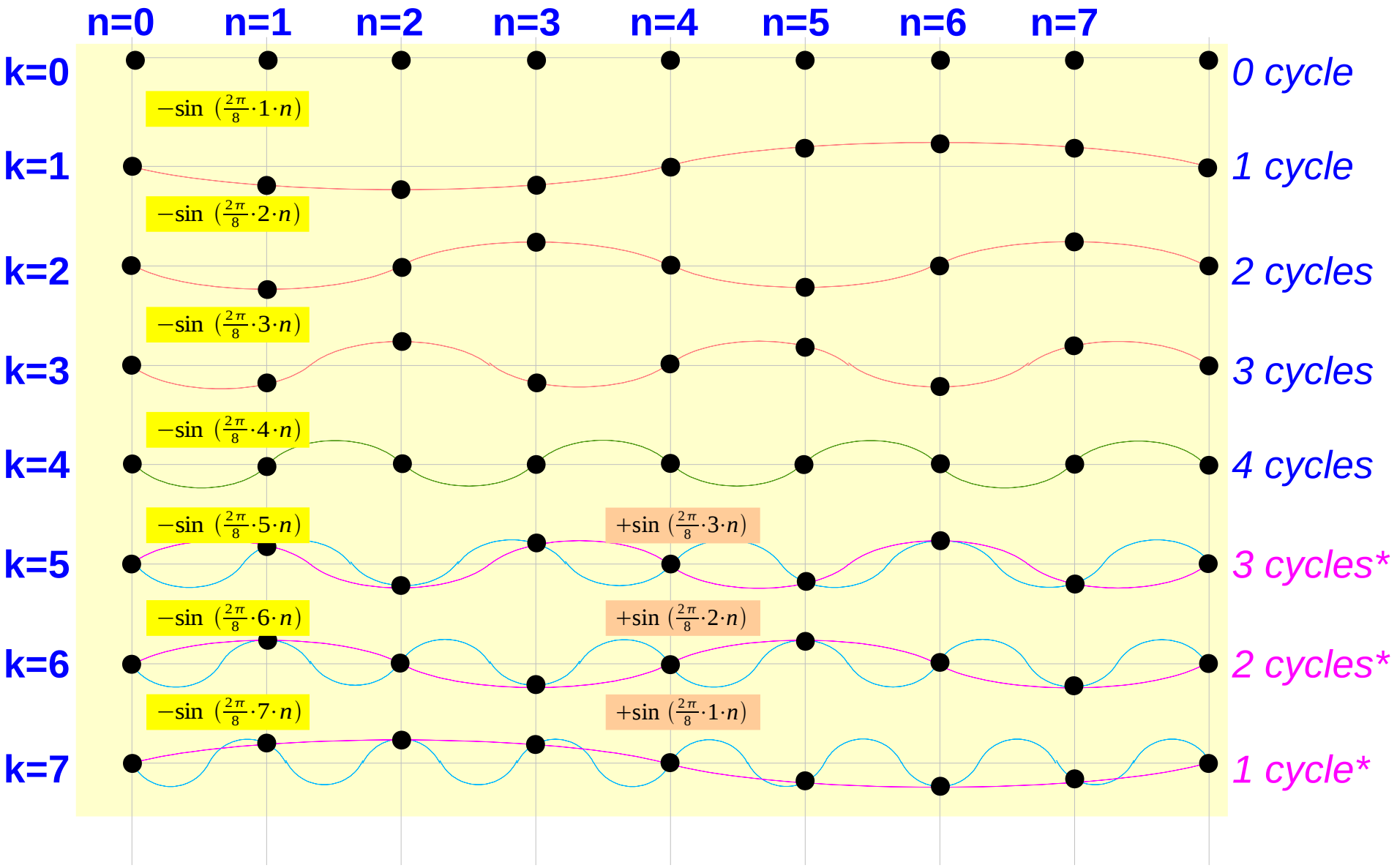
N=8 DFT Imaginary Factors



N=8 DFT Real Factor Symmetry



N=8 DFT Imaginary Factor Symmetry



N=8 IDFT

$$x[n] = \frac{1}{N} \sum_{k=0}^{7} W_8^{-kn} X[k] \qquad W_8^{-kn} = e^{+j(\frac{2\pi}{8})kn}$$

