Minimum Phase (3A)

Please send corrections (or suggestions) to youngwlim@hotmail.com.

This document was produced by using OpenOffice and Octave.

Properties of a Minimum Phase System

Lowest Time Delay

- Group Delay
- Energy Compaction

Invertible

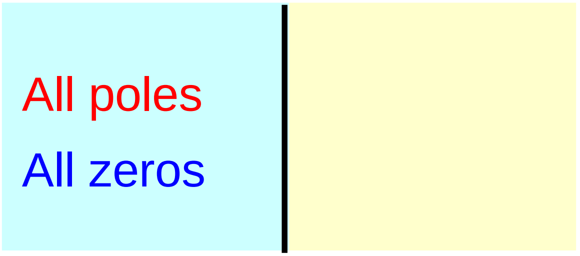
- Min Phase Filter
  - flat response
  - correct phase response
- Equalizer
  - flat response
  - incorrect phase response

Minimum Phase System

Stable Causal System

All its poles are in the left half of the s plane

Minimum Phase System



All poles

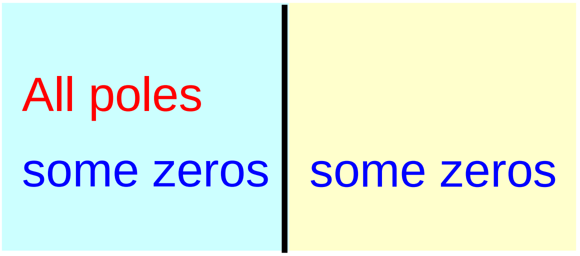
All zeros

Maximum Phase System

All poles

All zeros

Mixed Phase System



All poles

some zeros

some zeros

Minimum Phase System Properties (1)

Minimum Phase System

If an amplitude response is known

$$A(\omega) = |H(j\omega)| \qquad 0 \le \omega < \infty$$

⬇

the minimum phase response can be determined uniquely

$$\Phi_{min}(\omega) = arg\{H(j\omega)\}$$

Non-Minimum Phase System

some / all zeros in the right half s plane

With the same amplitude response

$$A(\omega) = |H(j\omega)| \qquad 0 \le \omega < \infty$$

The non-minimum phase response is always greater

$$\Phi(\omega) \ge \Phi_{min}(\omega)$$

Minimum Phase System Properties (2)

Minimum Phase System

Phase Response $\Phi(\omega)$ can be unambiguously determined from the amplitude response $A(\omega)$

→

Non-Minimum Phase System

Not valid

Verification of a Minimum Phase System

Check the progression of $\Phi(\omega)$ and $A(\omega)$ at high frequency

order m

$$H(\omega) = \frac{N(s)}{D(s)}$$

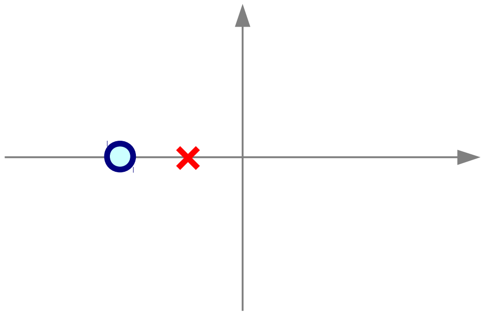
order n

	Minimum Phase System	Non-Minimum Phase System
Phase	$\Phi_{min}(\infty) = -90^\circ(n-m)$	$\lvert\Phi(\infty)\rvert \geq \lvert\Phi_{min}(\infty)\rvert$
Slope	$-20(n-m)dB\,/\,decade$	$-20(n-m)dB\,/\,decade$

Example

Minimum Phase System

$$H(s) = \frac{1+2s}{1+4s} \quad \begin{array}{l} \longrightarrow \; -0.5 \\ \longrightarrow \; -0.25 \end{array}$$



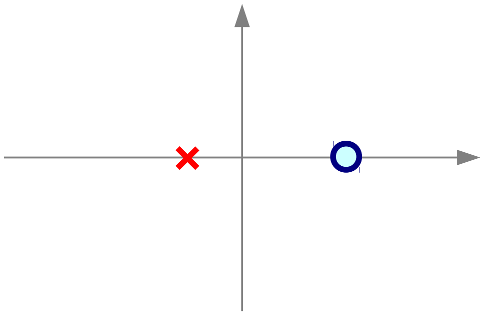
$$\frac{1+j2\omega}{1+j4\omega} = \frac{1+j2\omega}{1+j4\omega} \cdot \frac{1-j4\omega}{1-j4\omega}$$

$$= \frac{(1+8\omega^2) - j2\omega}{1+16\omega^2}$$

$$\Phi(\omega) = -\tan^{-1}\left(\frac{2\omega}{1+8\omega^2}\right)$$

Non-Minimum Phase System

$$H(s) = \frac{1-2s}{1+4s} \quad \begin{array}{l} \longrightarrow \; +0.5 \\ \longrightarrow \; -0.25 \end{array}$$



$$\frac{1-j2\omega}{1+j4\omega} = \frac{1-j2\omega}{1+j4\omega} \cdot \frac{1-j4\omega}{1-j4\omega}$$

