Hybrid CORDIC 1.A Sine/Cosine Generator Algorithms

20171125

.

The details moved to
https://en.wikiversity.org/wiki/Butterfly_Hardware_Implementations

.

Wilson ROM based Sine/Cosine Generation

[24] Fu & Willson Sine / Cosine Generation

ROM-based approach

For high resolution, ROM size grows exponentially

quater-wave symmetry

$$\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$$

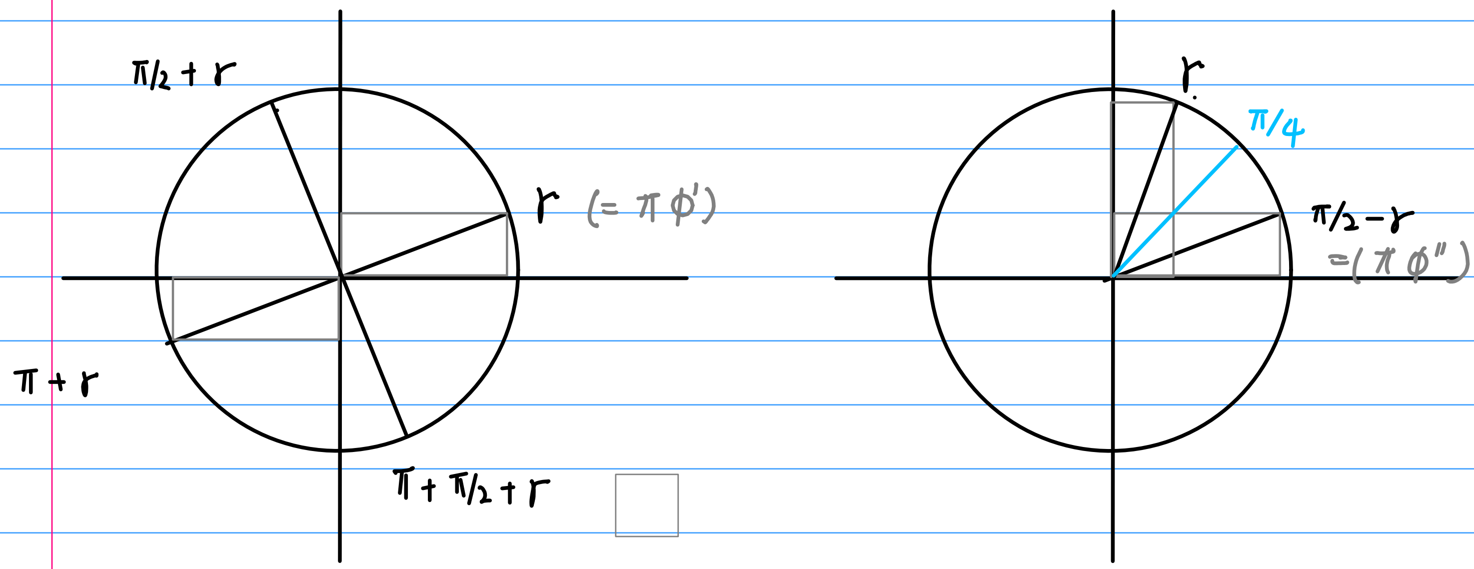
$$\phi \; [0, 2\pi] \longrightarrow [0, \boxed{\tfrac{\pi}{4}}]$$