$$\begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \\ x[4] \\ x[5] \\ x[6] \\ x[7] \end{bmatrix} = \frac{1}{N} \begin{bmatrix} W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 \\ W_8^0 & W_8^{-1} & W_8^{-2} & W_8^{-3} & W_8^{-4} & W_8^{-5} & W_8^{-6} & W_8^{-7} \\ W_8^0 & W_8^{-2} & W_8^{-4} & W_8^{-6} & W_8^{-8} & W_8^{-10} & W_8^{-12} & W_8^{-14} \\ W_8^0 & W_8^{-3} & W_8^{-6} & W_8^{-9} & W_8^{-12} & W_8^{-15} & W_8^{-18} & W_8^{-21} \\ W_8^0 & W_8^{-4} & W_8^{-8} & W_8^{-12} & W_8^{-16} & W_8^{-20} & W_8^{-24} & W_8^{-28} \\ W_8^0 & W_8^{-5} & W_8^{-10} & W_8^{-15} & W_8^{-20} & W_8^{-25} & W_8^{-30} & W_8^{-35} \\ W_8^0 & W_8^{-6} & W_8^{-12} & W_8^{-18} & W_8^{-24} & W_8^{-30} & W_8^{-36} & W_8^{-42} \\ W_8^0 & W_8^{-7} & W_8^{-14} & W_8^{-21} & W_8^{-28} & W_8^{-35} & W_8^{-42} & W_8^{-49} \end{bmatrix} \begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ X[3] \\ X[4] \\ X[5] \\ X[6] \\ X[7] \end{bmatrix}$$

N=8 IDFT Matrix (1)

$+n \cdot k \bmod 8$

$N = 8$

$$W_N^{-nk} = e^{+j(2\pi/N)nk}$$

$$\frac{W_N^{-k \pm N}}{W_N^{-k}} = \frac{e^{+j(\frac{2\pi}{N})(k \pm N)}}{e^{+j(\frac{2\pi}{N})k}} = e^{\pm j2\pi} = 1$$

example:
$49 \ mod \ 8 \equiv 1 \ mod \ 8$

n \ k	0	1	2	3	4	5	6	7
0	0 0	0 0	0 0	0 0	0 0	0 0	0 0	0 0
1	0 0	1 1	2 2	3 3	4 4	5 5	6 6	7 7
2	0 0	2 2	4 4	6 6	8 0	10 2	12 4	14 6
3	0 0	3 3	6 6	9 1	12 4	15 7	18 2	21 5
4	0 0	4 4	8 0	12 4	16 0	20 4	24 0	28 4
5	0 0	5 5	10 2	15 7	20 4	25 1	30 6	35 3
6	0 0	6 6	12 4	18 2	24 0	30 6	36 4	42 2
7	0 0	7 7	14 6	21 5	28 4	35 3	42 2	49 1

N=8 IDFT Matrix (2)

