CTFT (2B)

- Continuous Time Fourier Transform

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Conversion From CTFS to CTFT

$T_0 \rightarrow \infty$

$x_{T_0}(t) \rightarrow x(t)$

Periodic Aperiodic

$$\omega_0 = \left(\frac{2\pi}{T_0}\right) \rightarrow 0$$

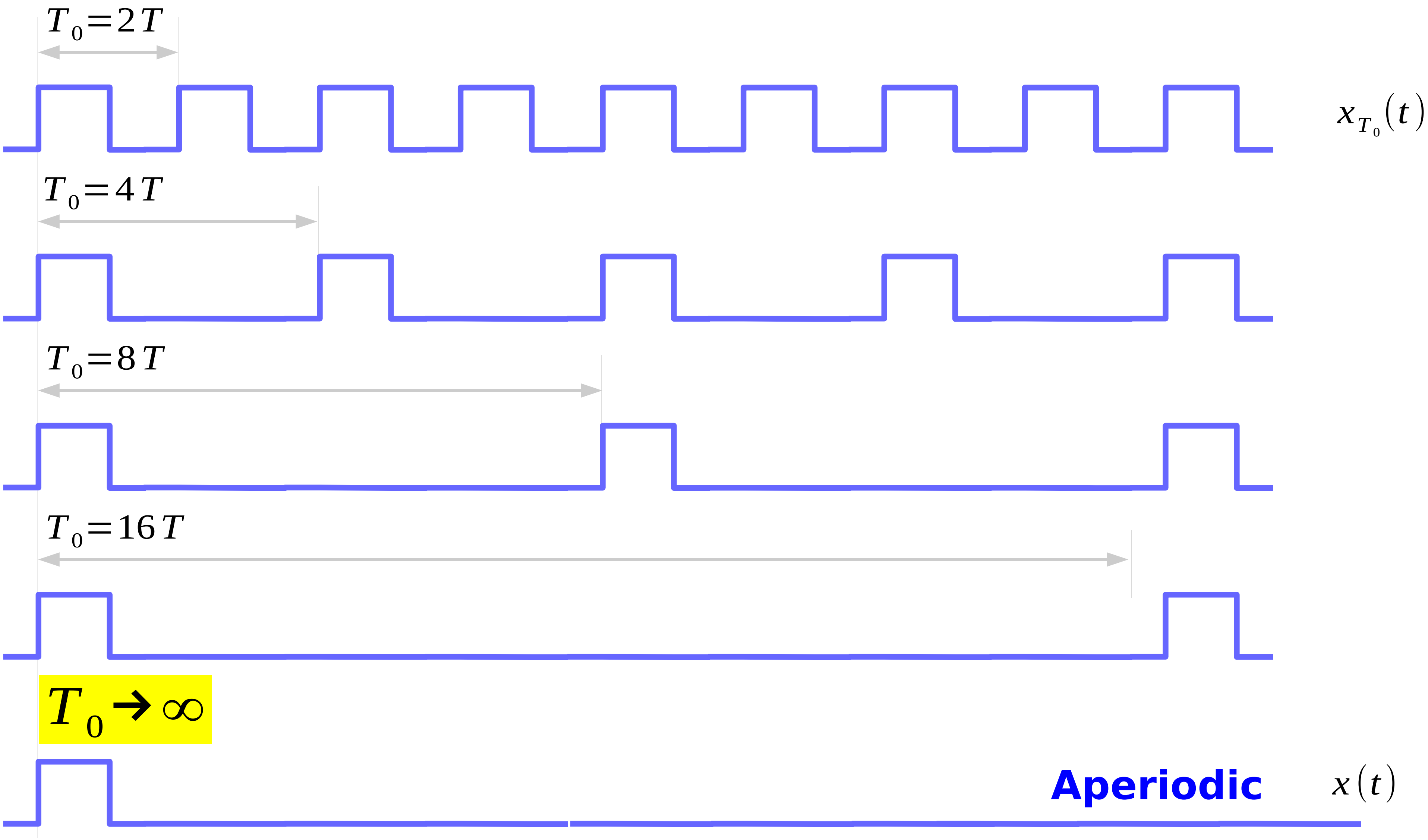
$\omega_0 \rightarrow d\omega$

$k\omega_0 \rightarrow \omega$

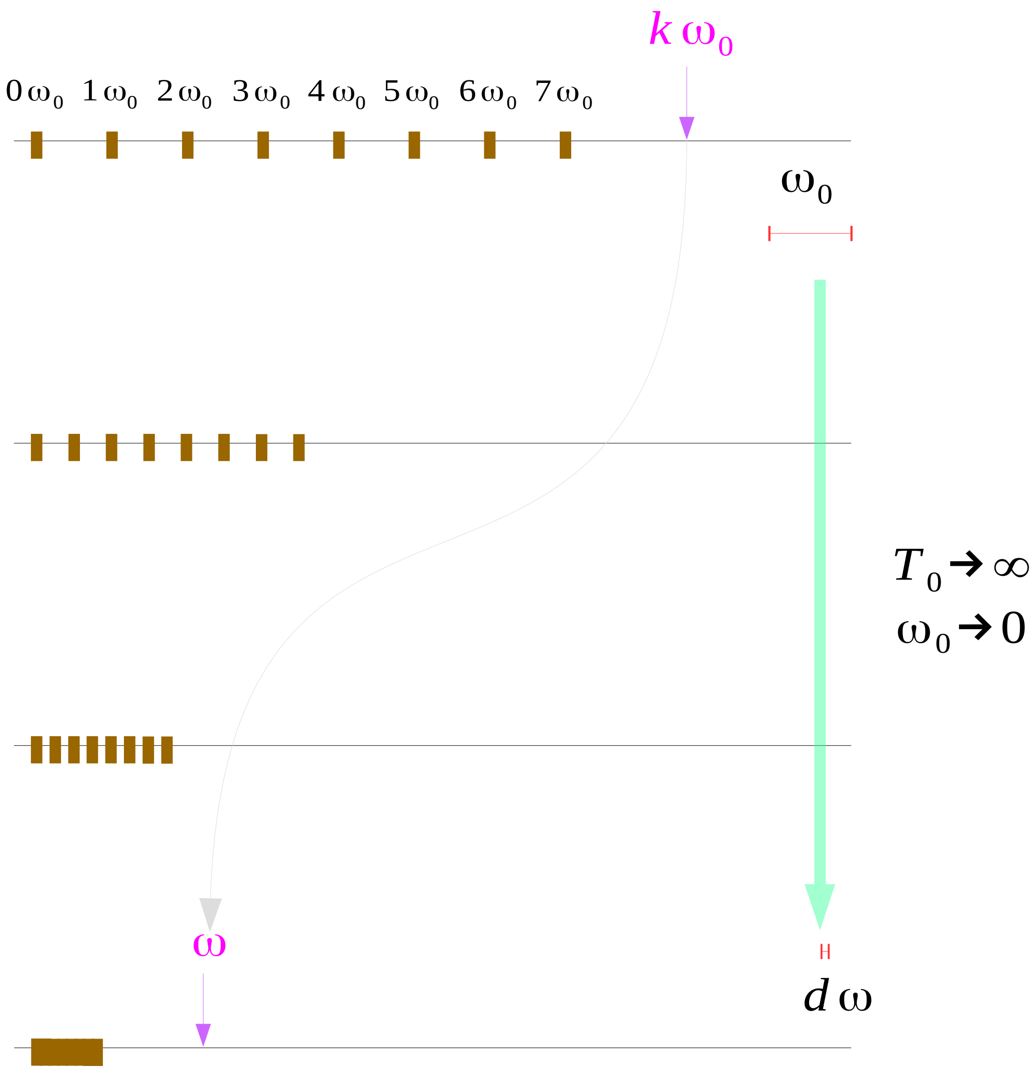
$C_k T_0 \rightarrow X(j\omega)$

Discrete Frequency Continuous Frequency

Aperiodic Signal Conversion $x(t)$



Limit Values ω , $d\omega$



C_k *at* $k\omega_0$

$T_0 \rightarrow \infty$

$$\omega_0 = \left(\frac{2\pi}{T_0}\right) \rightarrow 0$$

$$\omega_0 \rightarrow d\omega$$

$$k\omega_0 \rightarrow \omega$$

$X(j\omega)$ *at* ω

From CTFS to CTFT

$$x_{T_0}(t) = \sum_{k=-\infty}^{+\infty} C_k\, e^{+j\omega_0 k t} \cdot 1$$

$$= \sum_{k=-\infty}^{+\infty} C_k\, e^{+j\omega_0 k t} \cdot \left(\frac{T_0}{2\pi}\right) \cdot \left(\frac{2\pi}{T_0}\right)$$

$$= \frac{1}{2\pi} \sum_{k=-\infty}^{+\infty} C_k T_0\, e^{+j\omega_0 k t} \cdot \left(\frac{2\pi}{T_0}\right)$$

$$x_{T_0}(t) = \frac{1}{2\pi} \sum_{k=-\infty}^{+\infty} C_k T_0\, e^{+j\omega_0 k t} \cdot \omega_0$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega$$

$T_0 \rightarrow \infty$ $\quad \omega_0 = \left(\frac{2\pi}{T_0}\right) \rightarrow 0$