conditionally interchanging inputs X_0 & Y_0

Conditionally interchanging and negating outputs X & Y

$$X = X_0 \cos\phi - Y_0 \sin\phi$$
$$Y = Y_0 \cos\phi + X_0 \sin\phi$$

Madisetti VLSI arch



frequency Synthesis

argument: signed normalized by π angle ϕ $[-1, 1]$

binary representation of a radian angle required

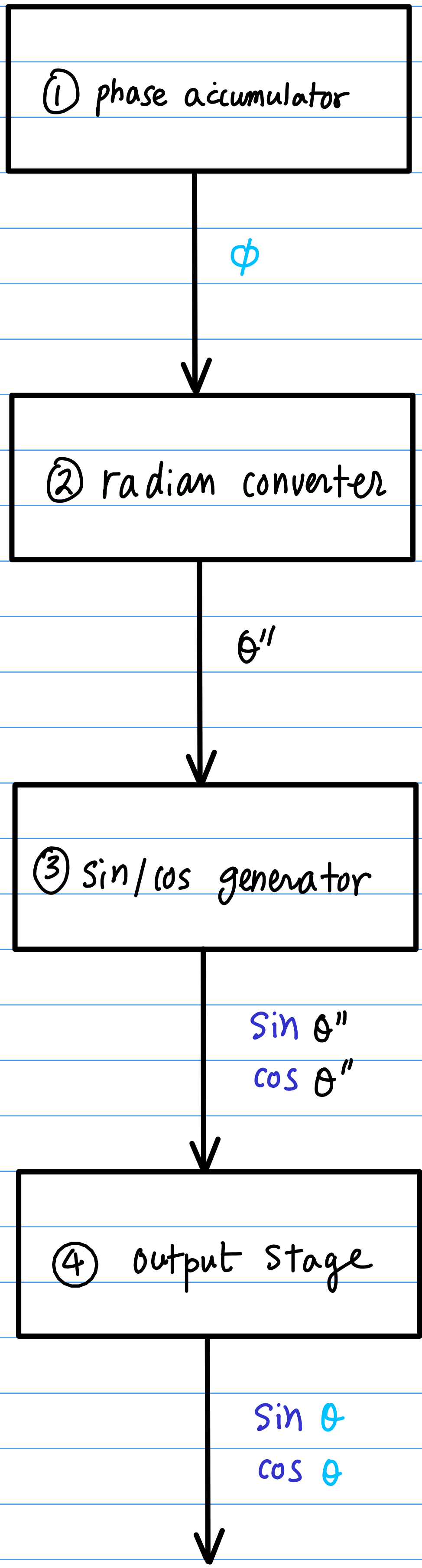
ϕ $[-1, 1] \longrightarrow [0, \pi/4] \longrightarrow$ Sine/cosine generator

$\pi\phi$ $[-\pi, +\pi]$

$\theta = \pi\phi$

1. a phase accumulator ϕ $[-1, 1]$
2. a radian converter $\phi \to \theta = \pi\phi$
3. a sine/cosine generator $\sin\theta$, $\cos\theta$
4. an output stage $\sin\theta$, $\cos\theta$

$\sin\theta \to \sin\pi\phi$, $\cos\theta \to \cos\pi\phi$



$\phi \in [-1, +1]$

normalized by π

angles must be in radian

for angle rotations

$MSB_1(\phi)$ $MSB_2(\phi)$ Quadrant

$MSB_3(\phi)$ $< \frac{\pi}{4}$

$-\pi < \theta = \pi\,\phi < \pi$

$0 < \theta'' = \pi\,\phi'' < \frac{\pi}{4}$

N-bit binary representation of θ''

the direction of subangle rotation

$b_k \in \{0, 1\} \longrightarrow r_k \in \{-1, +1\}$

angle recoding

$0 < \theta'' = \pi\,\phi'' < \frac{\pi}{4}$

$-\pi < \theta = \pi\,\phi < +\pi$

Radian Converter

normalized angle ϕ

ϕ	MSB1	MSB2	MSB3				...		

1st Quadrant

ϕ'	0	0	MSB3				...		

$\boxed{\text{MSB3}} = 0 \qquad \phi'' = \phi'$

$\boxed{\text{MSB3}} = 1 \qquad \phi'' = 0.5 - \phi'$

$0.5 - \phi'$	0	1	0	$\bar{a}_4$	$\bar{a}_5$	$\bar{a}_6$	...		$\bar{a}_N$
									+1

MSB1	MSB2	MSB3	$\rightarrow$	$x$inv	$y$inv	swap

$$0 < \theta'' = \pi \phi'' < \frac{\pi}{4}$$

Angle Recoding

$\boxed{MSB_1}\boxed{MSB_2}\boxed{MSB_3} \longrightarrow 0 < \theta'' < 1 \longrightarrow$ recoding $\{r_k\}$

$$\boxed{\sum_{k=1}^{N} b_k \cdot 2^{-k} = \phi_0 + \sum_{k=2}^{N+1} r_k \cdot 2^{-k}}$$

$$b_k \in \{0, 1\} \qquad\qquad r_k \in \{-1, +1\}$$

$$\begin{cases} b_k = 1 \longrightarrow r_{k+1} = +1 \\ b_k = 0 \longrightarrow r_{k+1} = -1 \end{cases} \qquad \boxed{r_k = (2b_{k-1} - 1)}$$