$+n \cdot k \mod 8$

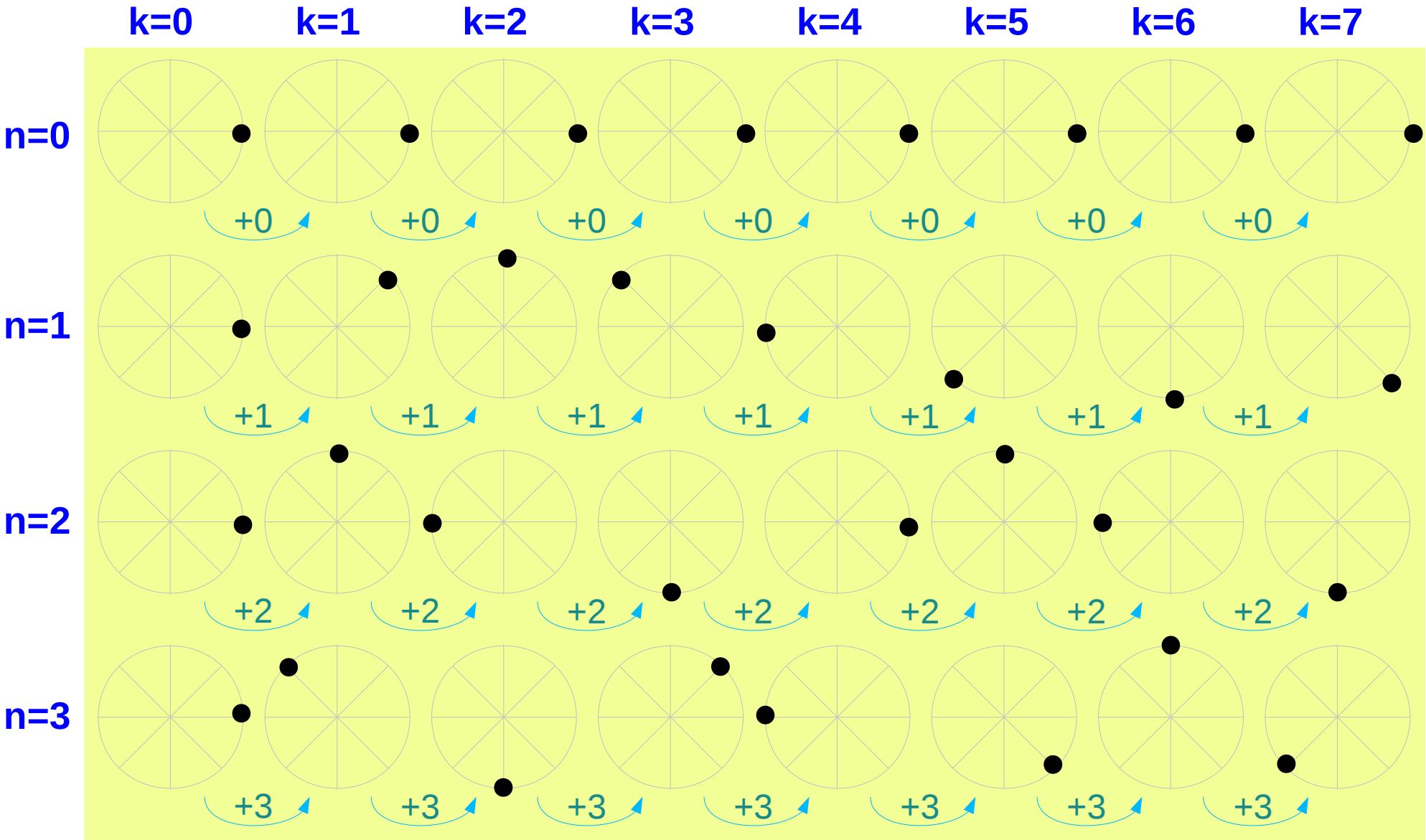
$N = 8$

$$W_N^{-nk} = e^{+j(2\pi/N)nk}$$

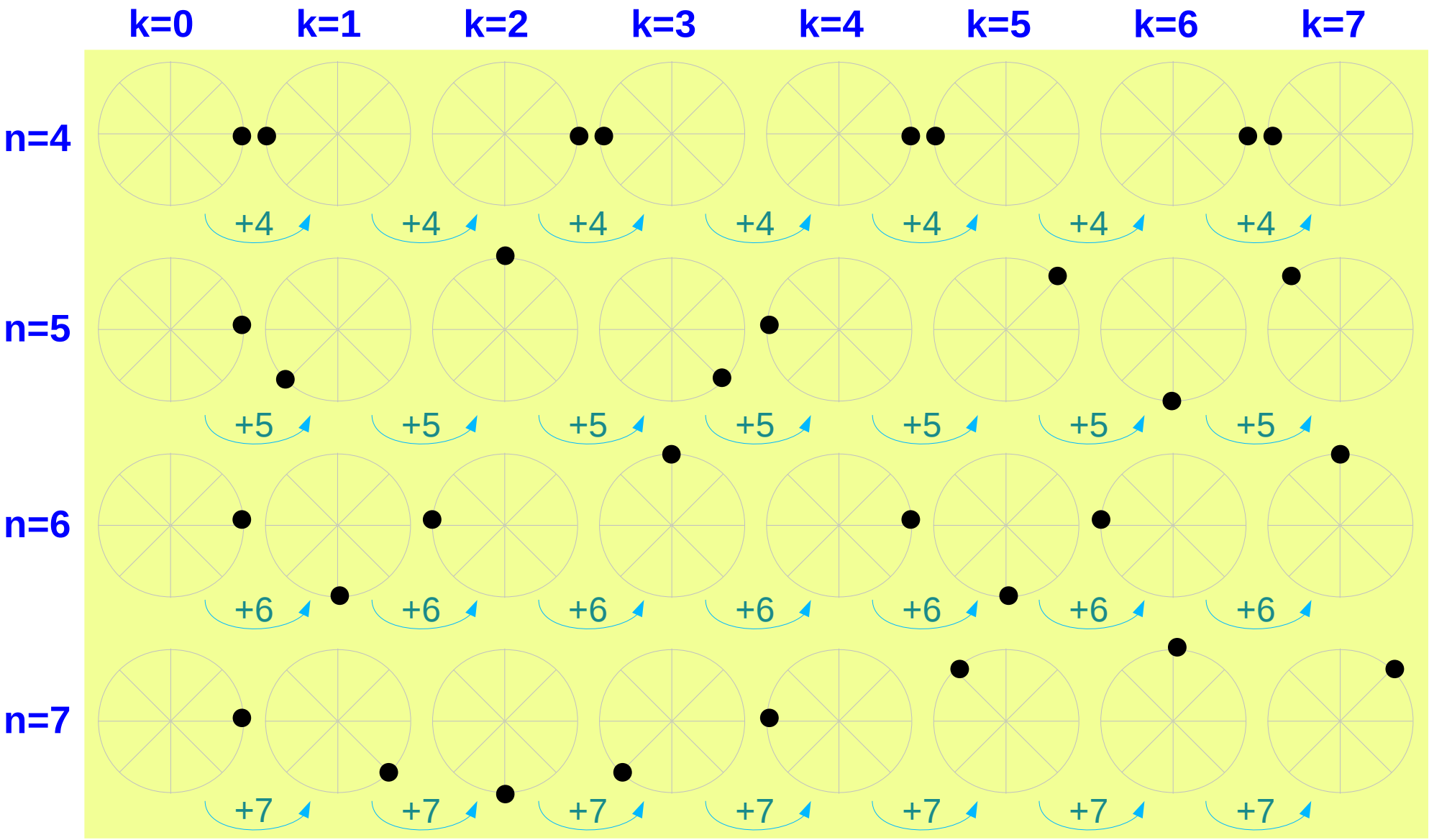
$$\frac{W_N^{-k\pm N}}{W_N^{-k}} = \frac{e^{+j(\frac{2\pi}{N})(k\pm N)}}{e^{+j(\frac{2\pi}{N})k}} = e^{\pm j2\pi} = 1$$

	k=0	k=1	k=2	k=3	k=4	k=5	k=6	k=7
n=0	0	0	0	0	0	0	0	0
	+0	+0	+0	+0	+0	+0	+0	
n=1	0	+1	+2	+3	+4	+5	+6	+7
	+1	+1	+1	+1	+1	+1	+1	
n=2	0	+2	+4	+6	0	+2	+4	+6
	+2	+2	+2	+2	+2	+2	+2	
n=3	0	+3	+6	+1	+4	+7	+2	+5
	+3	+3	+3	+3	+3	+3	+3	
n=4	0	+4	0	+4	0	+4	0	+4
	+4	+4	+4	+4	+4	+4	+4	
n=5	0	+5	+2	+7	+4	+1	+6	+3
	+5	+5	+5	+5	+5	+5	+5	
n=6	0	+6	+4	+2	0	+6	+4	+2
	+6	+6	+6	+6	+6	+6	+6	
n=7	0	+7	+6	+5	+4	+3	+2	+1
	+7	+7	+7	+7	+7	+7	+7	