$\omega_0 \rightarrow d\omega, \quad k\omega_0 \rightarrow \omega$

$x_{T_0}(t) \rightarrow x(t), \quad C_k T_0 \rightarrow X(j\omega)$

The Product $X(j\omega) \cdot d\omega$

$C_k T_0 \rightarrow X(j\omega)$

$$T_0 \omega_0 = T_0 \left(\frac{2\pi}{T_0} \right) = 2\pi$$

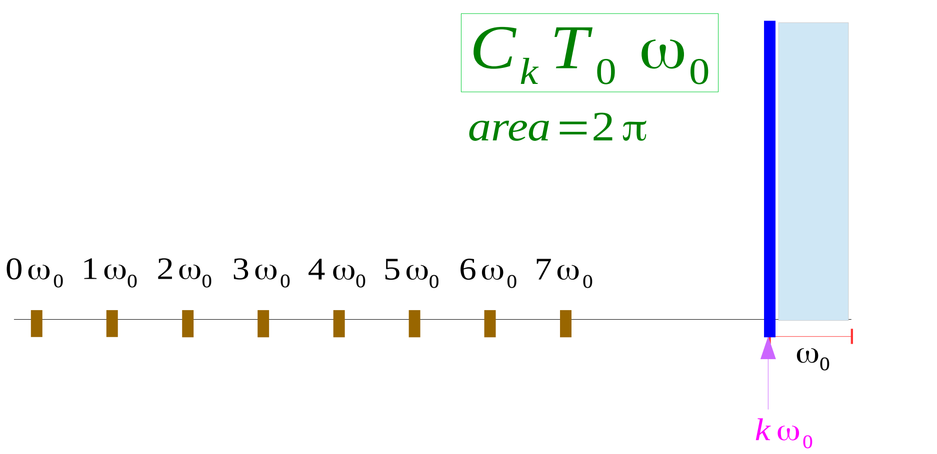
$C_k T_0$

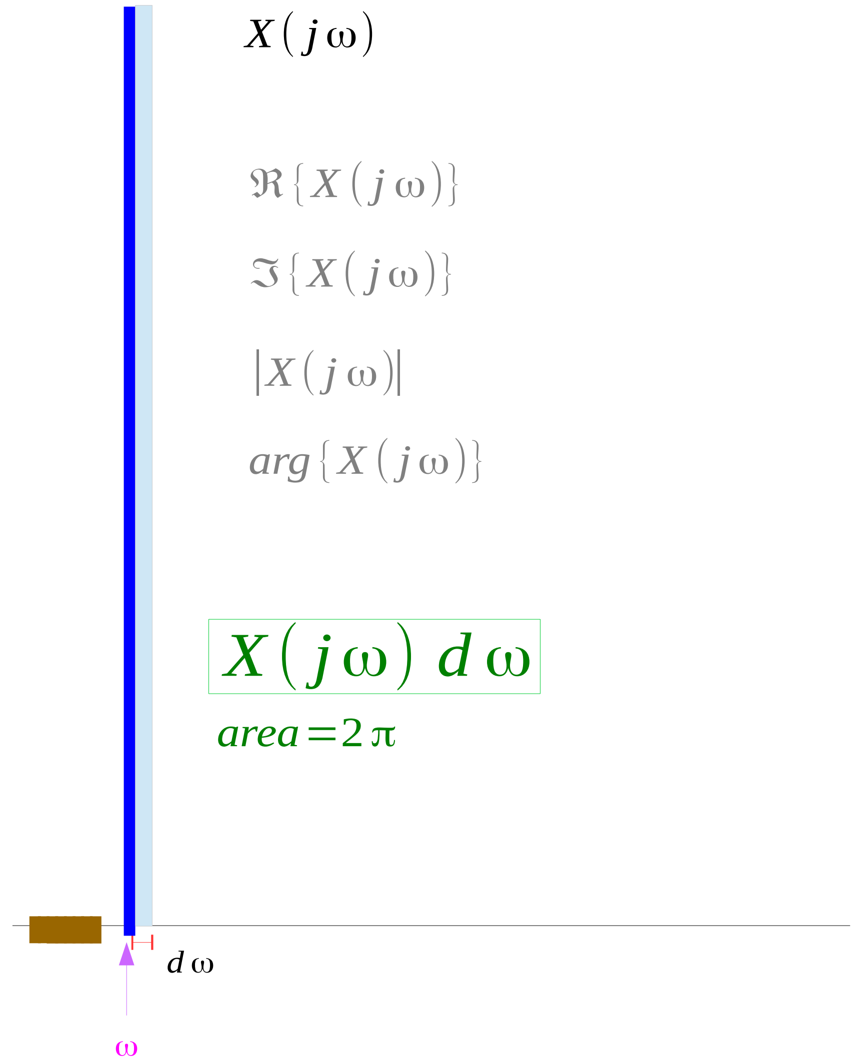
$\Re\{C_k T_0\}$

$\Im\{C_k T_0\}$

$|C_k T_0|$

$arg\{C_k T_0\}$





CTFS and CTFT

Continuous Time Fourier **Series**

$$C_k = \frac{1}{T}\int_0^T x(t)\, e^{-j\omega_0 k t}\, dt \quad \Leftrightarrow \quad x(t) = \sum_{k=-\infty}^{+\infty} C_k\, e^{+j\omega_0 k t}$$

$$C_k = \frac{1}{T_0}\int_{T_0} x_{T_0}(t)\, e^{-jk\omega_0 t}\, dt$$

$$C_k T_0 = \int_{T_0} x_{T_0}(t)\, e^{-jk\omega_0 t}\, dt$$

$$x_{T_0}(t) = \sum_{k=-\infty}^{\infty} C_k\, e^{+jk\omega_0 t} \cdot \frac{2\pi}{2\pi} \cdot \frac{T_0}{T_0}$$

$$x_{T_0}(t) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} C_k T_0\, e^{+jk\omega_0 t} \cdot \frac{2\pi}{T_0}$$

$$T_0 \to \infty, \quad \omega_0 \to d\omega \quad \left(\frac{2\pi}{T_0} \to 0\right), \quad k\omega_0 \to \omega \quad \Rightarrow \quad x_{T_0}(t) \to x(t), \quad C_k T_0 \to X(j\omega)$$

Continuous Time Fourier **Transform**

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt \quad \Leftrightarrow \quad x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega$$

(A) A Time Domain Impulse Function
(B) A Frequency Domain Impulse Function
(C) A Sinusoidal Function
(D) An Impulse Train
(E) A Periodic Signal

(A) A Time Domain Impulse

Continuous Time Fourier Transform

$$x(t) = A\delta(t) \quad \Longleftrightarrow \quad X(j\omega) = A$$

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt$$

$$X(j\omega) = \int_{-\infty}^{+\infty} A\delta(t) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{+\infty} A\delta(t) e^{0} dt$$

$$= A\int_{-\infty}^{+\infty} \delta(t) dt = A$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} A e^{+j\omega t} d\omega$$

$$= \frac{A}{2\pi}\int_{-\infty}^{+\infty} e^{+j\omega t} d\omega = A\delta(t)$$

(B) A Frequency Domain Impulse

Continuous Time Fourier <u>Transform</u>

$$X(j\omega) = 2\pi\,\delta(\omega) \iff x(t) = 1$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t} d\omega$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} 2\pi\,\delta(\omega) e^{+j\omega t} d\omega$$

$$= \int_{-\infty}^{+\infty} \delta(\omega) e^{0} d\omega = 1$$

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t} dt$$

$$X(j\omega) = \int_{-\infty}^{+\infty} e^{-j\omega t} dt = 2\pi\,\delta(\omega)$$

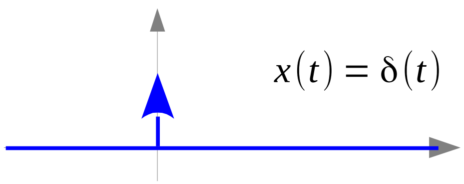
Time and Impulse Domain Impulse Functions

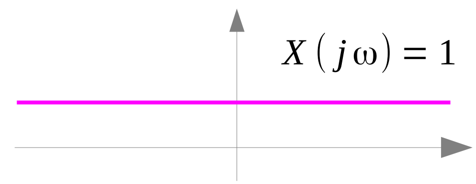
$$x(t) = \delta(t)$$

$$X(j\omega) = \int_{-\infty}^{+\infty} \delta(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{+\infty} \delta(t)\, e^{-j\omega 0}\, dt$$

$$= \int_{-\infty}^{+\infty} \delta(t)\, dt = 1$$



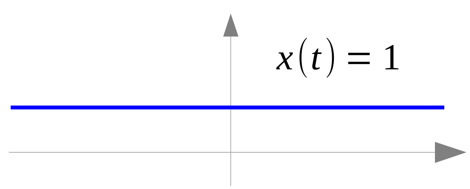


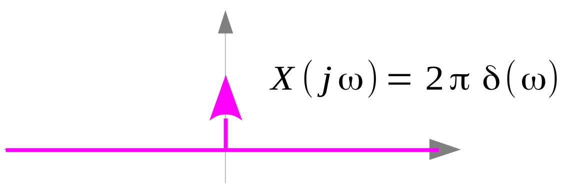
$$X(j\omega) = 2\pi\,\delta(\omega)$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} 2\pi\,\delta(\omega)\, e^{+j\omega t}\, d\omega$$

$$= \int_{-\infty}^{+\infty} \delta(\omega)\, e^{+j0t}\, d\omega$$

$$= \int_{-\infty}^{+\infty} \delta(\omega)\, d\omega = 1$$





Frequency Shifted Impulse

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} 2\pi\,\delta(\omega - \omega_0)\, e^{+j\omega t}\, d\omega$$