ϕ_0 depends only on bit widith N

for fixed N, $\boxed{\phi_0 = \frac{1}{2} - \frac{1}{2^{N+1}}}$ is a constant

Sine / Cosine Generator Overview

$$\begin{bmatrix} X_\theta \\ Y_\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} X_0 \\ Y_0 \end{bmatrix} \Rightarrow \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

a sequence of subrotations of the priori known angle

$$\boxed{\begin{gathered} \theta = \sigma_0\theta_0 + \sigma_1\theta_1 + \cdots + \sigma_N\theta_N \\ \sigma_k = \{-1, 0, +1\} \end{gathered}}$$

① $\boxed{\theta_k = \tan^{-1} 2^{-k}}$ traditional CORDIC

② $\boxed{\theta_k = 2^{-k}}$ possible because $\theta'' < 1$

$$\begin{bmatrix} X_\theta \\ Y_\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} X_0 \\ Y_0 \end{bmatrix}$$

$$= K \begin{bmatrix} 1 & -\tan(\sigma_0\theta_0) \\ \tan(\sigma_0\theta_0) & 1 \end{bmatrix} \begin{bmatrix} 1 & -\tan(\sigma_1\theta_1) \\ \tan(\sigma_1\theta_1) & 1 \end{bmatrix} \cdots \begin{bmatrix} 1 & -\tan(\sigma_N\theta_N) \\ \tan(\sigma_N\theta_N) & 1 \end{bmatrix}$$

$K = \boxed{\cos(\sigma_0\theta_0)}\boxed{\cos(\sigma_1\theta_1)}\cdots\boxed{\cos(\sigma_N\theta_N)}$ scale factor

$\sigma_k = +1$ positive angle rotation

$\sigma_k = -1$ negative angle rotation

The Scaling K.

The initial rotation $\phi_0 = \frac{1}{2} - \frac{1}{2^{N+1}}$

rotation starting point

$$(X_0, Y_0) = (K\cos\phi_0,\ K\sin\phi_0)$$

rotation always starts from this fixed point.

Cascade of feed forward rotational stages

$$\theta \rightarrow \boxed{MSB_1}\boxed{MSB_2}\boxed{MSB_3} \rightarrow \theta'' \rightarrow b_k\, 2^{-k} \rightarrow r_k\, 2^{-k}$$

binary representation

recoding

$\left\{\begin{array}{l}\text{no comparison}\\ \text{no error build up}\end{array}\right.$

① $\boxed{\theta_k = \tan^{-1} 2^{-k}}$ traditional CORDIC

☆ ② $\boxed{\theta_k = 2^{-k}}$ possible because $\theta'' < 1$

$$\begin{bmatrix} 1 & -\sigma_k \tan(2^{-k}) \\ \sigma_k \tan(2^{-k}) & 1 \end{bmatrix}$$

Subrotation angle θ_k

① $\boxed{\theta_k = \tan^{-1} 2^{-k}}$ traditional CORDIC

$$\begin{bmatrix} 1 & -\sigma_k \tan(\tan^{-1} 2^{-k}) \\ \sigma_k \tan(\tan^{-1} 2^{-k}) & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -\sigma_k 2^{-k} \\ \sigma_k 2^{-k} & 1 \end{bmatrix}$$

☆ ② $\boxed{\theta_k = 2^{-k}}$ possible because $\underline{\theta'' < 1}$

$$\begin{bmatrix} 1 & -\sigma_k \tan(2^{-k}) \\ \sigma_k \tan(2^{-k}) & 1 \end{bmatrix} \Rightarrow \left\{ \begin{array}{l} \begin{bmatrix} 1 & -\sigma_k \tan(2^{-k}) \\ \sigma_k \tan(2^{-k}) & 1 \end{bmatrix} \\ \\ \begin{bmatrix} 1 & -\sigma_k 2^{-k} \\ \sigma_k 2^{-k} & 1 \end{bmatrix} \end{array} \right.$$