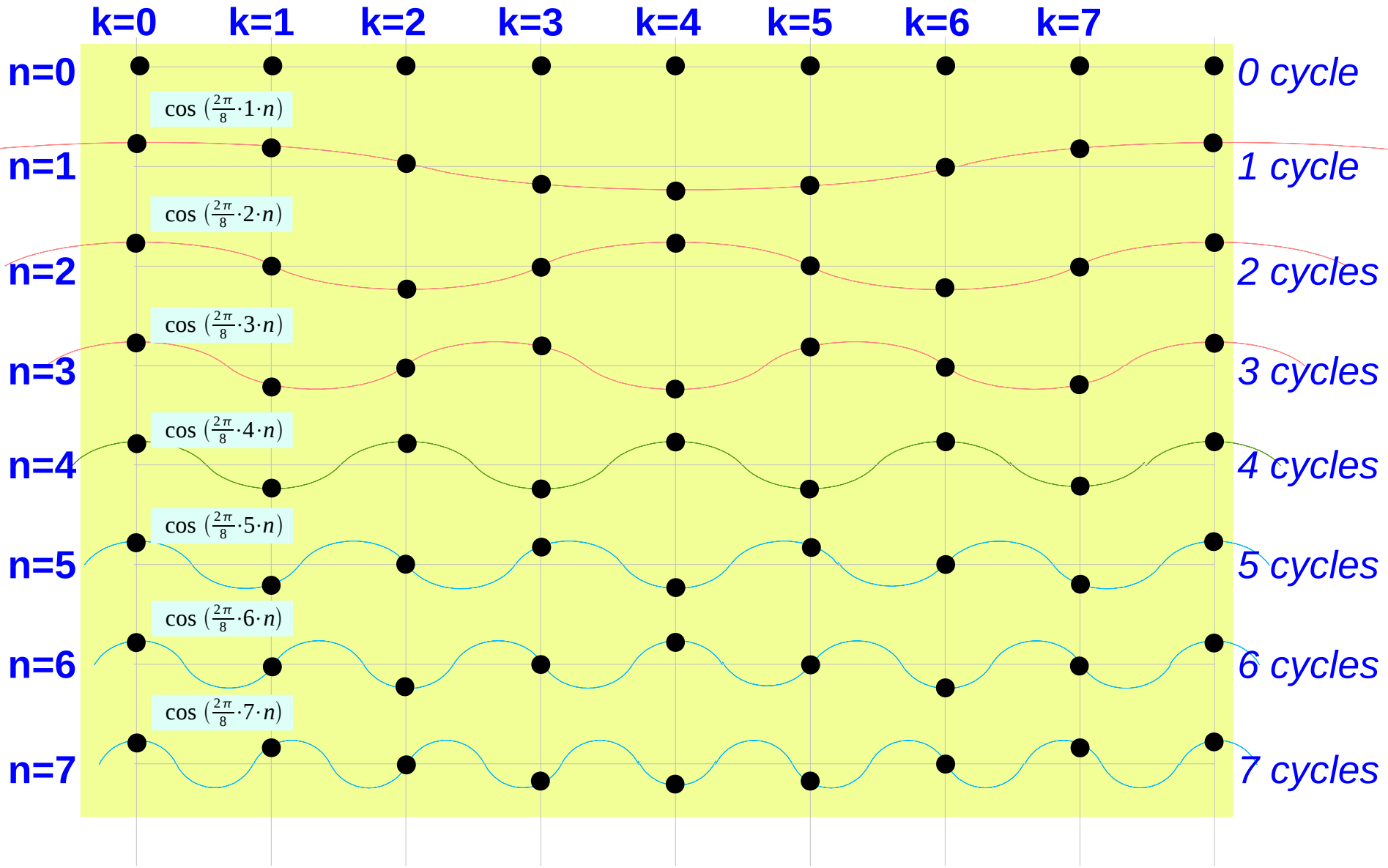
N=8 IDFT Complex Factors (1)