$$= \int_{-\infty}^{+\infty} \delta(\omega - \omega_0)\, e^{+j\omega_0 t}\, d\omega$$

$$= e^{+j\omega_0 t}\int_{-\infty}^{+\infty} \delta(\omega)\, d\omega$$

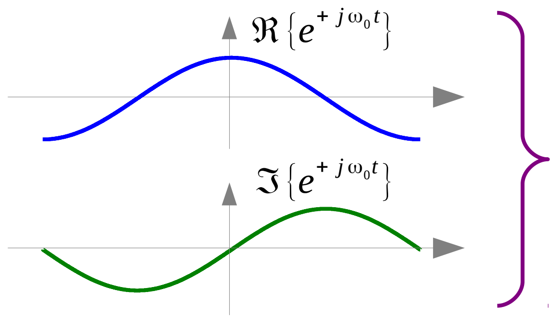
$$= e^{+j\omega_0 t}$$

$$= \cos\omega_0 t + j\sin\omega_0 t$$

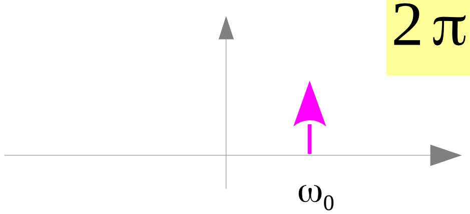
$$X(j\omega) = 2\pi\,\delta(\omega - \omega_0)$$

$$\omega_0 = \frac{2\pi}{T_0}$$

$$e^{+j\omega_0 t}$$

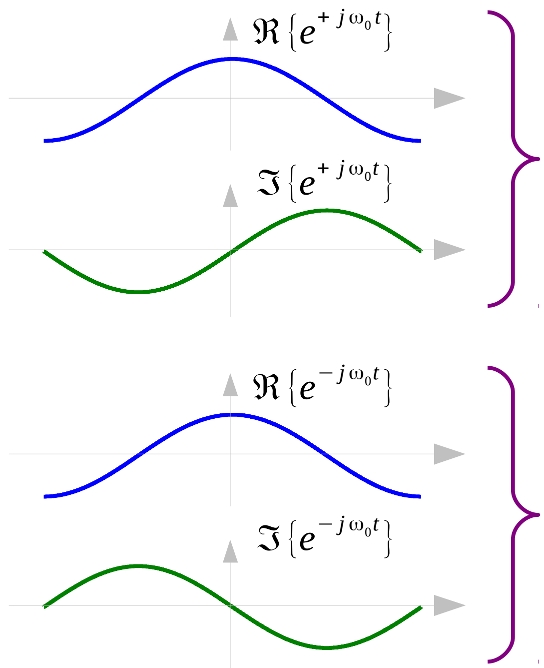


$$2\pi\,\delta(\omega - \omega_0)$$



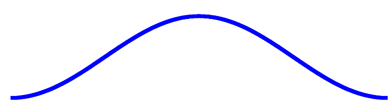
Euler Formula

$$\omega_0 = \frac{2\pi}{T_0}$$

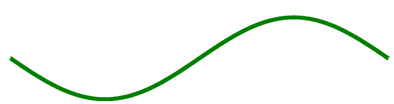


$$e^{+j\omega_0 t}$$

$$e^{-j\omega_0 t}$$



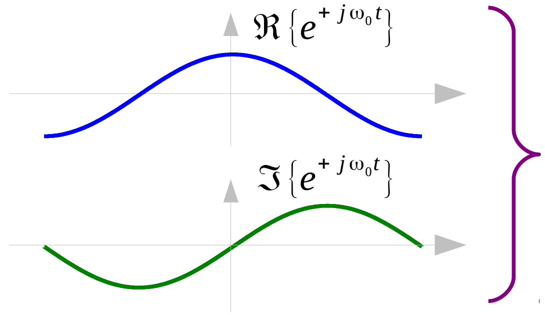
$$\cos(\omega_0 t) = \frac{e^{+j\omega_0 t} + e^{-j\omega_0 t}}{2}$$

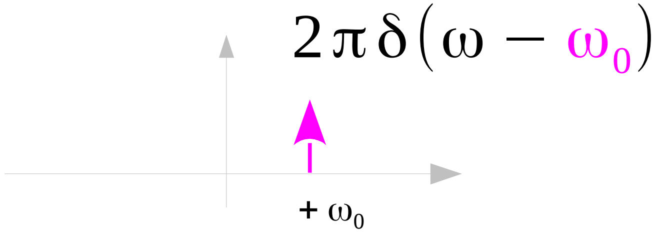


$$\sin(\omega_0 t) = \frac{e^{+j\omega_0 t} - e^{-j\omega_0 t}}{2j}$$

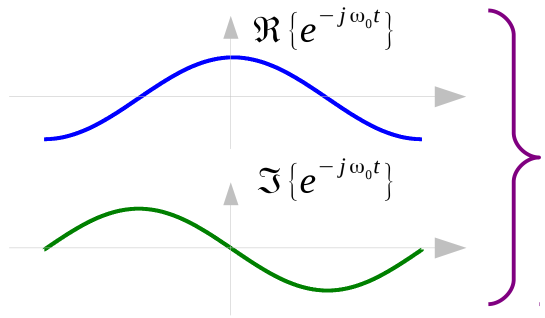
Complex Exponential Functions

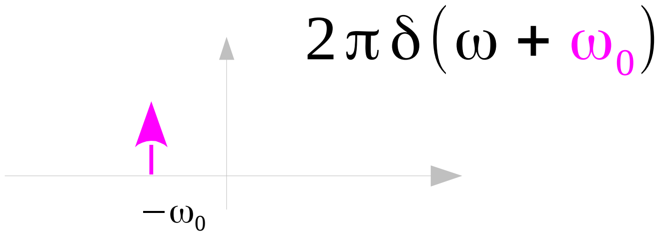
$e^{+j\omega_0 t}$



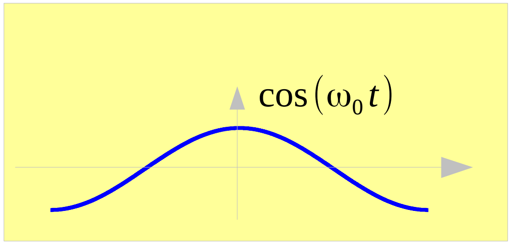


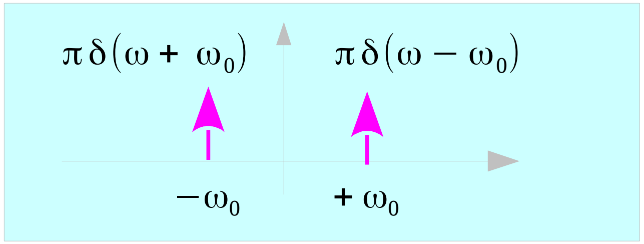
$e^{-j\omega_0 t}$



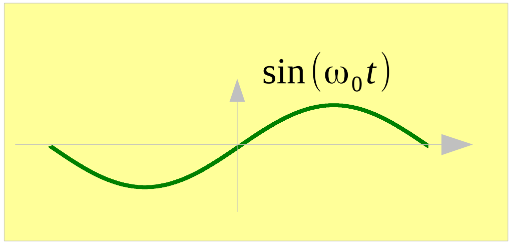


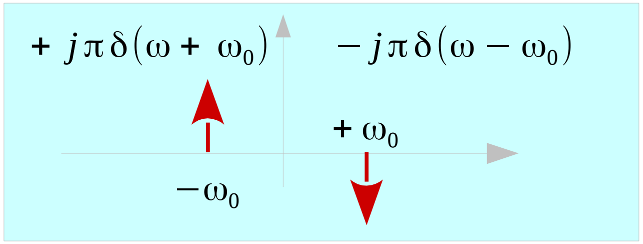
$$\frac{1}{2}\left(e^{+j\omega_0 t} + e^{-j\omega_0 t}\right)$$





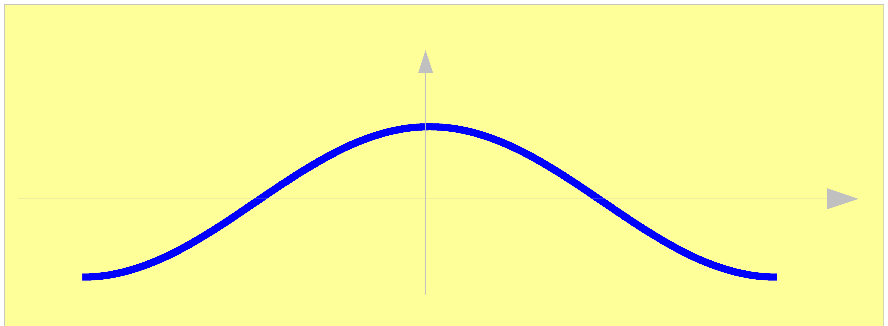
$$\frac{1}{2j}\left(e^{+j\omega_0 t} - e^{-j\omega_0 t}\right)$$

