

Stationary Random Processes - Examples

Young W Lim

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Based on
Probability, Random Variables and Random Signal Principles,
P.Z. Peebles, Jr. and B. Shi

Outline

- 1 Random Phase Oscillator
 - Problem definition
 - First order distribution
 - Uniform random variable Θ
 - Uniform random variable T
 - Second order distribution
 - Mean and variance
- 2 Stationary Process Examples
 - Examples - A
 - Examples - B

Outline

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$\sin(t)$, $A\sin(t)$

- $\sin(t)$
 - not **random process**.
- $x(t) = A\sin(t)$
 - a **random process** because A is a **random variable**
 - However, $x(t)$ is not **stationary**, but it is **cyclostationary**,
 - its statistical properties vary periodically.

<https://dsp.stackexchange.com/questions/32000/why-is-sint-a-stationary-process>

$A\sin(t + \phi)$

- $x(t) = A\sin(t + \phi)$
 - the $x(t)$ process is **stationary** because of the added **random phase**
 - the random phase $\phi \in [0, 2\pi]$ is a **uniformly distributed random variable** which is independent of A .
 - its statistical properties are independent of t , and hence, the process is **stationary**.

<https://dsp.stackexchange.com/questions/32000/why-is-sint-a-stationary-process>

Signals in an oscilloscope

When analyzing a signal with an oscilloscope,
it can be observed that

the signal's **amplitude spectrum**
does not vary over moving windows

so a sinusoidal wave is sort of **stationary** in frequency.

Additionally, the signal is itself **stationary** in envelope
(modulus 1 for the analytic version of the signal).

<https://dsp.stackexchange.com/questions/32000/why-is-sint-a-stationary-process>

Window function (1)

In signal processing and statistics, a **window function** is a mathematical function that is

- zero-valued outside of some chosen interval
- normally symmetric around the middle of the interval
- usually near a maximum in the middle
- usually tapering away from the middle.

https://en.wikipedia.org/wiki/Window_function

Window function (2)

when another function or waveform is
"multiplied" by a **window function**,

the product is also zero-valued outside the interval:
all that is left is the part where they overlap,
the "*view through the window*".

https://en.wikipedia.org/wiki/Window_function

Envelope

- the **envelope** of an oscillating signal is a smooth curve outlining its extremes.
- the envelope thus generalizes the concept of a constant amplitude into an instantaneous amplitude.
- a modulated sine wave varying between an upper envelope and a lower envelope.
- the **envelope function** may be a function of time, space, angle, or indeed of any variable

[https://en.wikipedia.org/wiki/Envelope_\(waves\)](https://en.wikipedia.org/wiki/Envelope_(waves))

Random Variable Definition

A random variable

a real function over a sample space $S = \{s_1, s_2, s_3, \dots, s_n\}$

$$s \rightarrow X(s)$$

$$x = X(s)$$

a random variable : a capital letter X

a particular value : a lowercase letter x

a sample space $S = \{s_1, s_2, s_3, \dots, s_n\}$

an element of S : s

Random Variable Example

Example

$$X(s_1) = x_1 \quad s_1 \longrightarrow x_1$$

$$X(s_2) = x_2 \quad s_2 \longrightarrow x_2$$

...

$$X(s_n) = x_n \quad s_n \longrightarrow x_n$$

$$S = \{s_1, s_2, s_3, \dots, s_n\}$$

$$X = \{x_1, x_2, x_3, \dots, x_n\}$$

a sample space
a random variable

Random Process (1)

A random process

a function of both **time** t and **outcome** θ

$$X(t, \theta)$$

assigning a **time function** to every **outcome** θ_i

$$\theta_i \rightarrow x_i(t)$$

where $x_i(t) = x(t, \theta_i)$

the family of such **time functions**
is called a **random process**
and denoted by $X(t, \theta)$

Random Process (2)

A random process

a random process $X(t, \theta)$
assigns a time function for a every outcome θ

$$x(t, \theta) = X(t, \theta)$$

a short notation

$$x(t) = X(t)$$

Ensemble of time functions

Time functions

A random process $X(t, \theta)$ represents
a family or ensemble of **time functions**

$$X(t, \theta_1) = x_1(t) \quad \theta_1 \longrightarrow x_1(t) = \cos(\omega t + \theta_1)$$

$$X(t, \theta_2) = x_2(t) \quad \theta_2 \longrightarrow x_2(t) = \cos(\omega t + \theta_2)$$

...

...

$$X(t, \theta_n) = x_n(t) \quad \theta_n \longrightarrow x_n(t) = \cos(\omega t + \theta_n)$$

$S = \{ \theta_1, \theta_2, \theta_3, \dots, \theta_n \}$ a sample space

$X(t) = \{ x_1(t), x_2(t), x_3(t), \dots, x_n(t) \}$ a random process

A sample function $x(t, \theta)$

A random process $X(t, \theta)$ represents
a family or ensemble of **time functions**

$$\theta \rightarrow x(t, \theta) = \cos(\omega t + \theta)$$

$x(t, \theta)$ represents

- a **sample function**
- an ensemble member
- a realization of the process

<http://ece-research.unm.edu/bsanthan/ece541/examp.pdf>

Random process $X(t, \theta)$

A random process $X(t, \theta)$ represents
a family or ensemble of **time functions**

$$\theta \rightarrow x(t, \theta) = \cos(\omega t + \theta)$$
$$x(t) = X(t, \theta)$$