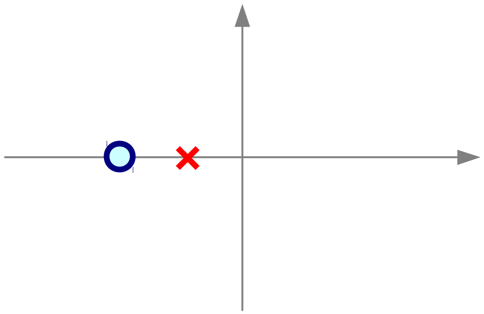
$$= \frac{(1-8\omega^2) - j6\omega}{1+16\omega^2}$$

$$\Phi(\omega) = -\tan^{-1}\left(\frac{6\omega}{1+8\omega^2}\right)$$

Example - Decomposition

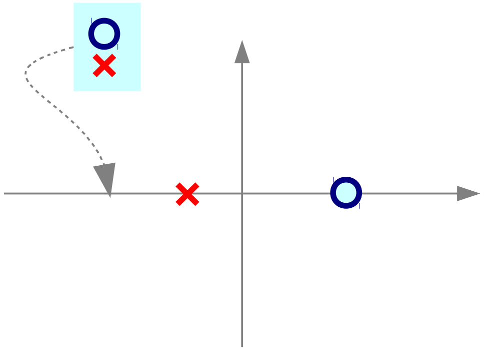
Minimum Phase System

$$H(s) = \frac{1+2s}{1+4s}$$



Non-Minimum Phase System

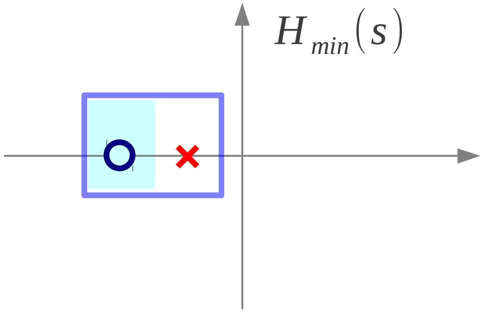
$$H(s) = \frac{1-2s}{1+4s} = \frac{1-2s}{1+4s} \cdot \frac{1+2s}{1+2s}$$

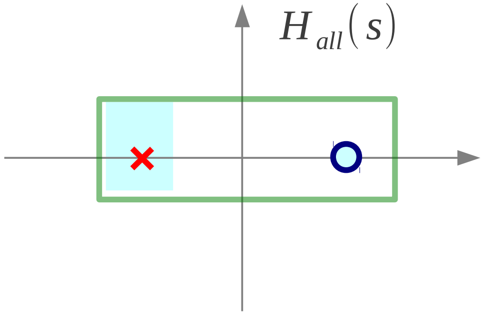


$$H(s) = \frac{1-2s}{1+4s} \cdot \frac{1+2s}{1+2s}$$

$$= \boxed{\frac{1+2s}{1+4s}} \cdot \boxed{\frac{1-2s}{1+2s}}$$

$$= H_{min}(s) \cdot H_{all}(s)$$





A non-minimum phase system can always be decomposed into $H_{min}(s) \cdot H_{all}(s)$

Example - All Pass Filter

$$H_{all}(s) = \frac{1-2s}{1+2s}$$

Flat Magnitude

$$\left|\frac{1-j2\omega}{1+j2\omega}\right| = \frac{|\ 1-j2\omega\ |}{|\ 1+j2\omega\ |}$$

$$= \frac{\sqrt{1+4\omega^2}}{\sqrt{1+4\omega^2}} = 1$$

$$|H_{all}(j\omega)| = \frac{\sqrt{1+4\omega^2}}{\sqrt{1+4\omega^2}} = 1$$

A Pure Phase Shifter

$$\frac{1-j2\omega}{1+j2\omega} = \frac{1-j2\omega}{1+j2\omega}\cdot\frac{1-j2\omega}{1-j2\omega}$$