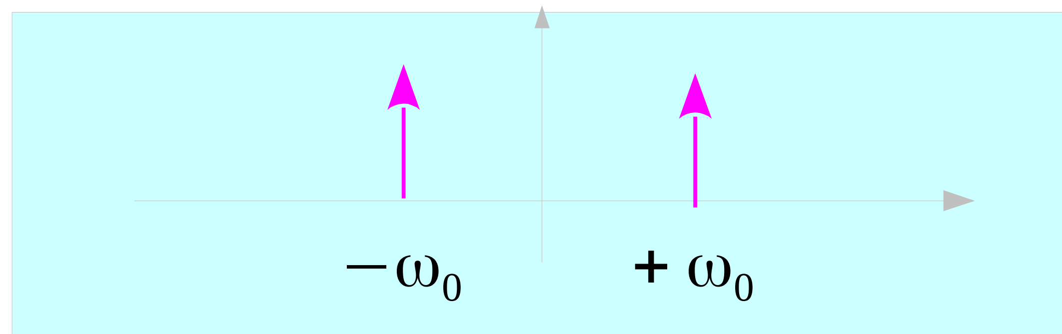


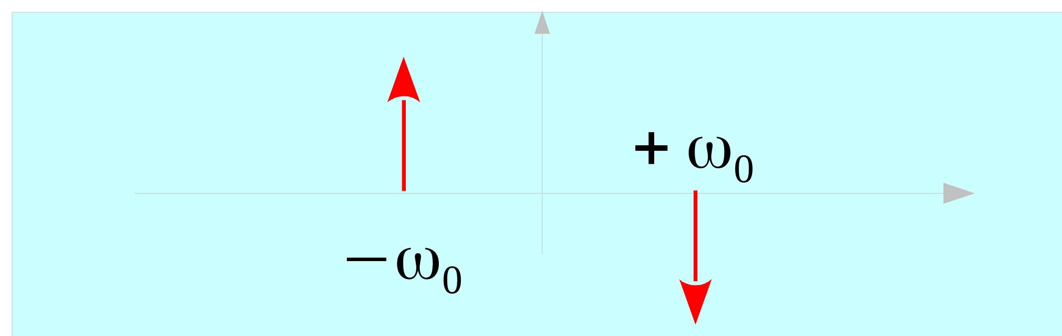


(C) Sinusoidal Functions

$$\cos(\omega_0 t) \qquad \pi\delta(\omega + \omega_0) \quad \pi\delta(\omega - \omega_0)$$

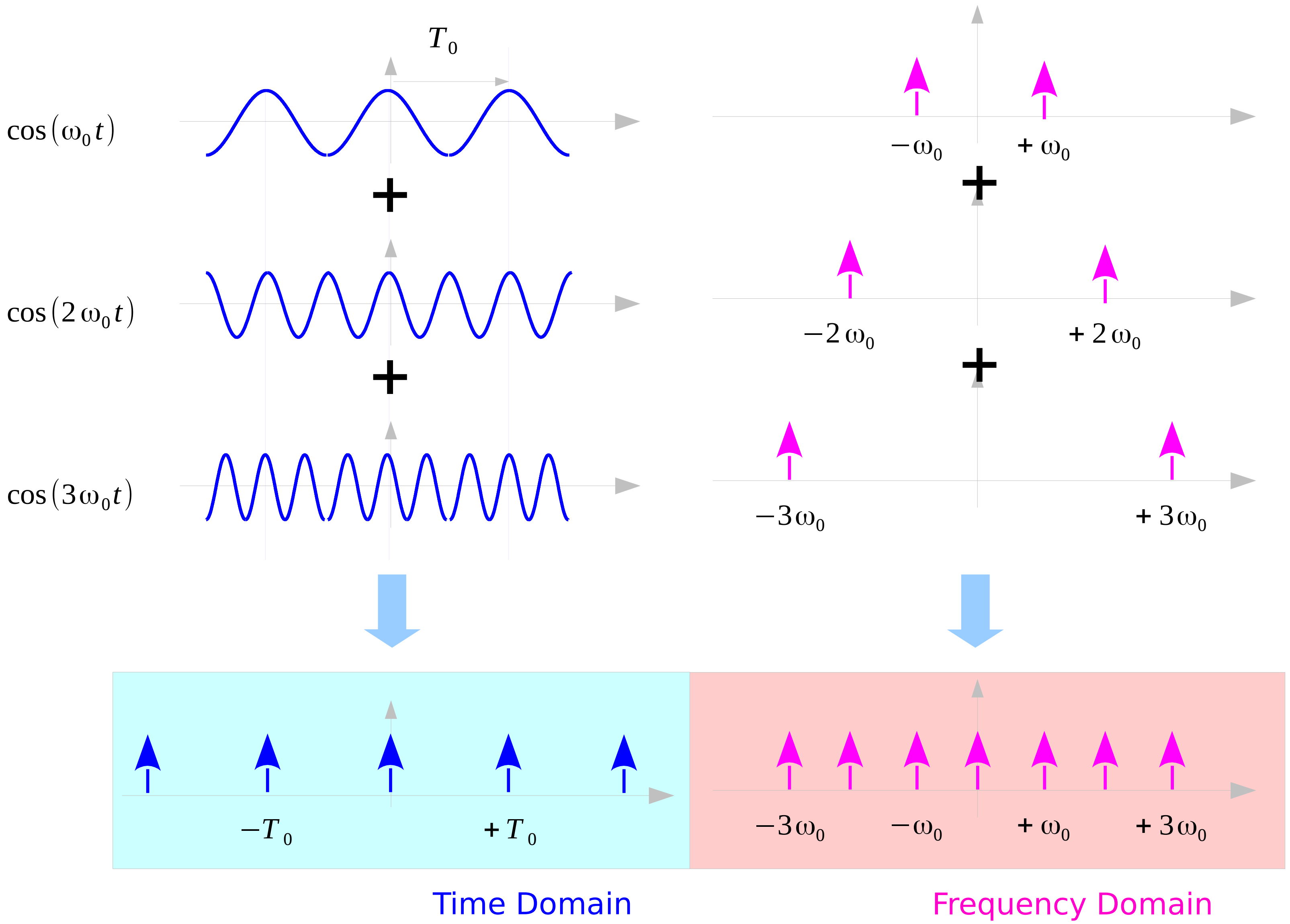




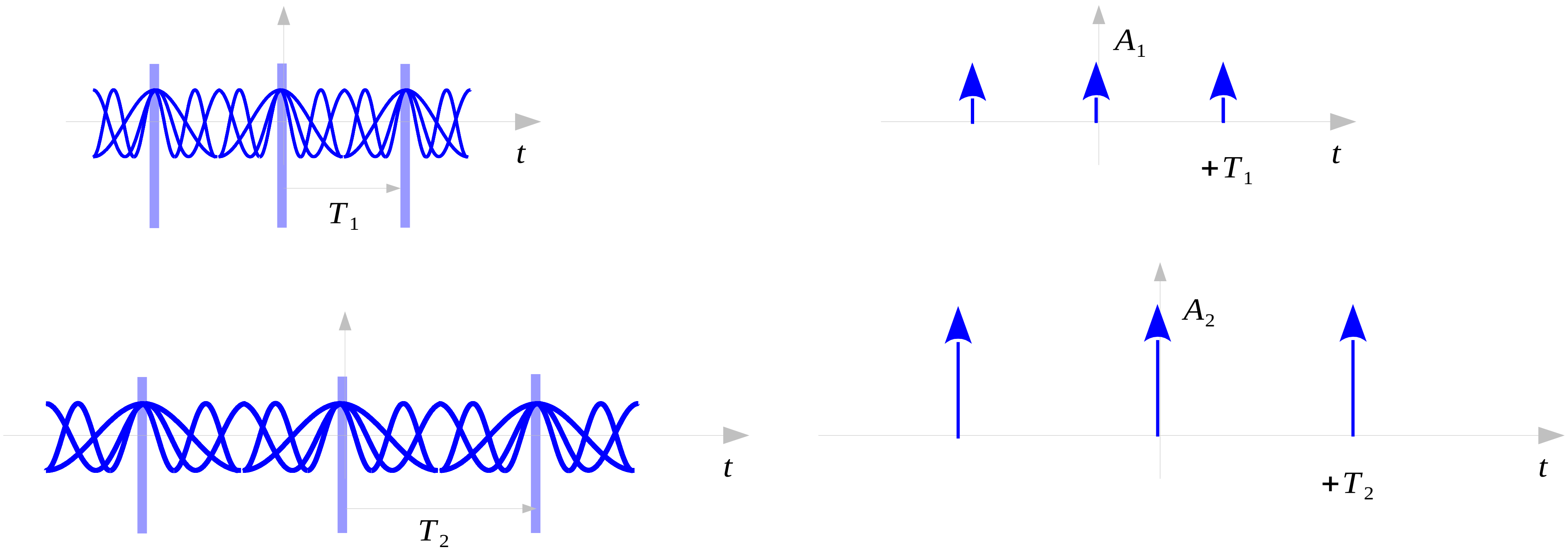


$$\sin(\omega_0 t) \qquad +j\pi\delta(\omega + \omega_0) \quad -j\pi\delta(\omega - \omega_0)$$

Sum of Cosine Functions : Impulse Train



Integer Multiples of a Fundamental Frequency



Period **T1 < T2**

# of integer multiples in T1 **<** # of integer muiples in T2

Amplitude **A1 < A2**

The # of integer multiples is proportional to the period
(the # of 1's to be added)