$X(t, \theta)$ becomes

- a **single time function** $x(t, \theta)$
- when t is a variable and θ is **fixed** at an **outcome**

<http://ece-research.unm.edu/bsanthan/ece541/examp.pdf>

Random variables with time

a **random process** $X(t, s)$ represents a **single time function** when t is a variable and s is fixed at an outcome

a random process $X(t, s)$ represents a **single random variable** when both t and s are fixed at a time and an outcome, respectively

$$X_i = X(t_i, s) = X(t_i)$$

random variable

$$X(t, s) = X(t)$$

random process

Random phase in $X(t) = \cos(\omega t + \Theta)$

Consider the output of a sinusoidal oscillator that has a **random phase** and an **amplitude** of the form:

$$X(t) = \cos(\omega t + \Theta)$$

where the **random variable** $\Theta \sim U([0, 2\pi])$

to specify the explicit dependence on the underlying **sample space** S the oscillator output can be written as

$$x(t, \Theta) = \cos(\omega t + \Theta)$$

Random variable $X_t(\theta)$

Consider the **random variable**

$$X(t, \theta) = \cos(\omega t + \theta)$$

where the time t is fixed

In other words,

$$X_t(\theta) = \cos(\omega t + \theta)$$

where $\theta_0 = \omega t$ is fixed (a *non-random* quantity)

thus the time t is fixed

<http://ece-research.unm.edu/bsanthan/ece541/examp.pdf>

Values of a time function

Consider the **random variable** for the fixed time t

$$X_t(\theta) = \cos(\omega t + \theta)$$

if the sample value θ as well as the time t is fixed,
then the values of the time function

$$x_1 = x(t_1) = \cos(\omega t_1 + \theta)$$

$$x_2 = x(t_2) = \cos(\omega t_2 + \theta)$$

where x is the **time function** for a fixed outcome θ and
let x_i denotes the value of the time function x at times t_i
(here x_i is not a sample function)

<http://ece-research.unm.edu/bsanthan/ece541/examp.pdf>

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First order distribution (1)

The **first order distribution** of the process $X(t) = \cos(\omega t + \Theta)$ can be found by looking at the **distribution** of the **random variable**

$$X_t(\Theta) = \cos(\theta_0 + \Theta)$$

where $\theta_0 = \omega t$ is fixed (a *non-random* quantity)
this can easily be shown via the **derivative method**
to be of the form:

$$f_X(x) = \frac{1}{\pi\sqrt{1-x^2}}, \quad |x| < 1$$

<http://ece-research.unm.edu/bsanthan/ece541/examp.pdf>

First order distribution (2)

The **first order distribution** of the process $X(t) = \cos(\omega t + \Theta)$

$$f_X(x) = \frac{1}{\pi\sqrt{1-x^2}}, \quad |x| < 1$$

- dependent only on the set of values x that the process $X(t)$ takes
- independent of
 - the particular **sampling instant** t
 - the constant **phase offset** $\theta_0 = \omega t$

<http://ece-research.unm.edu/bsanthan/ece541/examp.pdf>

pdf of $X(t) = \cos(\omega t + \Theta)$

- Uniform Random Variable Θ

<http://ece-research.unm.edu/bsanthan/ece541/examp.pdf>

pdf of $X(t) = \cos(\omega t + \Theta)$

Let Θ be a uniform random variable on $[0, 2\pi]$

Then $F_{\Theta}(\theta) = \frac{\theta}{2\pi}$,

$$X(t) = \cos(\omega t + \Theta)$$

be the random variable describing x in terms of Θ .

<http://ece-research.unm.edu/bsanthan/ece541/examp.pdf>

pdf of $X(t) = \cos(\omega t + \Theta)$

$$\begin{aligned}F_X(x) &= P(X \leq x) \\&= P(\cos(\omega t + \Theta) \leq x) \\&= P(\cos^{-1}(x) \leq \omega t + \Theta \leq 2\pi - \cos^{-1}(x)) \\&= P(\cos^{-1}(x) - \omega t \leq \Theta \leq 2\pi - \cos^{-1}(x) - \omega t) \\&= F_\Theta(2\pi - \cos^{-1}(x) - \omega t) - F_\Theta(\cos^{-1}(x) - \omega t)\end{aligned}$$

<https://math.stackexchange.com/questions/3456122/probability-density-function-of-harmonic-oscillation>

pdf of $X(t) = \cos(\omega t + \Theta)$

Differentiating both sides, we get:

$$\begin{aligned}\frac{d}{dx} F_X(x) &= \frac{d}{dx} \{ F_\Theta(2\pi - \cos^{-1}(x) - \omega t) - F_\Theta(\cos^{-1}(x) - \omega t) \} \\ &= \frac{d}{d\theta} F_\Theta(2\pi - \cos^{-1}(x) - \omega t) \frac{d}{dx} (-\cos^{-1}(x)) \\ &\quad - \frac{d}{d\theta} F_\Theta(\cos^{-1}(x) - \omega t) \frac{d}{dx} (\cos^{-1}(x))\end{aligned}$$

$$\begin{aligned}f_X(x) &= f_\Theta(\cos^{-1}(x) - \omega t) \frac{d}{dx} (-\cos^{-1}(x)) \\ &\quad - f_\Theta(2\pi - \cos^{-1}(x) - \omega t) \frac{d}{dx} (\cos^{-1}(x))\end{aligned}$$