$\tan(2^{-k}) \cong 2^{-k}$ $\quad k \geq k_0$

① Subrotation angle $\theta_k = \tan^{-1} 2^{-k}$ traditional

$$K = \cos(\theta_0)\cos(\theta_1)\cdots\cos(\theta_N)$$

$$K \begin{bmatrix} 1 & -\tan(\sigma_0\theta_0) \\ \tan(\sigma_0\theta_0) & 1 \end{bmatrix} \begin{bmatrix} 1 & -\tan(\sigma_1\theta_1) \\ \tan(\sigma_1\theta_1) & 1 \end{bmatrix} \cdots \begin{bmatrix} 1 & -\tan(\sigma_N\theta_N) \\ \tan(\sigma_N\theta_N) & 1 \end{bmatrix}$$

$$\boxed{\theta_k = \tan^{-1} 2^{-k}} \rightarrow \boxed{\tan\theta_k = 2^{-k}}$$

$$K \begin{bmatrix} 1 & -\sigma_0 2^{-0} \\ \sigma_0 2^{-0} & 1 \end{bmatrix} \begin{bmatrix} 1 & -\sigma_1 2^{-1} \\ \sigma_1 2^{-1} & 1 \end{bmatrix} \cdots \begin{bmatrix} 1 & -\sigma_N 2^{-N} \\ \sigma_N 2^{-N} & 1 \end{bmatrix}$$

↳ shift-and-add

② Subrotation angle $\theta_k = 2^{-k}$ recoding

$$K = \cos(\theta_0)\cos(\theta_1)\cdots\cos(\theta_N)$$

$$K \begin{bmatrix} 1 & -\tan(\sigma_0\theta_0) \\ \tan(\sigma_0\theta_0) & 1 \end{bmatrix} \begin{bmatrix} 1 & -\tan(\sigma_1\theta_1) \\ \tan(\sigma_1\theta_1) & 1 \end{bmatrix} \cdots \begin{bmatrix} 1 & -\tan(\sigma_N\theta_N) \\ \tan(\sigma_N\theta_N) & 1 \end{bmatrix}$$

☆ $\boxed{\theta_k = 2^{-k}} \rightarrow \boxed{\tan\theta_k = \tan 2^{-k}}$

$$K \begin{bmatrix} 1 & -\sigma_0\tan(2^{-0}) \\ \sigma_0\tan(2^{-0}) & 1 \end{bmatrix} \begin{bmatrix} 1 & -\sigma_1\tan(2^{-1}) \\ \sigma_1\tan(2^{-1}) & 1 \end{bmatrix} \cdots \begin{bmatrix} 1 & -\sigma_N\tan(2^{-N}) \\ \sigma_N\tan(2^{-N}) & 1 \end{bmatrix}$$

$\tan(2^{-k}) \cong 2^{-k} \quad k \geq k_0$

$$K \begin{bmatrix} 1 & -\sigma_0\tan(2^{-0}) \\ \sigma_0\tan(2^{-0}) & 1 \end{bmatrix} \cdots \begin{bmatrix} 1 & -\sigma_{k_0} 2^{-k_0} \\ \sigma_{k_0} 2^{-k_0} & 1 \end{bmatrix} \cdots \begin{bmatrix} 1 & -\sigma_N 2^{-N} \\ \sigma_N 2^{-N} & 1 \end{bmatrix}$$

simple shift-and-add

the $\tan\theta_k$ multipliers used in the first few subrotation stages cannot be implemented as simple shift-and-add operations the subangles $\boxed{\theta_k = 2^{-k}}$ used in recoding

the subangles $\boxed{\theta_k = \tan^{-1}(2^{-k})}$ used in CORDIC

$\tan \theta_k$ multipliers used
in the first few subrotation stages
cannot be implemented
as a simple shift-and-add operations

→ ROM implementation

reduced chip area
higher operating speed.

the rotations always start from the fixed point

the computational sequence

- a cascade of feed forward rotational stages

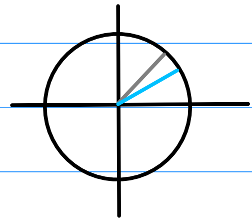
the desired output precision in bits
determines the number of stages

by always starting the sequence of rotations
from a fixed point,
the algorithm does not suffer from an error build up

which limits the accuracy of most recursive
digital oscillator structures

Architecture

1. phase accumulator $\phi \in [-1, +1]$
2. radian converter $\phi \to \theta \in [0, \frac{\pi}{4}]$
3. sine/cosine generator $\sin(\theta)$ $\cos(\theta)$
4. output stage $\sin(\pi\phi)$ $\cos(\pi\phi)$