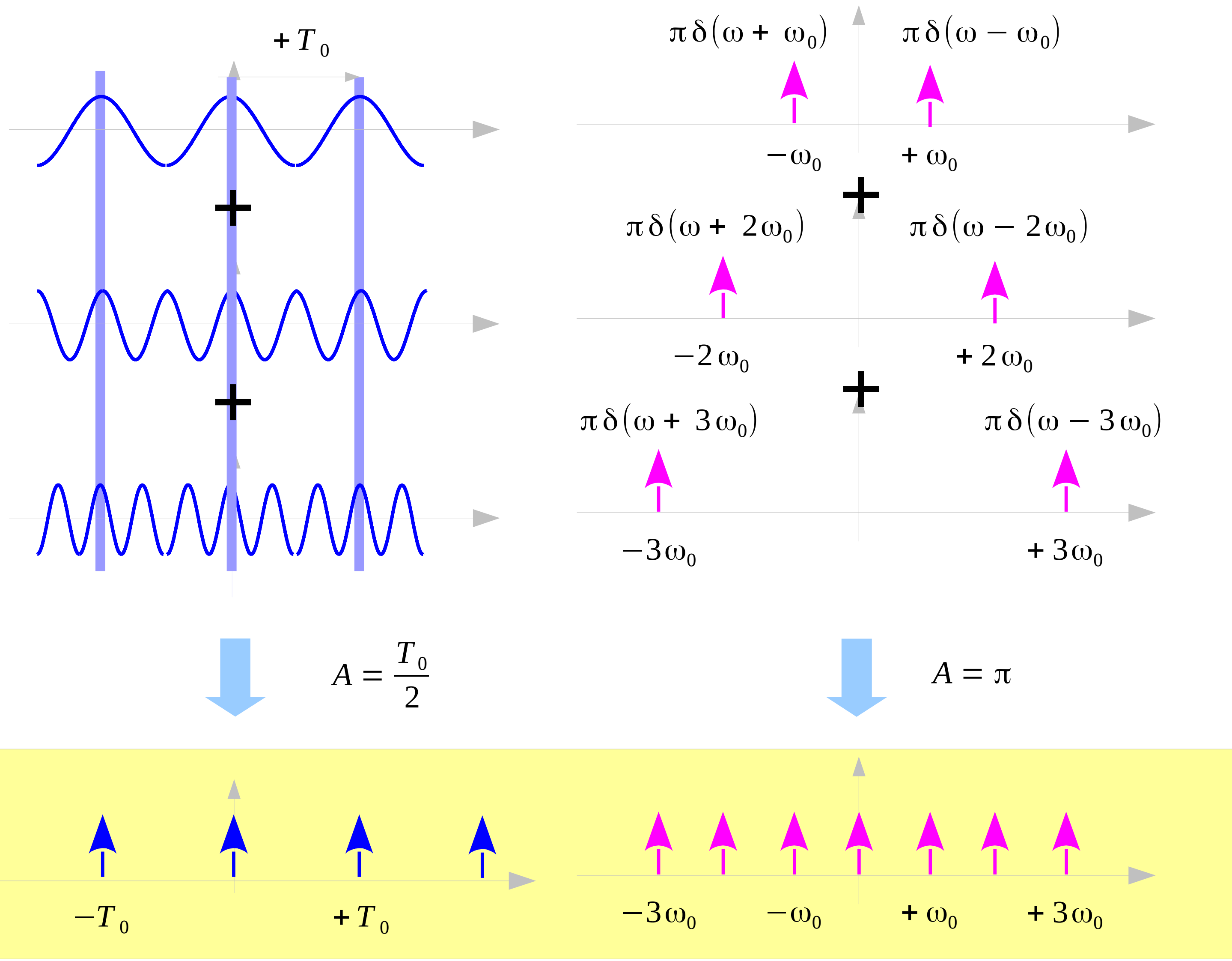
Value 1 is added at 0, $\pm T_0$, $\pm 2T_0$, ...

$$\cos(\omega_0 t) = \frac{e^{+j\omega_0 t} + e^{-j\omega_0 t}}{2}$$

$$\cos(2\omega_0 t) = \frac{e^{+j2\omega_0 t} + e^{-j2\omega_0 t}}{2}$$

$$\cos(3\omega_0 t) = \frac{e^{+j3\omega_0 t} + e^{-j3\omega_0 t}}{2}$$

$$\omega_0 = \frac{2\pi}{T_0}$$



Infinite sum of complex exponentials

$$\cos(\omega_0 t) = \frac{e^{+j\omega_0 t} + e^{-j\omega_0 t}}{2}$$

$\cos(1\omega_0 t)$ $\pi\delta(\omega + 1\omega_0) + \pi\delta(\omega - 1\omega_0)$

$$\cos(2\omega_0 t) = \frac{e^{+j2\omega_0 t} + e^{-j2\omega_0 t}}{2}$$

$\cos(2\omega_0 t)$ $\pi\delta(\omega + 2\omega_0) + \pi\delta(\omega - 2\omega_0)$

$$\cos(3\omega_0 t) = \frac{e^{+j3\omega_0 t} + e^{-j3\omega_0 t}}{2}$$

$\cos(3\omega_0 t)$ $\pi\delta(\omega + 3\omega_0) + \pi\delta(\omega - 3\omega_0)$

Time Domain → $\sum_{k=0}^{\infty} \cos(k\omega_0 t) = \frac{1}{2} \sum_{k=-\infty}^{\infty} e^{+jk\omega_0 t}$

Time Domain → $\left(\frac{T_0}{2}\right) \sum_{n=-\infty}^{\infty} \delta(t - nT_0)$

Frequency Domain → $\pi \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_o)$

CTFT of Complex Exponential Sum

$$p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT_s)$$

Time Shift CTFT

$$P(j\omega) = \sum_{k=-\infty}^{+\infty} e^{jk\omega T_s}$$

$$p(t) = \sum_{n=-\infty}^{+\infty} \frac{1}{T_s} e^{jn\omega_s t}$$

Freq Shift CTFT

$$P(j\omega) = \sum_{k=-\infty}^{\infty} \left(\frac{2\pi}{T_s}\right) \delta(\omega - k\omega_s)$$

$$\frac{T_s}{2} p(t) = \frac{1}{2} \sum_{n=-\infty}^{+\infty} e^{jn\omega_s t}$$

$$= \frac{T_s}{2} \sum_{n=-\infty}^{+\infty} \delta(t - nT_s)$$

$$\frac{T_s}{2} P(j\omega) = \sum_{k=-\infty}^{\infty} \pi\delta(\omega - k\omega_s)$$

$$\sum_{k=0}^{\infty} \cos(k\omega_0 t) = \frac{1}{2} \sum_{k=-\infty}^{\infty} e^{+jk\omega_0 t}$$

$$\left(\frac{T_0}{2}\right) \sum_{n=-\infty}^{\infty} \delta(t - nT_0)$$

$$\pi \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_o)$$

Exponential Sum and Impulse Train Relations

$$p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT_s)$$

Time Shift **CTFT**

$$P(j\omega) = \sum_{k=-\infty}^{+\infty} e^{jk\omega T_s}$$

CTFS

$$c_k = \frac{1}{T_s}$$

$$\omega_s = \frac{2\pi}{T_s}$$

Synthesis

$$p(t) = \sum_{n=-\infty}^{+\infty} \frac{1}{T_s} e^{jn\omega_s t}$$

Freq Shift **CTFT**

$$P(j\omega) = \sum_{k=-\infty}^{\infty} \left(\frac{2\pi}{T_s}\right) \delta(\omega - k\omega_s)$$

CTFT pairs of different forms

Impulse train

$$p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT_s)$$

=

$$p(t) = \sum_{n=-\infty}^{+\infty} \frac{1}{T_s} e^{jn\omega_s t}$$

$$\omega_s = \frac{2\pi}{T_s}$$

CTFT

CTFT

$$P(j\omega) = \sum_{k=-\infty}^{+\infty} e^{jk\omega T_s}$$

=

$$P(j\omega) = \sum_{k=-\infty}^{\infty} \omega_s \delta(\omega - k\omega_s)$$

Impulse train

CTFT pairs of similar forms

Impulse train

Infinite sum of complex exponentials

$n \omega_s t$

$$p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - n T_s)$$

=

$$p(t) = \sum_{n=-\infty}^{+\infty} \frac{1}{T_s} e^{j n \omega_s t}$$

$$\omega_s = \frac{2\pi}{T_s}$$

CTFT (↓)

CTFT (↓)

$$P(j\omega) = \sum_{k=-\infty}^{\infty} \omega_s \delta(\omega - k\omega_s)$$

=

$$P(j\omega) = \sum_{k=-\infty}^{+\infty} e^{j k \omega T_s}$$

$k T_s \omega$

Impulse train

Infinite sum of complex exponentials

Conversion of an Impulse Train

Continuous Time Fourier Series

$$p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT_s)$$

$$C_n = \frac{1}{T} \int_0^T x(t)\, e^{-jn\omega_0 t}\, dt$$

$$C_n = \frac{1}{T_s} \int_{-T_s/2}^{+T_s/2} \delta(t) e^{-jn\omega_s t}\, dt$$

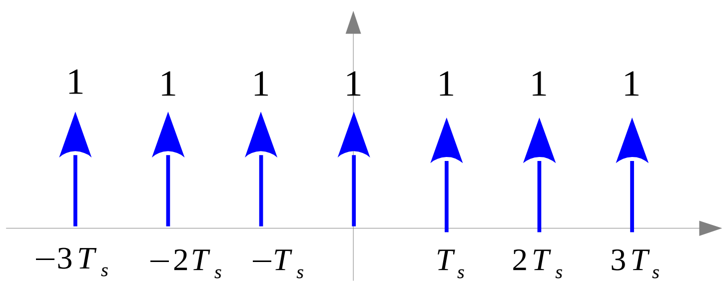
$$= \frac{1}{T_s} \int_{-T_s/2}^{+T_s/2} \delta(t)\, dt = \frac{1}{T_s}$$

$$x(t) = \sum_{n=0}^{\infty} C_n\, e^{+jn\omega_0 t}$$

$$p(t) = \sum_{n=-\infty}^{+\infty} C_n e^{jn\omega_s t}$$

$$= \sum_{n=-\infty}^{+\infty} \frac{1}{T_s} e^{jn\omega_s t}$$

Fourier Series Expansion



$$p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT_s) = \sum_{n=-\infty}^{+\infty} \frac{1}{T_s} e^{jn\omega_s t}$$