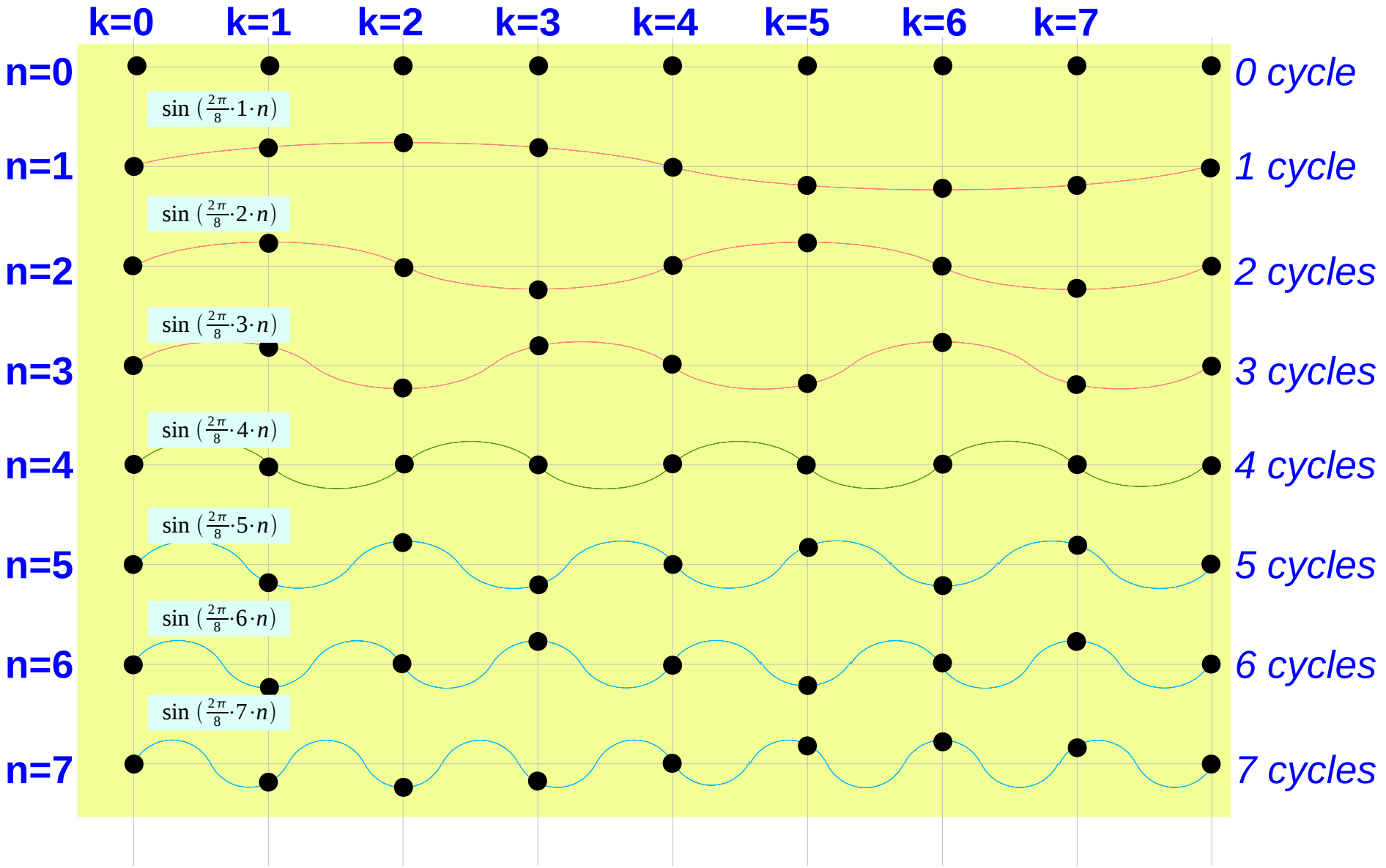
N=8 IDFT Complex Factors (2)