<https://math.stackexchange.com/questions/3456122/probability-density-function-of-harmonic-oscillation>

pdf of $X(t) = \cos(\omega t + \Theta)$

$$f_X(x) = f_\Theta(\cos^{-1}(x) - \omega t) \frac{d}{dx}(-\cos^{-1}(x)) \\ - f_\Theta(2\pi - \cos^{-1}(x) - \omega t) \frac{d}{dx}(\cos^{-1}(x))$$

Now, since $f_\Theta(\theta) = \frac{1}{2\pi}$ and $\frac{d}{dx} \cos^{-1}(x) = -\frac{1}{\sqrt{1-x^2}}$, we have:

$$f_X(x) = \frac{1}{2\pi} \left(\frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-x^2}} \right) \\ = \frac{1}{\pi\sqrt{1-x^2}}$$

<https://math.stackexchange.com/questions/3456122/probability-density-function-of-harmonic-oscillation>

pdf of $X(t) = \cos(\omega t + \Theta)$

Consider the output of a sinusoidal oscillator that has a random phase and an amplitude of the form:

$$X(t) = \cos(\omega t + \Theta)$$

where Θ is a uniform random variable on $[0, 2\pi]$ then the first order pdf of $X(t)$ is

$$f_X(x) = \frac{1}{\pi\sqrt{1-x^2}}, \quad x \in (-1, 1)$$

Note that the probability is unaffected by angular velocity and initial phase (ω, θ_0) , which is, intuitively, expected.

<https://math.stackexchange.com/questions/3456122/probability-density-function-of-harmonic-oscillation>

pdf of $X = \cos(\omega T + \phi)$

- Uniform Random Variable T

<http://ece-research.unm.edu/bsanthan/ece541/examp.pdf>

PDF of $X = \cos(\omega T + \phi)$

Let T be a uniform **random variable** on $[0, \frac{2\pi}{\omega}]$ that describes time. Then $F_T(t) = \frac{\omega}{2\pi} \cdot t = ft$, where f is the oscillation's frequency.

Now, let:

$$X = \cos(\omega T + \phi)$$

be the **random variable** describing x in terms of T .
not a time function

$$X(t) \neq \cos(\omega T + \phi)$$

<http://ece-research.unm.edu/bsanthan/ece541/examp.pdf>

PDF of $X = \cos(\omega T + \phi)$

$$\begin{aligned}F_X(x) &= P(X \leq x) \\&= P(\cos(\omega T + \phi) \leq x) \\&= P(\cos^{-1}(x) \leq \omega T + \phi \leq 2\pi - \cos^{-1}(x)) \\&= P\left(\frac{\cos^{-1}(x) - \phi}{\omega} \leq T \leq \frac{2\pi - \cos^{-1}(x) - \phi}{\omega}\right) \\&= F_T\left(\frac{2\pi - \cos^{-1}(x) - \phi}{\omega}\right) - F_T\left(\frac{\cos^{-1}(x) - \phi}{\omega}\right)\end{aligned}$$

<https://math.stackexchange.com/questions/3456122/probability-density-function-of-harmonic-oscillation>

pdf of $X(t) = \cos(\omega t + \Theta)$

Differentiating both sides, we get:

$$\begin{aligned}\frac{d}{dx} F_X(x) &= \frac{d}{dx} \left\{ F_T \left(\frac{2\pi - \cos^{-1}(x) - \phi}{\omega} \right) - F_T \left(\frac{\cos^{-1}(x) - \phi}{\omega} \right) \right\} \\ &= \frac{d}{dt} F_T \left(\frac{2\pi - \cos^{-1}(x) - \phi}{\omega} \right) \frac{d}{dx} \left(-\frac{\cos^{-1}(x)}{\omega} \right) \\ &\quad - \frac{d}{dt} F_T \left(\frac{\cos^{-1}(x) - \phi}{\omega} \right) \frac{d}{dx} \left(\frac{\cos^{-1}(x)}{\omega} \right) \\ f_X(x) &= f_T \left(\frac{\cos^{-1}(x) - \phi}{\omega} \right) \frac{d}{dx} \left(-\frac{\cos^{-1}(x)}{\omega} \right) \\ &\quad - f_T \left(\frac{2\pi - \cos^{-1}(x) - \phi}{\omega} \right) \frac{d}{dx} \left(\frac{\cos^{-1}(x)}{\omega} \right)\end{aligned}$$

PDF of $X = \cos(\omega T + \phi)$

Differentiating both sides, we get:

$$f_X(x) = f_T \left(\frac{\cos^{-1}(x) - \phi}{\omega} \right) \frac{(-\cos^{-1}(x))'}{\omega} \\ - f_T \left(\frac{2\pi - \cos^{-1}(x) - \phi}{\omega} \right) \frac{(\cos^{-1}(x))'}{\omega}$$

Now, since $f_T(t) = f = \frac{\omega}{2\pi}$ and $\frac{d}{dx} \cos^{-1}(x) = -\frac{1}{\sqrt{1-x^2}}$, we have:

$$f_X(x) = \frac{1}{2\pi} \left(\frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-x^2}} \right) \\ = \frac{1}{\pi\sqrt{1-x^2}}$$

<https://math.stackexchange.com/questions/3456122/probability-density-function-of-harmonic-oscillation>

PDF of $X = \cos(\omega T + \phi)$

$$f_X(x) = \frac{1}{\pi\sqrt{1-x^2}}, \quad x \in (-1, 1)$$

the probability is unaffected by angular velocity (ω) and initial phase (ϕ), which is, intuitively, expected.