CTFT of the CTFS synthesis

Continuous Time Fourier Transform

$$p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT_s) \quad \Leftrightarrow \quad P(j\omega)$$

$$P(j\omega) = \int_{-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} \delta(t - nT_s) e^{-jn\omega t} dt = \int_{-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} \frac{1}{T_s} e^{jn\omega_s t} e^{-jn\omega t} dt$$

$$= \sum_{n=-\infty}^{+\infty} \frac{1}{T_s} \int_{-\infty}^{+\infty} e^{-j(\omega - n\omega_s)t} dt$$

$$= \sum_{n=-\infty}^{+\infty} \left(\frac{2\pi}{T_s}\right) \delta(\omega - n\omega_s)$$

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t} dt$$

$$p(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} \left(\frac{2\pi}{T_s}\right) \delta(\omega - n\omega_s)\, e^{+j\omega t} d\omega = \sum_{n=-\infty}^{+\infty} \frac{1}{T_s} \int_{-\infty}^{+\infty} \delta(\omega - n\omega_s)\, e^{+j\omega t} d\omega$$

$$= \sum_{n=-\infty}^{+\infty} \frac{1}{T_s} e^{jn\omega_s t} = \sum_{n=-\infty}^{+\infty} \delta(t - nT_s)$$

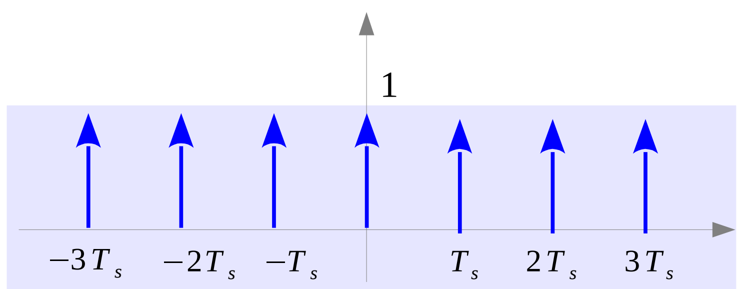
$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t} d\omega$$

Impulse train p(t), P(jω)

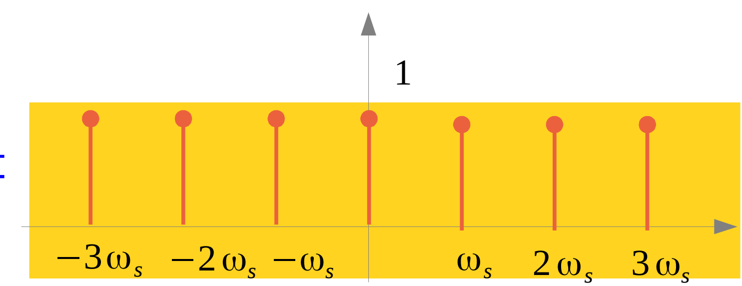
$$p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT_s)$$

Time Shift

$$P(j\omega) = \sum_{k=-\infty}^{+\infty} e^{jk\omega T_s}$$



Time Shift



T_s

$$\omega_s = \frac{2\pi}{T_s}$$

p(t), impulse train P(jω)

$$p(t) = \sum_{n=-\infty}^{+\infty} \frac{1}{T_s} e^{jn\omega_s t}$$

Freq Shift

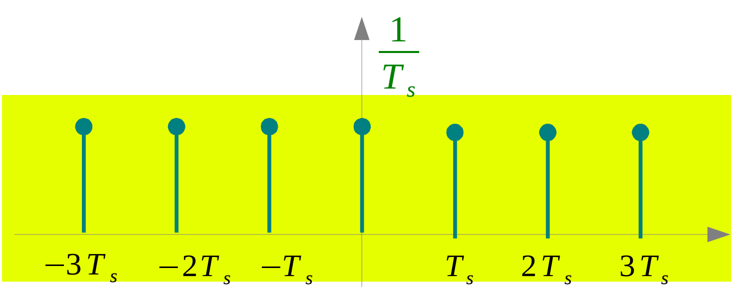
$$P(j\omega) = \sum_{k=-\infty}^{\infty} \left(\frac{2\pi}{T_s}\right) \delta(\omega - k\omega_s)$$

$$\frac{1}{T_s}$$

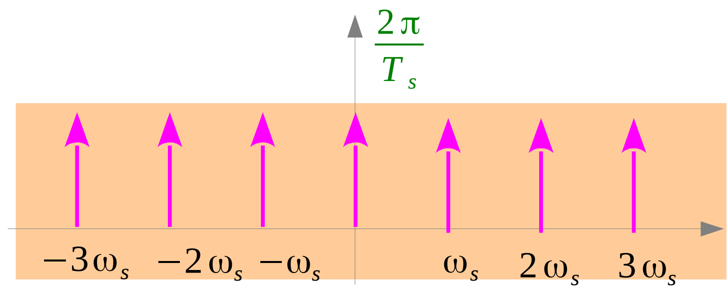
$$\frac{2\pi}{T_s}$$

T_s

$\omega_s = \frac{2\pi}{T_s}$



Freq Shift



(D) An Impulse Train - (1)

$$p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT_s)$$

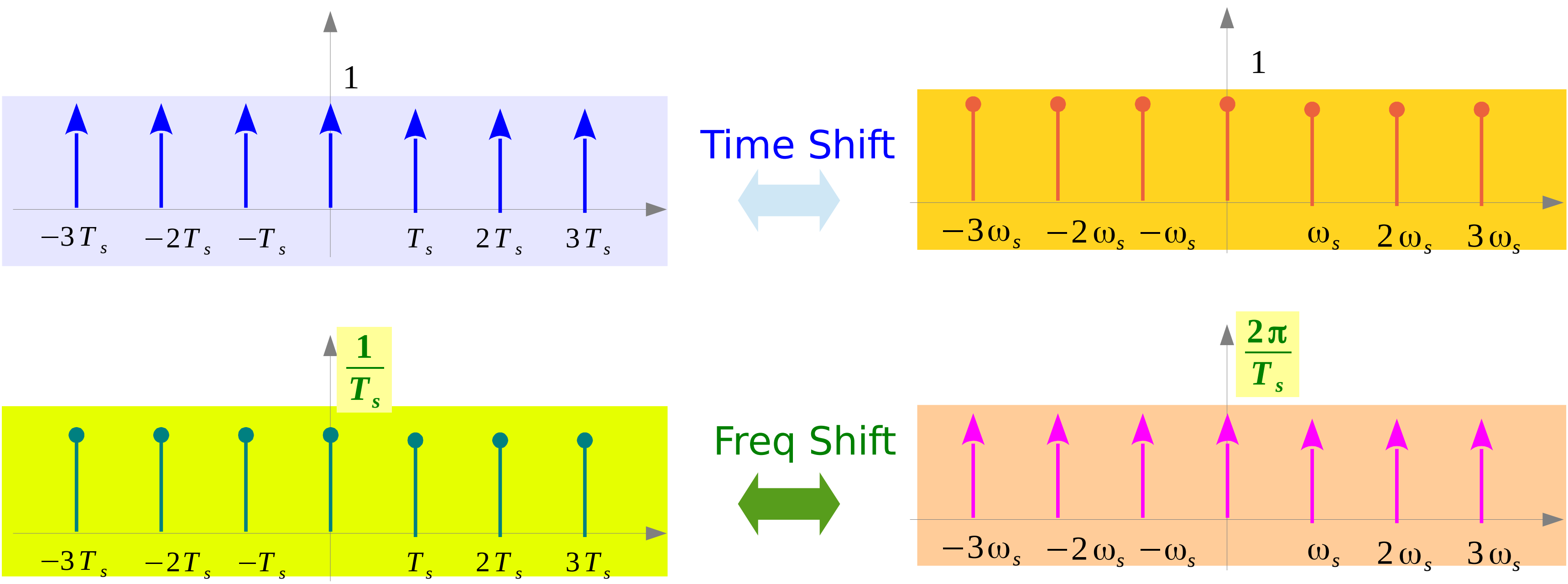
Time Shift CTFT

$$P(j\omega) = \sum_{k=-\infty}^{+\infty} e^{jk\omega T_s}$$

$$p(t) = \sum_{n=-\infty}^{+\infty} \frac{1}{T_s} e^{jn\omega_s t}$$

Freq Shift CTFT

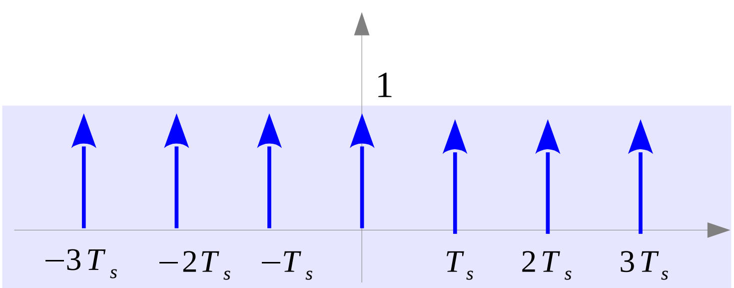
$$P(j\omega) = \sum_{k=-\infty}^{\infty} \left(\frac{2\pi}{T_s}\right) \delta(\omega - k\omega_s)$$

