Equivalence Relations (4A)

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Equivalence Relation

a binary relation that is at the same time

- a reflexive relation,
- a symmetric relation and
- a transitive relation.

The relation "**is equal to**" is a primary example of an equivalence relation.

Thus for any numbers a, b, and c:

a=a (reflexive property),
if **a=b** then **b=a** (symmetric property), and
if **a=b** and **b=c** then **a=c** (transitive property).

Any equivalence relation, as a consequence of
the reflexive, symmetric, and transitive properties,
provides a **partition** of a set into **equivalence classes**.

https://en.wikipedia.org/wiki/Equivalence_relation

Equivalence Relation Definition

A given binary relation **~** on a set **X**
is said to be an equivalence relation
if and only if it is reflexive, symmetric and transitive.

That is, for all a, b and c in **X**:

a ~ a. (Reflexivity)
a ~ b if and only if **b ~ a**. (Symmetry)
if **a ~ b** and **b ~ c** then **a ~ c**. (Transitivity)

X together with the relation **~** is called a **setoid**.
The **equivalence class** of **a** under **~**, denoted **[a]**,
is defined as $[a] = \{ b \in X \mid a \sim b \}$

https://en.wikipedia.org/wiki/Equivalence_relation

Congruent Modulo n

for a positive integer **n**, two numbers **a** and **b** are said to be **congruent modulo n**, if their difference **a − b** is an integer multiple of **n** (that is, if there is an integer **k** such that **a − b = kn**).
This congruence relation is typically considered when a and b are integers, and is denoted

$$a \equiv b \pmod{n}$$

(some authors use = instead of ≡)

a = b mod n **// a = b % n**

(this generally means that "mod" denotes the modulo operation, that is, that $0 \le a < n$).

The number n is called the **modulus** of the congruence.

https://en.wikipedia.org/wiki/Equivalence_relation

Congruent Modulo n : Examples

For example,

$$38 \equiv 14 \pmod{12}$$

because $38 - 14 = 24$, which is a multiple of 12,
or, equivalently, because both 38 and 14 have
the same remainder 2 when divided by 12.

The same rule holds for negative values:

$$-8 \equiv 7 \pmod{5}$$
$$2 \equiv -3 \pmod{5}$$
$$-3 \equiv -8 \pmod{5}$$

https://en.wikipedia.org/wiki/Equivalence_relation

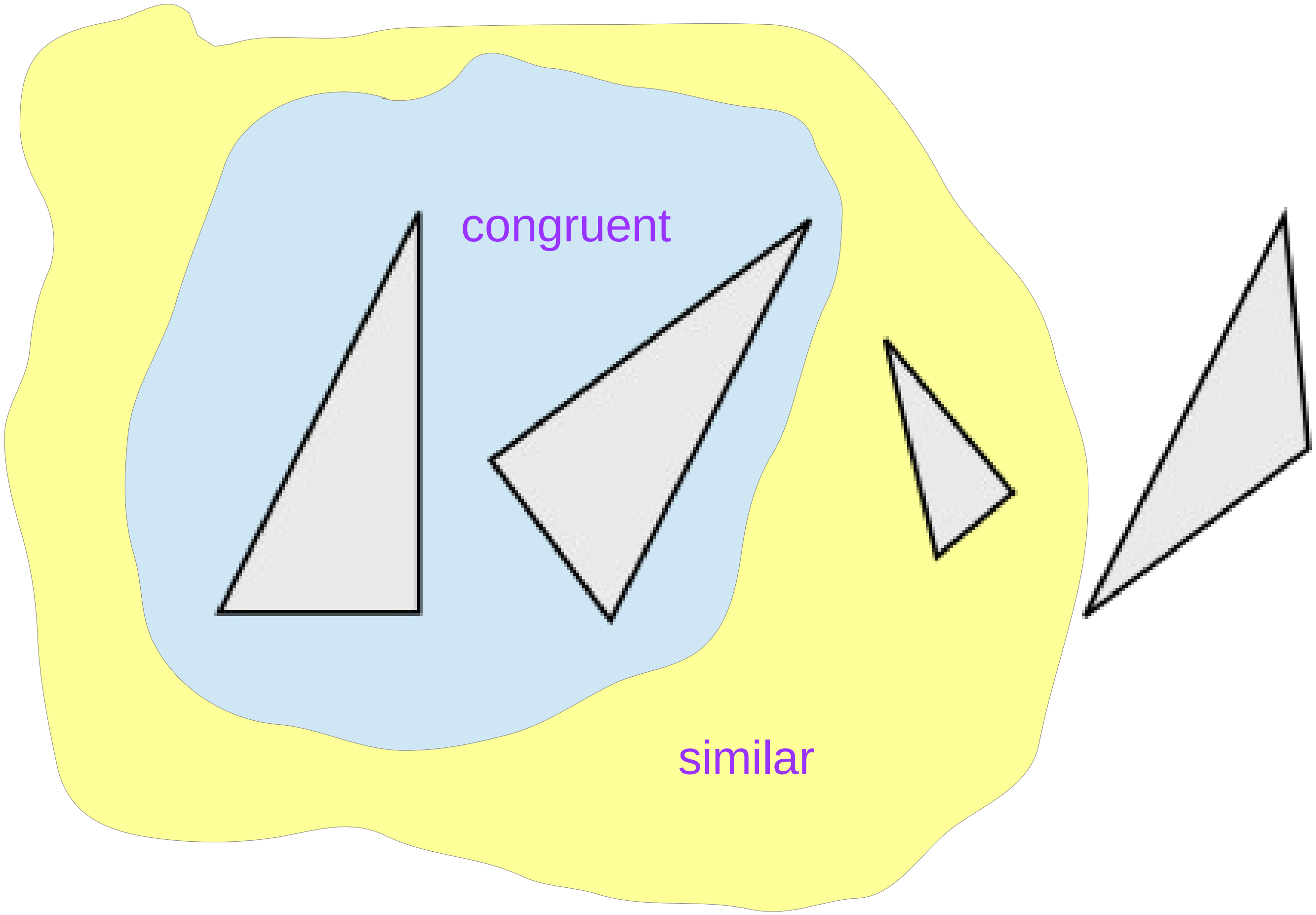
Congruent Modulo n : Properties

The congruence relation satisfies all the conditions of an equivalence relation:

Reflexivity: **a ≡ a (mod n)**
Symmetry: **a ≡ b (mod n)** if and only if **b ≡ a (mod n)**
Transitivity: If **a ≡ b (mod n)** and **b ≡ c (mod n)**, then **a ≡ c (mod n)**

https://en.wikipedia.org/wiki/Equivalence_relation

Congruence and Similarity in Geometry



https://en.wikipedia.org/wiki/Congruence_(geometry)

Equivalence Relation

Equivalence Relation ⟺ Reflexive Relation & Symmetric Relation & Transitive Relation

$$A = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$$

$$R \subset A \times A$$

$$R = \{ (a, b) \mid a \equiv b \ (mod\ 3) \}$$

Equivalence Class

$$A = \{0,1,2,3,4,5,6,7,8\}$$

$$R \subset A \times A$$

$$R = \{(a,b) \mid a \equiv b \ (mod\ 3)\}$$

$(0,0),(0,3),(0,6),$
$(3,0),(3,3),(3,6),$
$(6,0),(6,3),(6,6)$

$0 \sim 0,\ 0 \sim 3,\ 0 \sim 6,$
$3 \sim 0,\ 3 \sim 3,\ 3 \sim 6,$
$6 \sim 0,\ 6 \sim 3,\ 6 \sim 6$

$[0] = \{0,\ 3,\ 6\}$

$[0] \subset A$

$(1,1),(1,4),(1,7),$
$(4,1),(4,4),(4,7),$
$(7,1),(7,4),(7,7)$

$1 \sim 1,\ 1 \sim 4,\ 1 \sim 7,$
$4 \sim 1,\ 4 \sim 4,\ 4 \sim 7,$
$7 \sim 1,\ 7 \sim 4,\ 7 \sim 7$