$\phi \in [-1, +1]$ normalized angle

$\phi \in [-\pi, +\pi] \to \theta \in [0, \frac{\pi}{4}]$ 1st half quadrant

$\sin(\theta)$ $\cos(\theta)$

$\sin(\pi\phi)$ $\cos(\pi\phi)$

Overflowing 2's complement accumulator

normalized by π angle ϕ

need radian angle $\theta \in [0, \frac{\pi}{4}]$

$0 < \theta < 1$ rad

N-bit binary representation of θ

controls the direction of subrotation

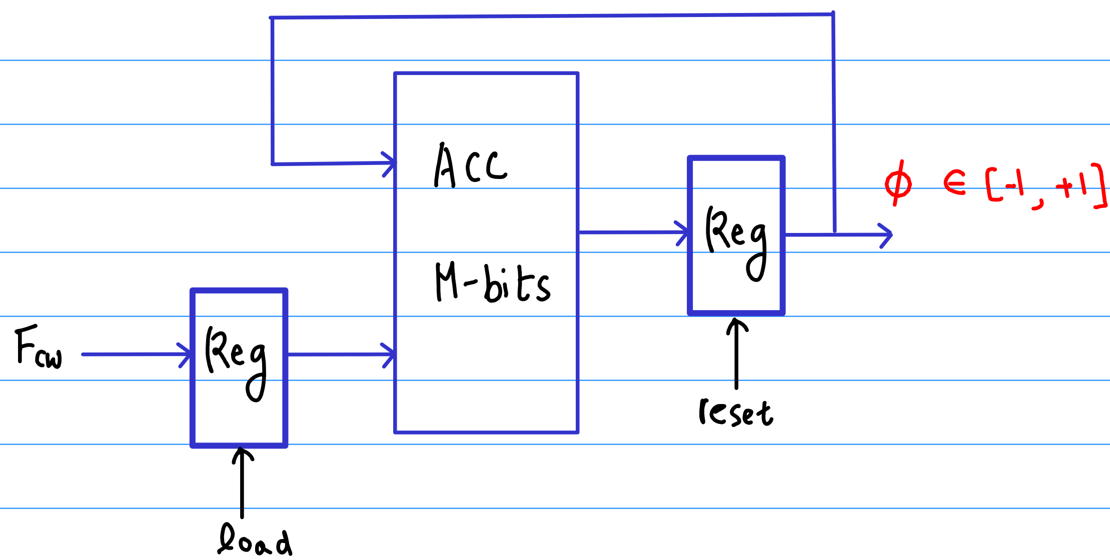
N-bit precision of $\cos\theta$ & $\sin\theta$