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Second order distribution (1)

to get the **second-order distribution**

use the **conditional distribution** $f_{X(t_1)|X(t_2)}(x_1|x_2)$

as in :

$$f_{X(t_1),X(t_2)}(x_1, x_2) = f_{X(t_2)}(x_2)f_{X(t_1)|X(t_2)}(x_1|x_2)$$

<http://ece-research.unm.edu/bsanthan/ece541/examp.pdf>

Second order distribution (2)

$$x_2 = x(t_2) = \cos(\omega t_2 + \theta)$$

This can happen only when :

$$\omega t_2 + \theta = \cos^{-1}(x_2)$$

$$\omega t_2 + \theta = 2\pi - \cos^{-1}(x_2)$$

where $0 \leq \cos^{-1}(x_2) \leq \pi$ and $0 \leq \theta \leq 2\pi$

<http://ece-research.unm.edu/bsanathan/ece541/examp.pdf>

Second order distribution (3)

given that $x(t_2) = x_2 = \cos(\omega t_2 + \theta)$:
find θ ,

$$\theta = \begin{cases} + (\cos^{-1}(x_2) - \omega t_2) \\ - (\cos^{-1}(x_2) + \omega t_2) \end{cases}$$

then $x(t_1) = x_1 = \cos(\omega t_1 + \theta)$ have two values

$$x(t_1) = \begin{cases} \cos(\omega t_1 + (\cos^{-1}(x(t_2)) - \omega t_2)) = x_{11} \\ \cos(\omega t_1 - (\cos^{-1}(x(t_2)) + \omega t_2)) = x_{12} \end{cases}$$

<http://ece-research.unm.edu/bsanthan/ece541/examp.pdf>

Second order distribution (4)

given that $x(t_2) = x_2 = \cos(\omega t_2 + \theta)$

find θ , then $x(t_1) = x_1 = \cos(\omega t_1 + \theta)$

has only two values with an equal probability 0.5

$$x(t_1) = \begin{cases} \cos(\omega t_1 + (\cos^{-1}(x(t_2)) - \omega t_2)) = x_{11} \\ \cos(\omega t_1 - (\cos^{-1}(x(t_2)) + \omega t_2)) = x_{12} \end{cases}$$

<http://ece-research.unm.edu/bsanthan/ece541/examp.pdf>

Second order distribution (5)

the **conditional distribution** of $x(t_1) = x_1$ given that $x(t_2) = x_2$:

$$\begin{aligned} f_{X(t_1)|X(t_2)}(x_1|x_2) &= \left(\frac{1}{2} \delta(x_1 - x_{11}) + \frac{1}{2} \delta(x_1 - x_{12}) \right) \\ &= \frac{1}{2} \delta \left(x_1 - \cos \left[\omega t_1 + \left(\cos^{-1}(x_2) - \omega t_2 \right) \right] \right) \\ &\quad + \frac{1}{2} \delta \left(x_1 - \cos \left[\omega t_1 - \left(\cos^{-1}(x_2) + \omega t_2 \right) \right] \right) \end{aligned}$$

$$\begin{aligned} f_{X(t_1)|X(t_2)}(x(t_1)|x(t_2)) &= \left(\frac{1}{2} \delta(x(t_1) - x_{11}) + \frac{1}{2} \delta(x(t_1) - x_{12}) \right) \\ &= \frac{1}{2} \delta \left(x(t_1) - \cos \left[\omega t_1 + \left(\cos^{-1}(x(t_2)) - \omega t_2 \right) \right] \right) \\ &\quad + \frac{1}{2} \delta \left(x(t_1) - \cos \left[\omega t_1 - \left(\cos^{-1}(x(t_2)) + \omega t_2 \right) \right] \right) \end{aligned}$$

First order distribution $f_X(x)$ (1)

the first order distribution of $x(t_2) = x_2 = \cos(\omega t_2 + \theta)$:

$$f_{X(t_2)}(x_2) = \frac{1}{2\pi\sqrt{1-x_2^2}}$$

$$f_{X(t_2)}(x(t_2)) = \frac{1}{2\pi\sqrt{1-x^2(t_2)}}$$

<http://ece-research.unm.edu/bsanthan/ece541/examp.pdf>

First order distribution $f_X(x)$ (2)

the **first order distribution** $f_X(x)$ of $X(t, \theta) = \cos(\omega t + \theta)$

- dependent only on the set of values x ($-1 \leq x \leq 1$) that the process $X(t, \theta)$ takes
- independent of
 - the particular sampling instant t
 - the constant phase offset $\theta_0 = \omega t$

<http://ece-research.unm.edu/bsanthan/ece541/examp.pdf>

Second order distribution (7)