$[1] = \{1,\ 4,\ 7\}$

$[1] \subset A$

$(2,2),(2,5),(2,8),$
$(5,2),(5,5),(5,8),$
$(8,2),(8,5),(8,8)$

$2 \sim 2,\ 2 \sim 5,\ 2 \sim 8,$
$5 \sim 2,\ 5 \sim 5,\ 5 \sim 8,$
$8 \sim 2,\ 8 \sim 5,\ 8 \sim 8$

$[2] = \{2,\ 5,\ 8\}$

$[2] \subset A$

Partitions

$$A = \{0,1,2,3,4,5,6,7,8\}$$
$$R \subset A \times A$$
$$R = \{(a,b) \mid a \equiv b \pmod 3\}$$

$[0] = \{0, 3, 6\}$ $\quad$ $[1] = \{1, 4, 7\}$ $\quad$ $[2] = \{2, 5, 8\}$

$[0] \subset A$ $\quad$ $[1] \subset A$ $\quad$ $[2] \subset A$

$[0] \cap [1] = \emptyset$ $\quad$ $[1] \cap [2] = \emptyset$ $\quad$ $[2] \cap [0] = \emptyset$

$$[0] \cup [1] \cup [2] = \{0, 3, 6\} \cup \{1, 4, 7\} \cup \{2, 5, 8\} = \{0,1,2,3,4,5,6,7,8\} = A$$

