

# Power Density Spectrum - Continuous Time

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Based on  
Probability, Random Variables and Random Signal Principles,  
P.Z. Peebles,Jr. and B. Shi



# Fourier Transform

$N$  Gaussian random variables

## Definition

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

# Average Power

$N$  Gaussian random variables

## Definition

$$x_T(t) = \begin{cases} x(t) & -T < t < T \\ 0 & \text{otherwise} \end{cases}$$

$$E(T) = \int_{-T}^{+T} x^2(t) dt$$

$$P(T) = \frac{1}{2T} \int_{-T}^{+T} x^2(t) dt$$

# Measuring Average Power

$N$  Gaussian random variables

## Definition

$$P(T) = \frac{1}{2T} \int_{-T}^{+T} x^2(t) dt$$

$$\begin{aligned} P_{XX} &= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^{+T} E[x^2(t)] dt \\ &= A[E[x^2(t)]] \end{aligned}$$

# Fourier Transform of $x_T(t)$

$N$  Gaussian random variables

## Definition

$$\int_{-T}^{+T} |x^2(t)| dt < \infty$$

$$X_T(\omega) = \int_{-\infty}^{+\infty} x_T(t) e^{-j\omega t} dt$$

$$= \int_{-T}^{+T} x(t) e^{-j\omega t} dt$$

# Energy and Power of $x_T(t)$

$N$  Gaussian random variables

## Definition

$$E(T) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |X_T(\omega)| d\omega$$

$$P(T) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{|X_T(\omega)|}{2T} d\omega$$



# Power Spectrum of $x_T(t)$

$N$  Gaussian random variables

## Definition

$$P_{XX} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \lim_{T \rightarrow \infty} \frac{|X_T(\omega)|^2}{2T} d\omega$$

$$S_{XX} = \lim_{T \rightarrow \infty} \frac{|X_T(\omega)|^2}{2T}$$

$$P_{XX} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{XX}(\omega) d\omega$$

# Properties of Power Spectrum

$N$  Gaussian random variables

- $S_{XX}(\omega) \geq 0$
- $S_{XX}(-\omega) = S_{XX}(\omega)$   $X(t)$  real
- $S_{XX}(\omega)$  real
- $\frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{XX}(\omega) d\omega = A [E [X^2(t)]]$
- $S_{XX}(\omega) = \omega^2 S_{XX}(\omega)$
- $\frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{XX}(\omega) e^{j\omega t} d\omega = A [R_{XX}(t, t + \tau)]$
- $S_{XX}(\omega) = \int_{-\infty}^{+\infty} A [R_{XX}(t, t + \tau)] e^{-j\omega \tau} d\tau$

# Power Spectrum and Auto-Correlation Functions

$N$  Gaussian random variables

## Definition

$$S_{XX} = \int_{-\infty}^{+\infty} R_{XX}(\tau) e^{-j\omega\tau} d\tau$$

$$R_{XX} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{XX}(\omega) e^{+j\omega\tau} d\omega$$