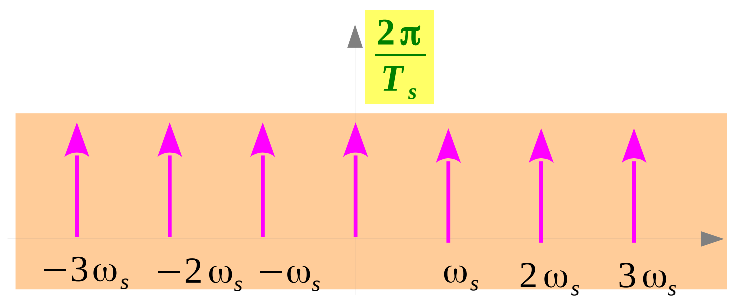


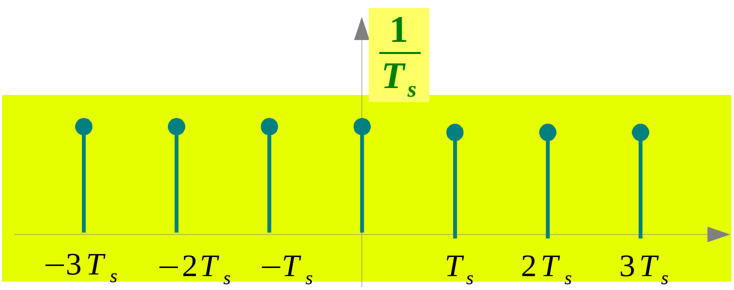
$$p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT_s) \quad \Longleftrightarrow \quad P(j\omega) = \sum_{k=-\infty}^{\infty} \left(\frac{2\pi}{T_s}\right) \delta(\omega - k\omega_s)$$

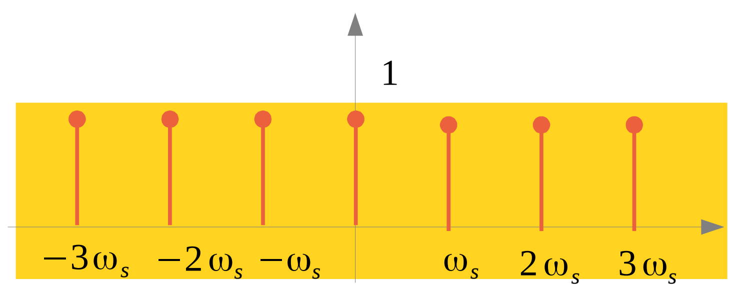
$$\|\qquad\qquad\qquad\qquad\|$$

$$p(t) = \sum_{n=-\infty}^{+\infty} \frac{1}{T_s} e^{jn\omega_s t} \quad \Longleftrightarrow \quad P(j\omega) = \sum_{k=-\infty}^{+\infty} e^{jk\omega T_s}$$









(D) An Impulse Train – (3)

$$p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT_s)$$

Time Shift ⟺

$$P(j\omega) = \sum_{k=-\infty}^{+\infty} e^{jk\omega T_s}$$

||

||

$$p(t) = \sum_{n=-\infty}^{+\infty} \frac{1}{T_s} e^{jn\omega_s t}$$

Freq Shift ⟺

$$P(j\omega) = \sum_{k=-\infty}^{\infty} \left(\frac{2\pi}{T_s}\right) \delta(\omega - k\omega_s)$$

"*an equation that relates the Fourier series coefficients of the periodic summation of a function to values of the function's continuous Fourier transform.*

Consequently, the periodic summation of a function is completely defined by discrete samples of the original function's Fourier transform.

And conversely, the periodic summation of a function's Fourier transform is completely defined by discrete samples of the original function."

CTFS Coefficients and an Impulse Train

A general formula for the **CTFT** of any periodic function for which a **CTFS** exists

Its **energy** is not finite but its **power** is finite : **Power Signals**

Discrete Coefficients

$$C_k$$

Weighted Impulse Train

$$\sum_{k=-\infty}^{\infty} 2\pi C_k \delta(\omega - k\omega_s)$$

Fourier Series Expansion

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k \, e^{+jk\omega_s t}$$

CTFT

Fourier Transform

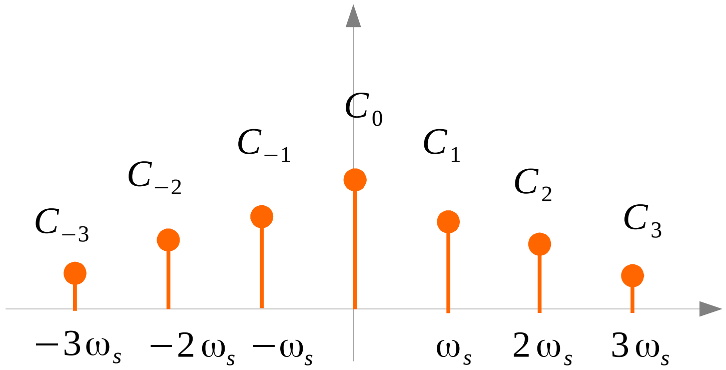
$$X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi C_k \delta(\omega - k\omega_s)$$

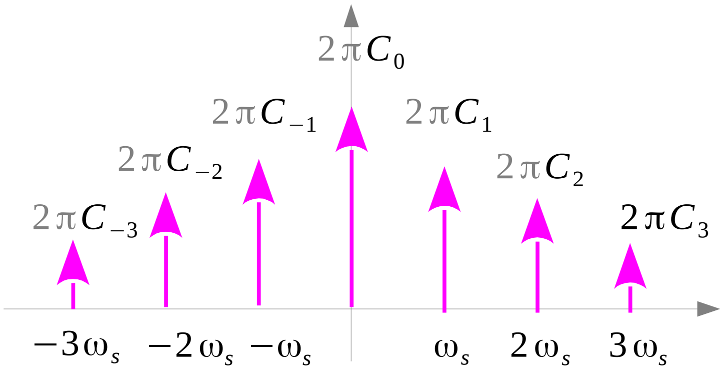
CTFS

Fourier Series Coefficients

$$C_k = \frac{1}{T_s} \int_{-T_s/2}^{+T_s/2} x(t) \, e^{-jk\omega_s t} \, dt$$

CTFT of a CTFS Synthesis Signal





Fourier Series Coefficients

$$C_k = \frac{1}{T_s} \int_{-T_s/2}^{+T_s/2} x(t)\, e^{-jk\omega_s t}\, dt$$

Fourier Series Expansion

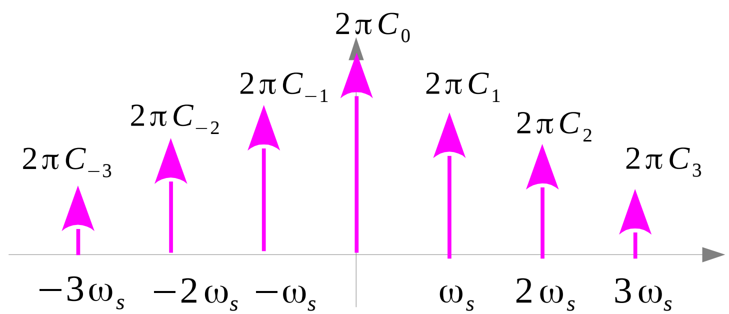
$$x(t) = \sum_{k=-\infty}^{+\infty} C_k\, e^{+jk\omega_s t}$$

Fourier Transform

$$X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi C_k\, \delta(\omega - k\omega_s)$$

CTFT of any periodic signal

A general formula for the CTFT of any periodic function for which a CTFS exists



Period

$T_s \Rightarrow \omega_s = \frac{2\pi}{T_s}$

Fourier Series Expansion

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k e^{+jk\omega_s t}$$

Fourier Series Coefficients

$$C_k = \frac{1}{T_s}\int_{-T_s/2}^{+T_s/2} x(t)\, e^{-jk\omega_s t}\, dt$$

Fourier Transform

$$X(j\omega) = \sum_{k=-\infty}^{\infty} C_k \int_{-\infty}^{+\infty} e^{jk\omega_s t}\, e^{-j\omega t}\, dt$$

$$= \sum_{k=-\infty}^{\infty} C_k \int_{-\infty}^{+\infty} e^{-j(\omega - k\omega_s)t}\, dt$$

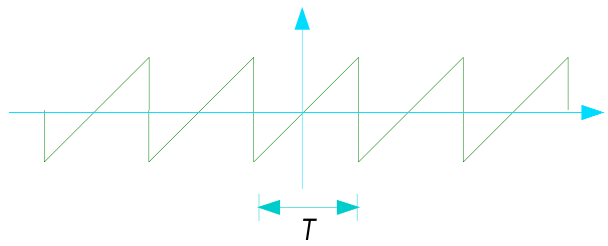
$$= \sum_{k=-\infty}^{\infty} 2\pi C_k \delta(\omega - k\omega_s)$$