The **second order pdf** of the process $X(t) = \cos(\omega t + \Theta)$

$$\begin{aligned} f_{X(t_1), X(t_2)}(x_1, x_2) &= f_{X(t_1)}(x_1) f_{X(t_2)|X(t_1)}(x_2|x_1) \\ &= f_{X(t_1)}(x_1) \left(\frac{1}{2} \delta(x_2 - x_{21}) + \frac{1}{2} \delta(x_2 - x_{22}) \right) \end{aligned}$$

$$\begin{aligned} f_{X(t_1), X(t_2)}(x_1, x_2) &= f_{X(t_2)}(x_2) f_{X(t_1)|X(t_2)}(x_1|x_2) \\ &= f_{X(t_1)}(x_2) \left(\frac{1}{2} \delta(x_1 - x_{11}) + \frac{1}{2} \delta(x_1 - x_{12}) \right) \end{aligned}$$

where $x(t_1) = x_1$ and $x(t_2) = x_2$

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Second order distribution (8)

$$\begin{aligned} f_{X(t_1), X(t_2)}(x_1, x_2) &= f_{X(t_2)}(x_2) f_{X(t_1)|X(t_2)}(x_1|x_2) \\ &= \left\{ \frac{1}{2\pi\sqrt{1-x_2^2}} \right\} \delta(x_1 - \cos[\omega t_1 + (\cos^{-1}(x_2) - \omega t_2)]) \\ &\quad + \left\{ \frac{1}{2\pi\sqrt{1-x_2^2}} \right\} \delta(x_1 - \cos[\omega t_1 - (\cos^{-1}(x_2) + \omega t_2)]) \end{aligned}$$

<http://ece-research.unm.edu/bsanthan/ece541/examp.pdf>

Second order distribution (9)

$$\begin{aligned}
 f_{X(t_1), X(t_2)}(x(t_1), x(t_2)) &= f_{X(t_2)}(x(t_2))f_{X(t_1)|X(t_2)}(x(t_1)|x(t_2)) \\
 &= \left\{ \frac{1}{2\pi\sqrt{1-x^2(t_2)}} \right\} \delta(x(t_1) - \cos[\omega t_1 + (\cos^{-1}(x(t_2)) - \omega t_2)]) \\
 &+ \left\{ \frac{1}{2\pi\sqrt{1-x^2(t_2)}} \right\} \delta(x(t_1) - \cos[\omega t_1 - (\cos^{-1}(x(t_2)) + \omega t_2)])
 \end{aligned}$$

<http://ece-research.unm.edu/bsanthan/ece541/examp.pdf>

Second order distribution (10)

The second order pdf can thus be written as

$$\begin{aligned}f_{X(t_1), X(t_2)}(x_1, x_2) &= f_{X(t_2)}(x_2) f_{X(t_2)|X(t_1)}(x_1|x_2) \\ &= f_{X(t_2)}(x_2) \left(\frac{1}{2} \delta(x_1 - x_{11}) + \frac{1}{2} \delta(x_1 - x_{12}) \right)\end{aligned}$$

$$\begin{aligned}f_{X(t_1), X(t_2)}(x(t_1), x(t_2)) &= f_{X(t_2)}(x(t_2)) f_{X(t_2)|X(t_1)}(x(t_1)|x(t_2)) \\ &= f_{X(t_2)}(x(t_2)) \left(\frac{1}{2} \delta(x(t_1) - x_{11}) + \frac{1}{2} \delta(x(t_1) - x_{12}) \right)\end{aligned}$$

These depend only on $t_2 - t_1$,
and thus the second order pdf is stationary

<http://ece-research.unm.edu/bsanthan/ece541/examp.pdf>

Second order distribution (11)

given that $x(t_2) = x_2 = \cos(\omega t_2 + \theta)$

find θ , then $x(t_1) = x_1 = \cos(\omega t_1 + \theta)$

has only two values with an equal probability 0.5

$$x(t_1) = \begin{cases} x_{11} = \cos(\omega t_1 + (\cos^{-1}(x(t_2)) - \omega t_2)) \\ x_{12} = \cos(\omega t_1 - (\cos^{-1}(x(t_2)) + \omega t_2)) \end{cases}$$

$$\begin{aligned} f_{X(t_1), X(t_2)}(x(t_1), x(t_2)) &= f_{X(t_2)}(x(t_2)) f_{X(t_2)|X(t_1)}(x(t_1)|x(t_2)) \\ &= f_{X(t_2)}(x(t_2)) \left(\frac{1}{2} \delta(x(t_1) - x_{11}) + \frac{1}{2} \delta(x(t_1) - x_{12}) \right) \end{aligned}$$

These depend only on $t_2 - t_1$,
and thus **the second order pdf is stationary**

<http://ece-research.unm.edu/bsanthan/ece541/examp.pdf>

Second order distribution (12)

$\delta(x(t_1) - x_{11})$ when $x(t_1)$ is equal to $x_{11} = \cos(\omega t_1 + \theta_1)$

$\delta(x(t_1) - x_{12})$ when $x(t_1)$ is equal to $x_{12} = \cos(\omega t_1 + \theta_2)$

$$f_{X(t_2)}(x(t_2)) = \frac{1}{2\pi\sqrt{1-x^2(t_2)}}$$

These depend only on $t_2 - t_1$,
 and thus **the second order pdf is stationary**

<http://ece-research.unm.edu/bsanthan/ece541/examp.pdf>

Second-Order Stationary Process

$$f_X(x_1, x_2; t_1, t_2)$$

if $X(t)$ is to be a **second-order stationary**

$$f_X(x_1, x_2; t_1, t_2) = f_X(x_1, x_2; t_1 + \Delta, t_2 + \Delta)$$

must be true for any time t_1, t_2 and any real number Δ

the **second order density function**

does not change with a shift in time origin

Second-Order Stationary Process

$f_X(x_1, x_2; t_1, t_2)$

- $f_X(x_1, x_2; t_1, t_2)$ is independent of t_1 and t_2
the **second order density function**
does not change with a shift in time origin