Output stage $\theta \rightarrow \pi\phi$

$\sin\theta \rightarrow \sin\pi\phi$

$\cos\theta \rightarrow \cos\pi\phi$

① phase accumulator



M-bit adder — repeatedly increments the phase a

by F_{cw} at each clock cycle

frequency control word

at time n, $\boxed{\phi = n F_{cw}/2^M}$

$$\cos\phi = \cos(n F_{cw}/2^M)$$
$$\sin\phi = \sin(n F_{cw}/2^M)$$

$-1 < \phi < +1$

normalized angle

Normalized Angle

at time n,

$$\boxed{\phi = n F_{cw}/2^M}$$

$$\cos\phi = \cos(n F_{cw}/2^M)$$
$$\sin\phi = \sin(n F_{cw}/2^M)$$

$-1 < \phi < +1$ $\quad\quad$ ϕ radians$/\pi$

$-\pi < \pi\phi < +\pi$ $\quad\quad$ $\pi\phi$ radians

$-\pi < \theta < +\pi$ $\quad\quad$ θ radians

$$\theta = \pi\phi$$

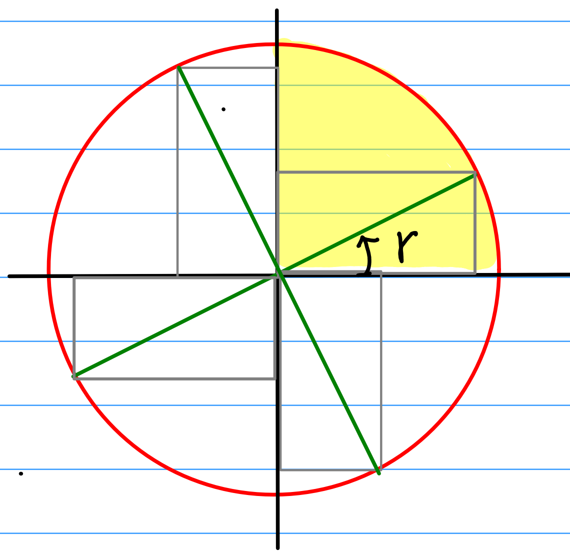
$$\theta \in [-\pi, +\pi]$$
$$\phi \in [-1, +1]$$

② Radian Converter

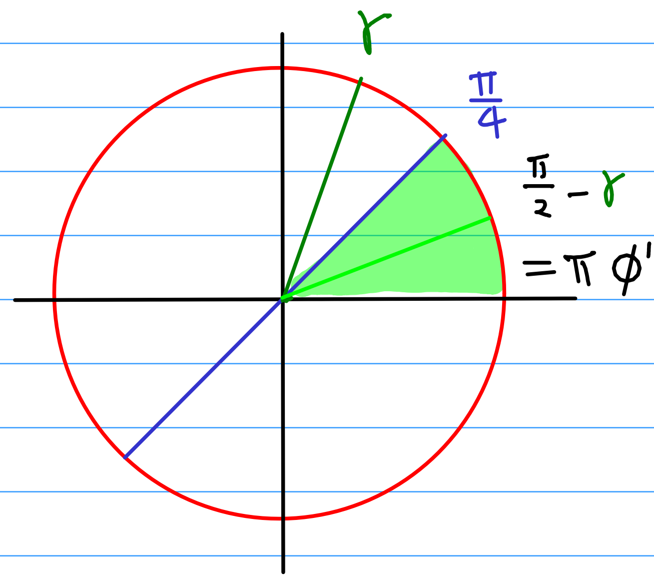
$$\theta = \pi\phi \rightarrow \theta' = \pi\phi' \rightarrow \theta'' = \pi\phi''$$

$$\theta \in [-\pi, +\pi] \rightarrow \theta' = [0, \pi/2] \rightarrow \theta'' = [0, \pi/4]$$

$$\phi \in [-1, +1] \rightarrow \phi' = [0, 0.5] \rightarrow \phi'' = [0, 0.25]$$