Periodic Time Signal → **CTFS** → **CTFT** → **Sampled in Frequency**

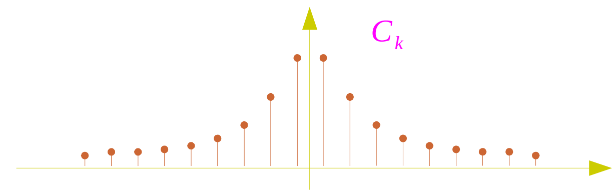
CTFS & CTFT

Continuous Time
Fourier Series
CTFS

Continuous Time



Discrete Frequency



$$x(t) = \sum_{k=-\infty}^{+\infty} C_k \, e^{+jk\omega_s t}$$

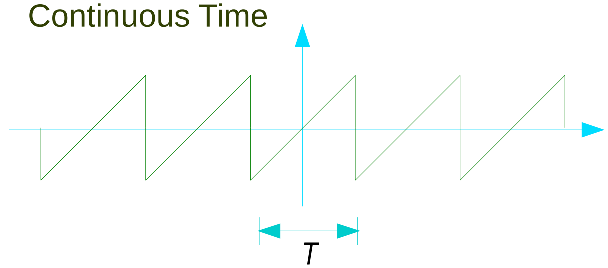
$$C_k = \frac{1}{T_s} \int_{-T_s/2}^{+T_s/2} x(t) \, e^{-jk\omega_s t} \, dt$$

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k \, e^{+jk\omega_s t}$$

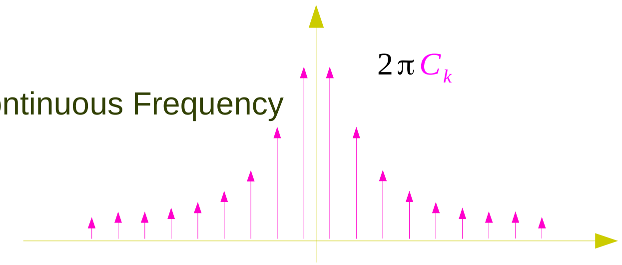
$$X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi C_k \delta(\omega - k\omega_s)$$

Continuous Time
Fourier Transform
CTFT

Continuous Time



Continuous Frequency



Dirichlet Condition

Sufficient Condition for Fourier Transform Pair

1. On any finite interval
   a. f(t) is bounded
   b. f(t) has a finite number of maxima and minima
   c. f(t) has a finite number of discontinuities
2. f(t) is a absolutely integrable $\int_{-\infty}^{+\infty} |f(t)|\, dt < \infty$

→ Fourier Transform exists $F(j\omega)$

Energy signals

$$E = \int_{-\infty}^{+\infty} |f(t)|^2\, dt < \infty$$

The voltage across a 1 Ohm resistor

$$p(t) = |f(t)|^2 / R = |f(t)|^2$$

There are functions that does not meet Dirichlet condition but still have Fourier Transform Pair

1. Unit Step function

2. Periodic functions

Power Condition

Less stringent requirement for Fourier Transform Pair

1. On any finite interval
   a. f(t) is bounded
   b. f(t) has a finite number of maxima and minima
   c. f(t) has a finite number of discontinuities

2. ~~f(t) is a absolutely integrable~~ $\int_{-\infty}^{+\infty} |f(t)|\, dt < \infty$

⇒ Fourier Transform exists $F(j\omega)$

A finite amount of power **Power Signals**

$$P = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{+T/2} |f(t)|^2\, dt < \infty$$

$$p(t) = |f(t)|^2 / R = |f(t)|^2$$

1. The unit step function
2. Periodic functions
3. The signum function

Other Convention (1)

x

Continuous Time Fourier Transform {unitary, angular frequency}

$$X(j\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt \quad \Leftrightarrow$$

$$x(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega$$

Continuous Time Fourier Transform {non-unitary, angular frequency}

$$X(j\omega) = 1 \cdot \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt \quad \Leftrightarrow$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega$$

Other Convention (2)

Continuous Time Fourier Transform {unitary, frequency}

$$X(f) = \int_{-\infty}^{+\infty} x(t)\, e^{-j2\pi f t}\, dt \quad \Leftrightarrow$$

$$x(t) = \int_{-\infty}^{+\infty} X(f)\, e^{+j2\pi f t}\, df$$

Continuous Time Fourier Transform {non-unitary, angular frequency}

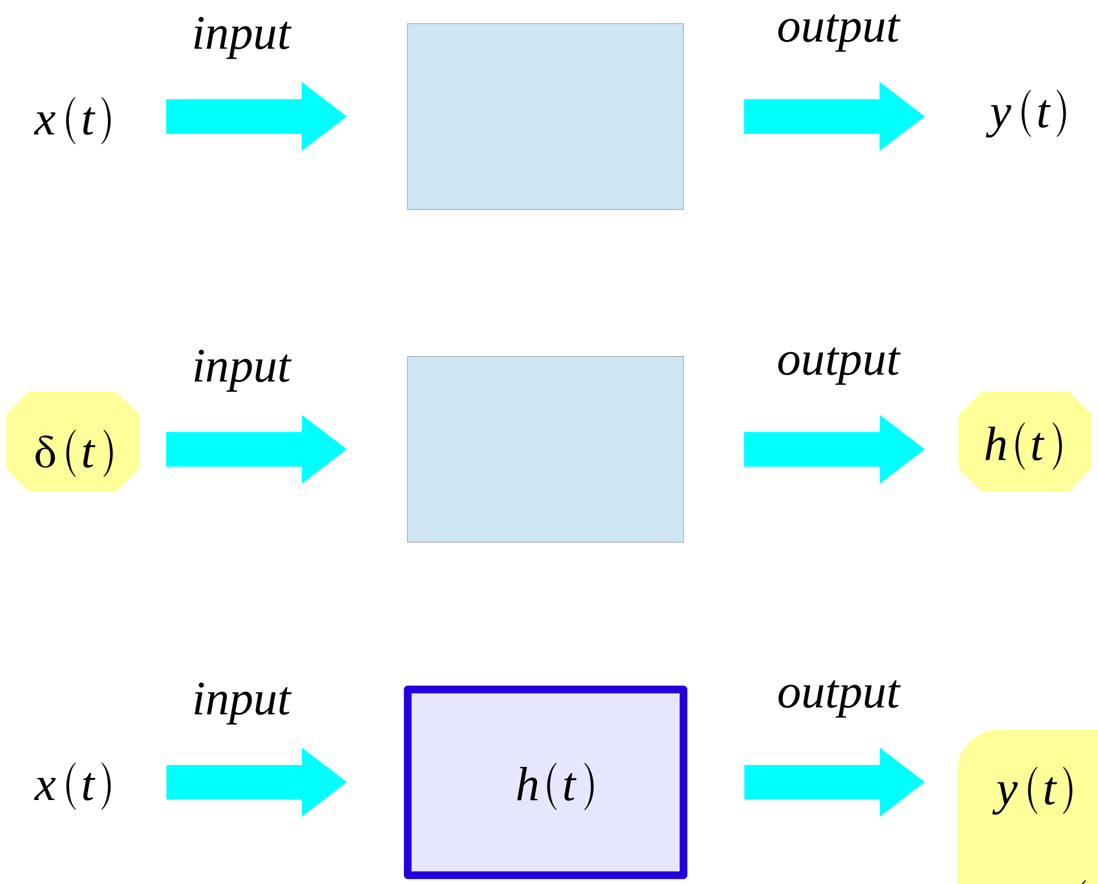
$$X(j\omega) = 1 \cdot \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt \quad \Leftrightarrow$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega$$

$$\omega = 2\pi f$$

$$d\omega = 2\pi\, df$$

Impulse Response

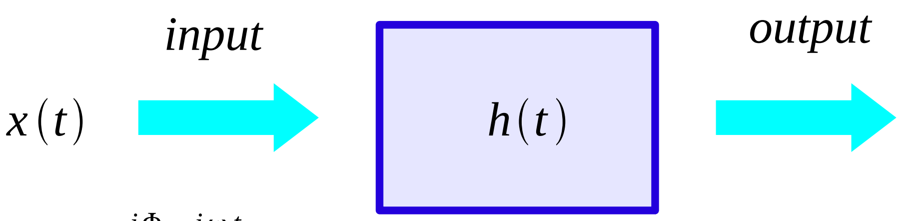


$$y(t)$$

$$= h(t) * x(t)$$

$$= \int_{-\infty}^{+\infty} h(\tau)\, x(t-\tau)\, d\tau$$

Frequency Response



$$x(t) = A e^{j\Phi} e^{j\omega t}$$

$$y(t) = h(t) * x(t) = \int_{-\infty}^{+\infty} h(\tau) x(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{+\infty} h(\tau) A e^{j\Phi} e^{j\omega(t-\tau)}\, d\tau$$

$$= \int_{-\infty}^{+\infty} h(\tau) A e^{j\Phi} e^{j\omega t} e^{-j\omega\tau}\, d\tau$$

$$= A e^{j\Phi} e^{j\omega t} \int_{-\infty}^{+\infty} h(\tau) e^{-j\omega\tau}\, d\tau$$

$$= A e^{j\Phi} e^{j\omega t}\, H(j\omega)$$

$$H(j\omega) = \int_{-\infty}^{+\infty} h(\tau) e^{-jw\tau}\, d\tau$$

$$x(t) = A e^{j\Phi} e^{j\omega t}$$

$$y(t) = H(j\omega) A e^{j\Phi} e^{j\omega t}$$

References

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