- the **autocorrelation function**

$$R_{XX}(t, t + \tau) = E[X(t)X(t + \tau)] = R_{XX}(\tau)$$

Outline

- 1 Random Phase Oscillator
 - Problem definition
 - First order distribution
 - Uniform random variable Θ
 - Uniform random variable T
 - Second order distribution
 - Mean and variance
- 2 Stationary Process Examples
 - Examples - A
 - Examples - B

Example: $X(t) = \cos(\omega t + \Theta)$

- the random process $X(t)$
- the **first-order** moments μ_X
- the **second-order** moments σ_X^2

<http://ece-research.unm.edu/bsanthan/ece541/examp.pdf>

Example: $X(t) = \cos(\omega t + \Theta)$

The **mean** of the process is obtained
by taking the **expectation** operator
with respect to the **random** parameter Θ on both sides

$$X_t(\Theta) = \cos(\omega t + \Theta)$$
$$E_{\Theta}[X_t(\Theta)] = E_{\Theta}[\cos(\omega t + \Theta)]$$

note that the **expectation** integral is a linear operation:

<http://ece-research.unm.edu/bsanthan/ece541/examp.pdf>

Example: $X(t) = \cos(\omega t + \Theta)$

$$\begin{aligned}\mu_X &= E_{\Theta}[X_t(\Theta)] = E_{\Theta}[\cos(\omega t + \Theta)] \\ &= E_{\Theta}[\cos(\omega t)\cos(\Theta) - \sin(\omega t)\sin(\Theta)] \\ &= E_{\Theta}[\cos(\Theta)]\cos(\omega t) - E_{\Theta}[\sin(\Theta)]\sin(\omega t)\end{aligned}$$

Since the random parameter Θ is uniformly distributed

$$\begin{aligned}\mu_X &= E_{\Theta}[\cos(\Theta)]\cos(\omega t) - E_{\Theta}[\sin(\Theta)]\sin(\omega t) \\ &= \cos(\omega t) \left(\frac{1}{2\pi}\right) \int_0^{2\pi} \cos(\theta) d\theta - \sin(\omega t) \left(\frac{1}{2\pi}\right) \int_0^{2\pi} \sin(\theta) d\theta \\ &= 0\end{aligned}$$

Example: $X(t) = \cos(\omega t + \Theta)$

The variance of the random process $X(t)$

$$\sigma_X^2 = E_{\Theta}[(x_t(\Theta) - \mu_X)^2] = E_{\Theta} [[x_t(\Theta)]^2] - \mu_X^2$$

Substituting the mean of the process

$$\begin{aligned}\sigma_X^2 &= \left(\frac{1}{2\pi}\right) \int_0^{2\pi} \cos^2(\omega t + \theta) d\theta \\ &= \left(\frac{1}{2\pi}\right) \int_0^{2\pi} [1 + \cos(2\omega t + 2\theta)] d\theta \\ &= \frac{1}{2}\end{aligned}$$

<http://ece-research.unm.edu/bsanthan/ece541/examp.pdf>

Example: $X(t) = \cos(\omega t + \Theta)$

the average power of the random sinusoidal signal $X(t)$

$$P_{ave}^X = \sigma_X^2 = \frac{1}{2}$$

.
the same as the average power of a sinusoid the phase is not random

<http://ece-research.unm.edu/bsanthan/ece541/examp.pdf>

Example: $X(t) = \cos(\omega t + \Theta)$

the correlation between the R.Vs $x(t_1)$ and $x(t_2)$ denoted as $R_{XX}(t_1, t_2)$

$$\begin{aligned} R_{XX}(t_1, t_2) &= E_{\Theta}[x(t_1)x(t_2)] = \int_0^{2\pi} \cos[\omega t_1 + \theta] \cos[\omega t_2 + \theta] d\theta \\ &= \left(\frac{1}{4\pi}\right) \int_0^{2\pi} \cos[\omega(t_1 + t_2) + 2\theta] d\theta \\ &\quad + \left(\frac{1}{4\pi}\right) \int_0^{2\pi} \cos[\omega(t_1 - t_2)] d\theta \\ &= \left(\frac{1}{2}\right) \cos[\omega(t_1 - t_2)] \end{aligned}$$

<http://ece-research.unm.edu/bsanthan/ece541/examp.pdf>

Example: $X(t) = \cos(\omega t + \Theta)$

The covariance of R.Vs $X(t_1)$ and $X(t_2)$ denoted $C_{XX}(t_1, t_2)$

$$C_{XX}(t_1, t_2) = R_{XX}(t_1, t_2) - \mu_X(t_1)\mu_X(t_2) = \left(\frac{1}{2}\right) \cos[\omega(t_1 - t_2)]$$

The correlation coefficient of the R.Vs $X(t_1)$ and $X(t_2)$ denoted $\rho_{XX}(t_1, t_2)$

$$\rho_{XX}(t_1, t_2) = \cos[\omega(t_1 - t_2)]$$

<http://ece-research.unm.edu/bsanthan/ece541/examp.pdf>

Example: $X(t) = \cos(\omega t + \Theta)$

Looking at the **mean** and the **variance** of the random process $X(t)$

we can see that they are shift-invariant and consequently the process is **first-order stationary**.

The ACF and other second-order statistics of the process are dependent only on the variable $\tau = t_1 - t_2$.