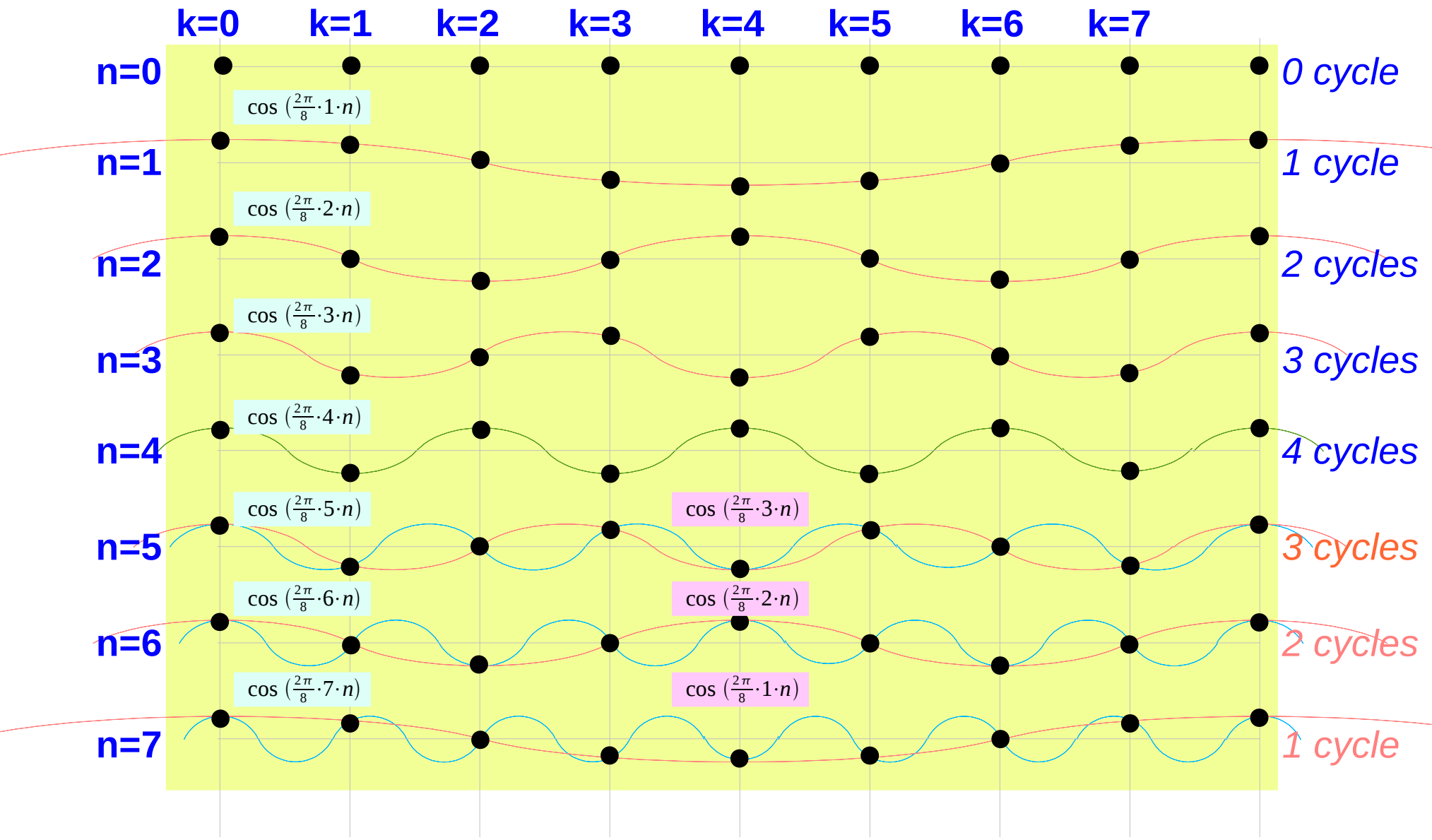
N=8 IDFT Real Factors



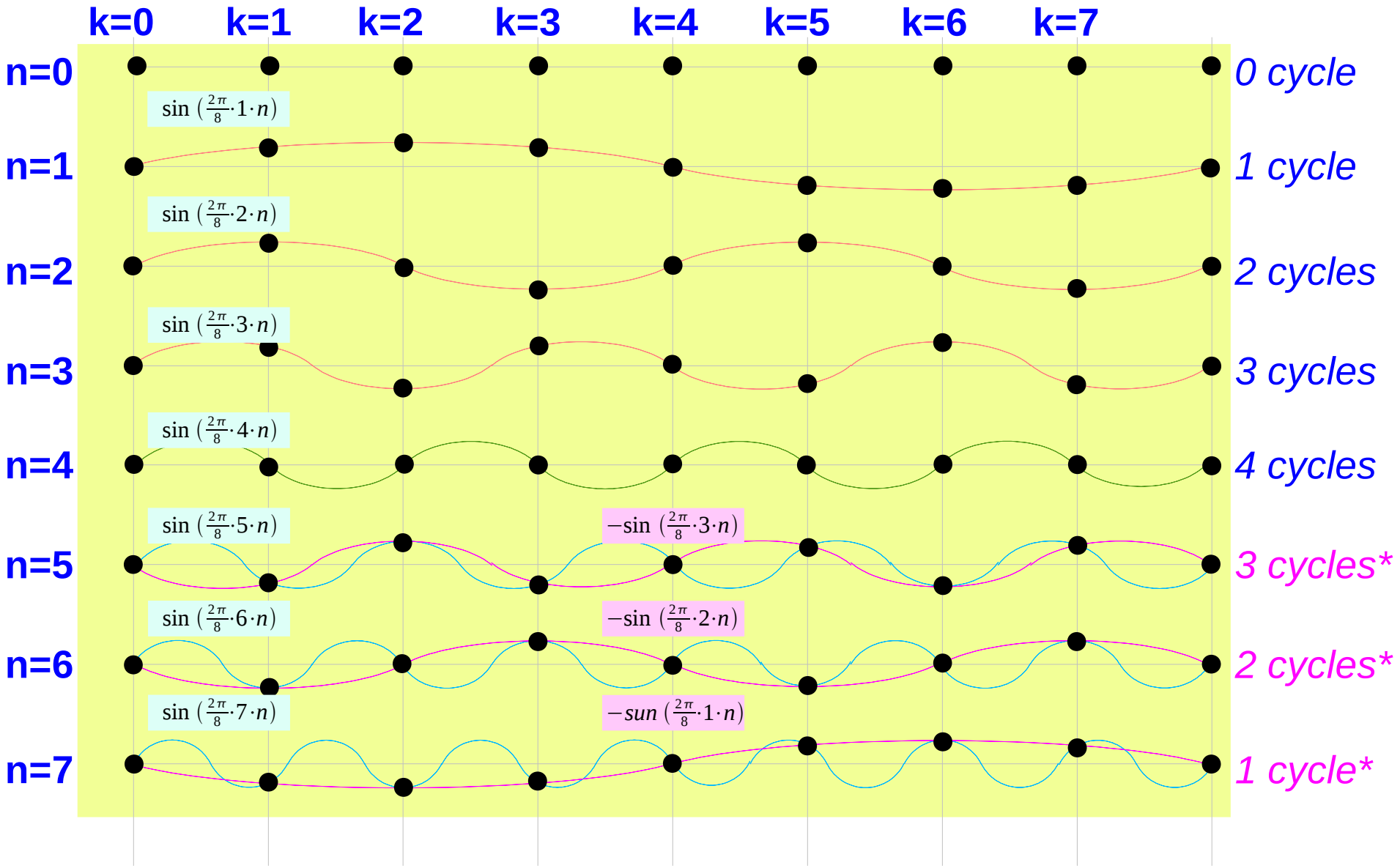
N=8 IDFT Imaginary Factors



N=8 IDFT Real Factor Symmetry



N=8 IDFT Imaginary Factor Symmetry



References

[1] http://en.wikipedia.org/
[2] J.H. McClellan, et al., Signal Processing First, Pearson Prentice Hall, 2003