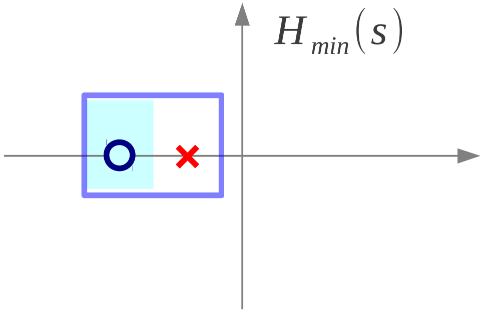
$$= \frac{(1-4\omega^2)-j4\omega}{1+4\omega^2}$$

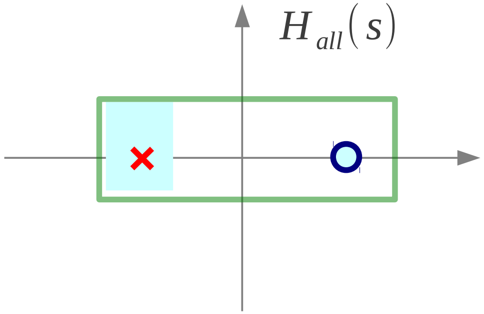
$$arg\{H_{all}(j\omega)\} = -\tan^{-1}\left(\frac{4\omega}{1-4\omega^2}\right)$$

$$H(s) = \frac{1-2s}{1+4s}\cdot\frac{1+2s}{1+2s}$$

$$= \boxed{\frac{1+2s}{1+4s}}\cdot\boxed{\frac{1-2s}{1+2s}}$$

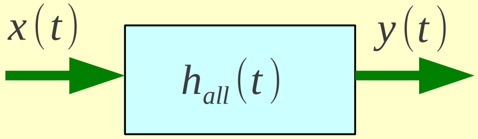
$$= H_{min}(s)\ \cdot\ H_{all}(s)$$





A non-minimum phase system can always be decomposed into $H_{min}(s)\cdot H_{all}(s)$

All Pass Filter – Energy Compaction

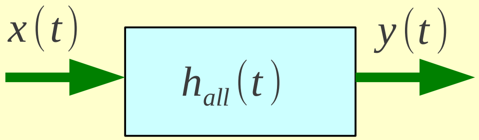


Parseval's Theorem

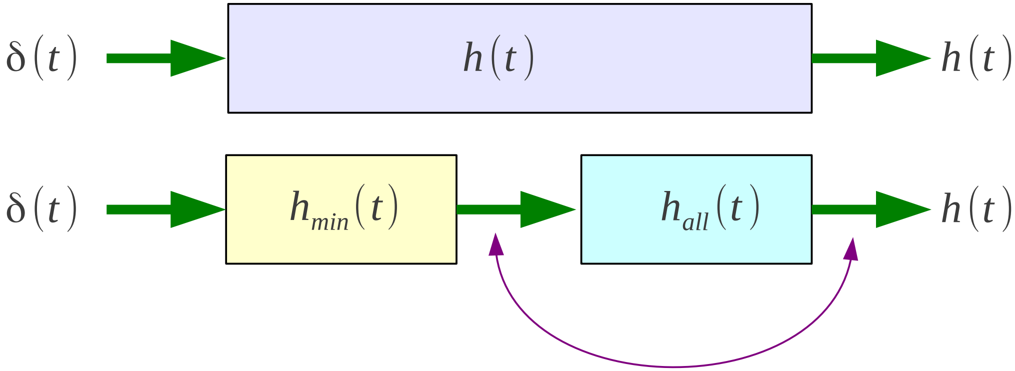
$$\int_{-\infty}^{+\infty} |x(t)|^2 \, dt = \int_{-\infty}^{+\infty} |y(t)|^2 \, dt$$

Energy Compaction

$$\int_{-\infty}^{t_0} |x(t)|^2 \, dt \geq \int_{-\infty}^{t_0} |y(t)|^2 \, dt$$



The energy build-up in the input is more **rapid** than in the output



$$\int_{-\infty}^{t_0} |h_{min}(t)|^2 \, dt \geq \int_{-\infty}^{t_0} |h(t)|^2 \, dt$$

The signal energy until t_0 of the minimum phase

≥ any other causal signal
with the same magnitude response

Thus minimum phase signals
➡ maximally concentrated toward time 0
when compared against all causal signals
having the same magnitude response

minimum phase signals
➡ minimum delay signals

Properties of a Minimum Phase System

Properties of a Minimum Phase System

References

[1] http://en.wikipedia.org/
[2] J.H. McClellan, et al., Signal Processing First, Pearson Prentice Hall, 2003
[3] http://www.libinst.com/tpfd.htm
[4] K.Shin, J.K. Hammond, “Fundamentals of Signal Processing for Sound and Vibration Engineers”
[5] www.radiolab.com.au/DesignFile/DN004.pdf