The random process $X(t)$ is therefore a **WSS** process also.

The ACF can then expressed in terms of the variable $\tau = t_1 - t_2$ as:

$$R_{XX}(\tau) = \left(\frac{1}{2}\right) \cos(\omega\tau)$$

<http://ece-research.unm.edu/bsanthan/ece541/examp.pdf>

Outline

- 1 Random Phase Oscillator
 - Problem definition
 - First order distribution
 - Uniform random variable Θ
 - Uniform random variable T
 - Second order distribution
 - Mean and variance
- 2 Stationary Process Examples
 - Examples - A
 - Examples - B

Example A.1: $X(t) = \cos(\omega t)$

A **white noise** is not necessarily strictly stationary.

Let ω be a **random variable uniformly distributed** in the interval $(0, 2\pi)$

define the time series $\{X(t)\}$

$$X(t) = \cos(\omega t) \quad (t = 1, 2, \dots)$$

https://en.wikipedia.org/wiki/Stationary_process

Example A.1: $X(t) = \cos(\omega t)$

Then

$$E[X(t)] = \frac{1}{2\pi} \int_0^{2\pi} \cos(t\omega) d\omega = 0$$

$$\text{Var}(X(t)) = \frac{1}{2\pi} \int_0^{2\pi} \cos^2(t\omega) d\omega = 1/2$$

$$\text{Cov}(x(t), x(s)) = \frac{1}{2\pi} \int_0^{2\pi} \cos(t\omega) \cos(s\omega) d\omega = 0 \quad \forall t \neq s$$

So $\{X(t)\}$ is a **white noise**,
however it is not strictly stationary.

https://en.wikipedia.org/wiki/Stationary_process

Example A.2: $X(t) = \cos(t + U)$

a **stationary process** example
for which any single realisation has
an apparently noise-free structure,

Let U have a uniform distribution on $(0, 2\pi]$ and
define the time series $\{X(t)\}$ by

$$X(t) = \cos(t + U) \quad \text{for } t \in \mathbb{R}$$

then $\{X(t)\}$ is **strictly stationary (SSS)**.

https://en.wikipedia.org/wiki/Stationary_process

Example A.2: $X(t) = \cos(t + U)$

Show that $X(t)$ is a **WSS** process.

We need to check two conditions:

$$\mu_X(t) = \mu_X \quad \text{for } t \in \mathbb{R}$$

$$R_X(t_1, t_2) = R_X(t_1 - t_2) \quad \text{for } t_1, t_2 \in \mathbb{R}$$

https://www.probabilitycourse.com/chapter10/10_1_4_stationary_processes.php

Example A.2: $X(t) = \cos(t + U)$

$$\begin{aligned}\mu_X(t) &= E[X(t)] \\ &= E[\cos(t + U)] \\ &= \frac{1}{2\pi} \int_0^{2\pi} \cos(t + u) du \\ &= 0, \quad \text{for all } t \in \mathbb{R}.\end{aligned}$$

https://www.probabilitycourse.com/chapter10/10_1_4_stationary_processes.php

Example A.2: $X(t) = \cos(t + U)$

$$\begin{aligned}R_X(t_1, t_2) &= E[X(t_1)X(t_2)] \\&= E[\cos(t_1 + U)\cos(t_2 + U)] \\&= E\left[\frac{1}{2}\cos(t_1 + t_2 + 2U) + \frac{1}{2}\cos(t_1 - t_2)\right] \\&= E\left[\frac{1}{2}\cos(t_1 + t_2 + 2U)\right] + E\left[\frac{1}{2}\cos(t_1 - t_2)\right] \\&= \frac{1}{2\pi} \int_0^{2\pi} \cos(t_1 + t_2 + u) du + \frac{1}{2}\cos(t_1 - t_2) \\&= 0 + \frac{1}{2}\cos(t_1 - t_2) = \frac{1}{2}\cos(t_1 - t_2), \quad \text{for all } t_1, t_2 \in \mathbb{R}.\end{aligned}$$

Example A.3: $X(t) = \alpha \cos(\omega t + \Theta)$

The random phase signal $X(t) = \alpha \cos(\omega t + \Theta)$
where $\Theta \in U[0, 2\pi]$ is **SSS**

it is known that the **first order pdf** is

$$f_{X(t)}(x) = \frac{1}{\pi\alpha\sqrt{1-(x/\alpha)^2}}, \quad -\alpha < x < +\alpha$$

which is independent of t , and is therefore **stationary**

<http://isl.stanford.edu/~abbas/ee278/lect07.pdf>

Example A.3: $X(t) = \alpha \cos(\omega t + \Theta)$

To find the **second order pdf**,
note that if we are given the value of $X(t)$ at one point, say t_1 ,
there are (at most) two possible **sample functions**

- $X(t_1) = x_1$
 - at t_1 , two sinusoid waves intersect with each other
- $X(t_2) = x_{21}$ or x_{22}
 - at t_2 , two sinusoid waves do not intersect with each other

<http://isl.stanford.edu/~abbas/ee278/lect07.pdf>

Example A.3: $X(t) = \alpha \cos(\omega t + \Theta)$

The **second order pdf** can thus be written as

$$\begin{aligned} f_{X(t_1), X(t_2)}(x_1, x_2) &= f_{X(t_1)}(x_1) f_{X(t_2)|X(t_1)}(x_2|x_1) \\ &= f_{X(t_1)}(x_1) \left(\frac{1}{2} \delta(x_2 - x_{21}) + \frac{1}{2} \delta(x_2 - x_{22}) \right) \end{aligned}$$

which depends only on $t_2 - t_1$,
and thus the second order pdf is **stationary**