$[0] = [3] = [6]$ $\quad$ $[1] = [4] = [7]$ $\quad$ $[2] = [5] = [8]$

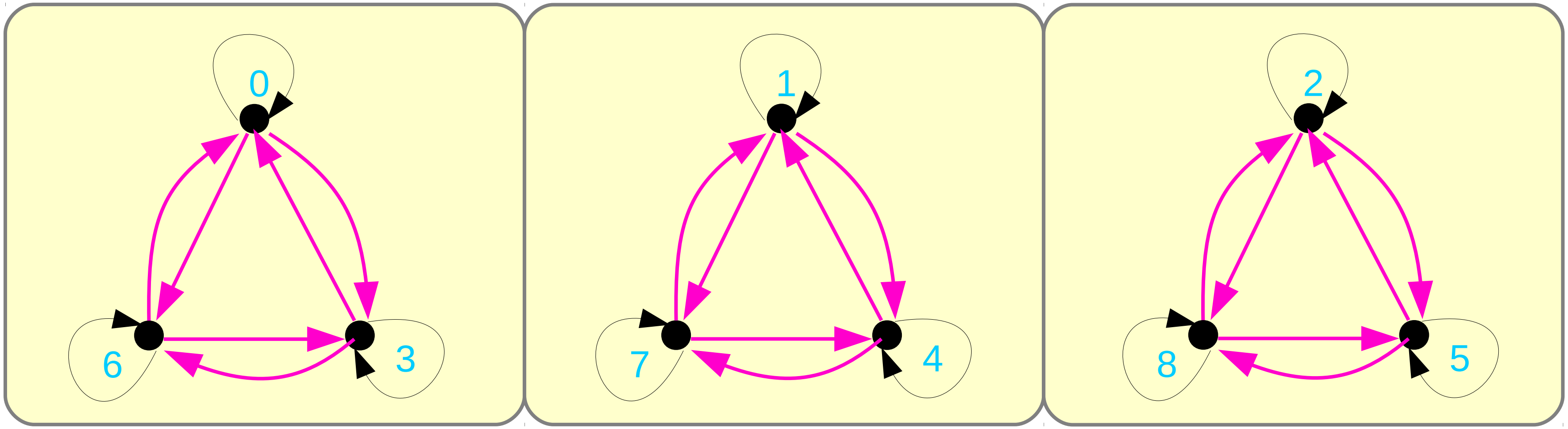
Equivalence Relation Examples

$A = \{0,1,2,3,4,5,6,7,8\}$

$R \subset A \times A$

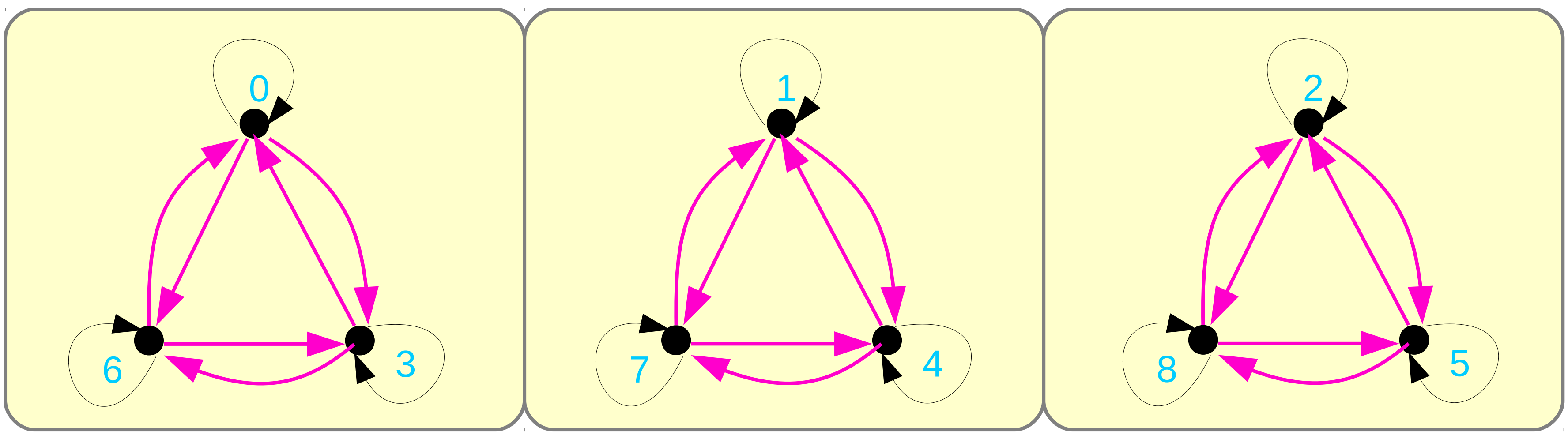
$R = \{(a,b) \mid a \equiv b \ (mod\ 3)\}$

$$RR = \begin{array}{c|ccccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \hline 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 3 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 4 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 5 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 6 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 7 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 8 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{array}$$



Equivalence Classes

Class #1 Class #2 Class #3



$$\begin{array}{c|ccc} & 0 & 3 & 6 \\ \hline 0 & 1 & 1 & 1 \\ 3 & 1 & 1 & 1 \\ 6 & 1 & 1 & 1 \end{array} \qquad \begin{array}{c|ccc} & 1 & 4 & 7 \\ \hline 1 & 1 & 1 & 1 \\ 4 & 1 & 1 & 1 \\ 7 & 1 & 1 & 1 \end{array} \qquad \begin{array}{c|ccc} & 2 & 5 & 8 \\ \hline 2 & 1 & 1 & 1 \\ 5 & 1 & 1 & 1 \\ 8 & 1 & 1 & 1 \end{array}$$

partition $P_1 = \{0,3,6\}$ partition $P_2 = \{1,4,7\}$ partition $P_3 = \{2,5,8\}$

Equivalence Class

$A = \mathbf{Z}^{+} = \{0, 1, 2, 3, 4, 5, 6, \cdots\}$

$R \subset A \times A$

$R = \{(a, b) \mid a \equiv b \ (mod\ 3)\}$

$\{0,\ 3,\ 6,\ 9,\ \cdots\}$	$[0]$	$[33]$
$\{1,\ 4,\ 7,\ 10,\ \cdots\}$	$[1]$	$[331]$
$\{2,\ 5,\ 8,\ 11,\ \cdots\}$	$[2]$	$[3332]$

https://www.cse.iitb.ac.in/~nutan/courses/cs207-12/notes/lec7.pdf

References

[1] http://en.wikipedia.org/
[2]