<http://isl.stanford.edu/~abbas/ee278/lect07.pdf>

Example A.3: $X(t) = \alpha \cos(\omega t + \Theta)$

- if we know that $X(t_1) = x_1$ and $X(t_2) = x_2$,
the sample path is totally determined
except when $x_1 = x_2 = 0$,
- when $x_1 = x_2 = 0$,
two paths may be possible
- thus all **n-th order pdfs are stationary**

<http://isl.stanford.edu/~abbas/ee278/lect07.pdf>

Outline

- 1 Random Phase Oscillator
 - Problem definition
 - First order distribution
 - Uniform random variable Θ
 - Uniform random variable T
 - Second order distribution
 - Mean and variance
- 2 Stationary Process Examples
 - Examples - A
 - Examples - B

Example B.1: $X(t) = Y$

Let Y be any scalar **random variable**,
and define a time-series $\{X(t)\}$, by

$$X(t) = Y \quad \text{for all } t.$$

Then $\{X(t)\}$ is a **stationary** time series

- **realisations** consist of a series of **constant** values,
- a different **constant** value for each **realisation**.

https://en.wikipedia.org/wiki/Stationary_process

Example B.1: $X(t) = Y$

$$X(t) = Y \quad \text{for all } t.$$

$X(t)$ is a **first-order stationary**

$$f_X(x_1; t_1) = f_X(x_1; t_1 + \Delta) = \text{const}$$

$X(t)$ is a **second-order stationary**

$$f_X(x_1, x_2; t_1, t_2) = f_X(x_1, x_2; t_1 + \Delta, t_2 + \Delta) = \text{const}$$

$X(t)$ is to be a **N^{th} -order stationary**

$$f_X(x_1, \dots, x_N; t_1, \dots, t_N) = f_X(x_1, \dots, x_N; t_1 + \Delta, \dots, t_N + \Delta) = \text{const}$$

Example B.2: $Z(t) = X(t) + Y(t)$

Let $X(t)$ and $Y(t)$ be two jointly **WSS** random processes.

Consider the random process $Z(t)$

$$Z(t) = X(t) + Y(t)$$

Show that $Z(t)$ is **WSS**.

https://www.probabilitycourse.com/chapter10/10_1_4_stationary_processes.php

Example B.2: $Z(t) = X(t) + Y(t)$

Since $X(t)$ and $Y(t)$ are jointly WSS, we conclude

$$\mu_{X(t)} = \mu_X$$

$$\mu_{Y(t)} = \mu_Y$$

$$R_X(t_1, t_2) = R_X(t_1 - t_2)$$

$$R_Y(t_1, t_2) = R_Y(t_1 - t_2)$$

$$R_{XY}(t_1, t_2) = R_{XY}(t_1 - t_2)$$

https://www.probabilitycourse.com/chapter10/10_1_4_stationary_processes.php

Example B.2: $Z(t) = X(t) + Y(t)$

Since $X(t)$ and $Y(t)$ are jointly WSS, we conclude

$$\begin{aligned}\mu_Z(t) &= E[X(t) + Y(t)] \\ &= E[X(t)] + E[Y(t)] \\ &= \mu_X + \mu_Y.\end{aligned}$$

https://www.probabilitycourse.com/chapter10/10_1_4_stationary_processes.php

Example B.2: $Z(t) = X(t) + Y(t)$

Since $X(t)$ and $Y(t)$ are jointly WSS, we conclude

$$\begin{aligned}R_Z(t_1, t_2) &= E[(X(t_1) + Y(t_1))(X(t_2) + Y(t_2))] \\ &= E[X(t_1)X(t_2)] + E[X(t_1)Y(t_2)] \\ &\quad + E[Y(t_1)X(t_2)]E[Y(t_1)Y(t_2)] \\ &= R_X(t_1 - t_2) + R_{XY}(t_1 - t_2) \\ &\quad + R_{YX}(t_1 - t_2) + R_Y(t_1 - t_2).\end{aligned}$$

https://www.probabilitycourse.com/chapter10/10_1_4_stationary_processes.php

Example B.3: $X(t) = \pm \sin t, \pm \cos t$

Let

$$X(t) = \begin{cases} +\sin t & p_0 = \frac{1}{4} \\ -\sin t & p_1 = \frac{1}{4} \\ +\cos t & p_2 = \frac{1}{4} \\ -\cos t & p_3 = \frac{1}{4} \end{cases}$$

$$E[X(t)] = 0$$

$$R_X(t_1, t_2) = \frac{1}{2} \cos(t_2 - t_1)$$

thus $X(t)$ is WSS

Example B.3: $X(t) = \pm \sin t, \pm \cos t$

Let

$$X(t) = \begin{cases} +\sin t & p_0 = \frac{1}{4} \\ -\sin t & p_1 = \frac{1}{4} \\ +\cos t & p_2 = \frac{1}{4} \\ -\cos t & p_3 = \frac{1}{4} \end{cases}$$

But $X(0)$ and $X(\frac{\pi}{4})$ do not have the same pmf (different ranges), so the first order pmf is not stationary, and the process is not **SSS**

<http://isl.stanford.edu/~abbas/ee278/lect07.pdf>