$$\theta' = \pi\phi'$$



$$\theta'' = \pi\phi''$$

$$-1 < \phi < +1$$

$$-\pi < \pi\phi < +\pi$$

$$-\pi < \theta < +\pi$$

radian

MSB_1

MSB_2

MSB_3

$$0 < \phi'' < 1/4$$

$$0 < \pi\phi'' < \pi/4$$

$$0 < \theta'' < 1$$

radian

normalized angle ϕ

ϕ

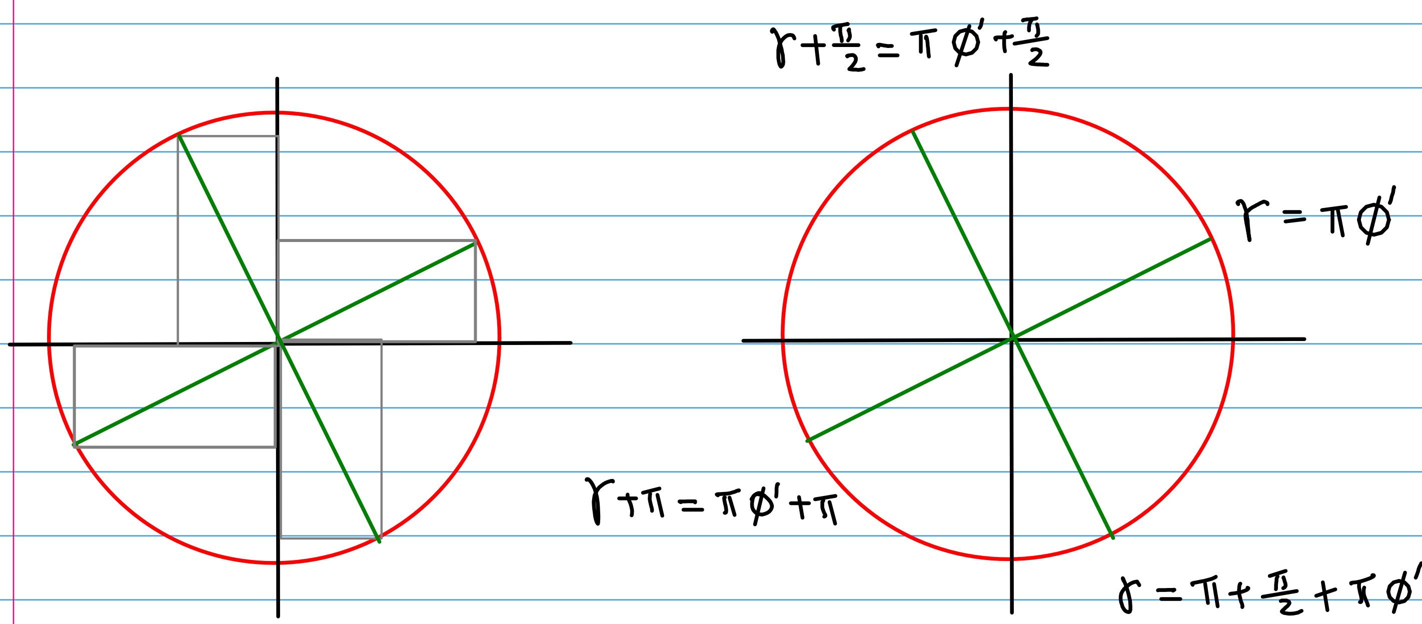
MSB1	MSB2	MSB3					...	

$\longleftarrow$ Quadrant of $\pi\phi$ $\longrightarrow$ $\longleftarrow$ U. half / L. half $\longrightarrow$

$\longleftarrow$ Stored to interchange / negate $\longrightarrow$

ϕ'

0	0	MSB3					...	

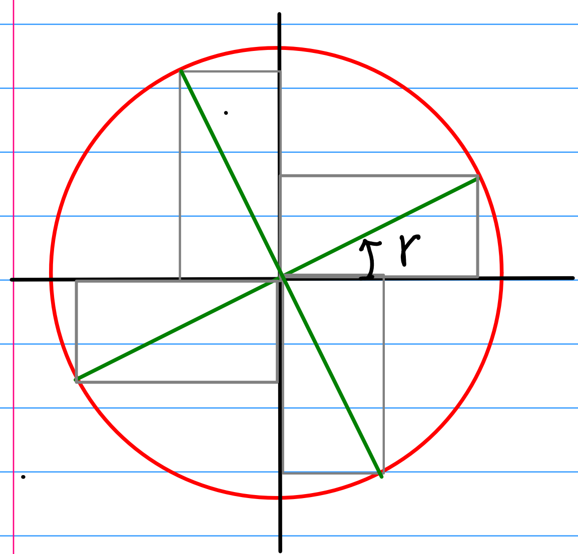


$\phi \rightarrow \phi' \rightarrow$

$\pi\phi'$	$+\ 0 \cdot \frac{\pi}{2}$	00
$\pi\phi'$	$+\ 1 \cdot \frac{\pi}{2}$	01
$\pi\phi'$	$+\ 2 \cdot \frac{\pi}{2}$	10
$\pi\phi'$	$+\ 3 \cdot \frac{\pi}{2}$	11

$\uparrow$
1st Quad

Quadrant Symmetry



0 1	0 0
$\gamma + \frac{\pi}{2} = \pi\phi' + \frac{\pi}{2}$	$\gamma = \pi\phi'$
$\gamma + \pi = \pi\phi' + \pi$	$\gamma + \frac{3\pi}{2} = \pi\phi' + \frac{3\pi}{2}$
1 0	1 1

0	1	+1
0	0	0
1	1	-1
1	0	-2

$$\theta = \pi\phi \Rightarrow \theta' = \pi\phi'$$

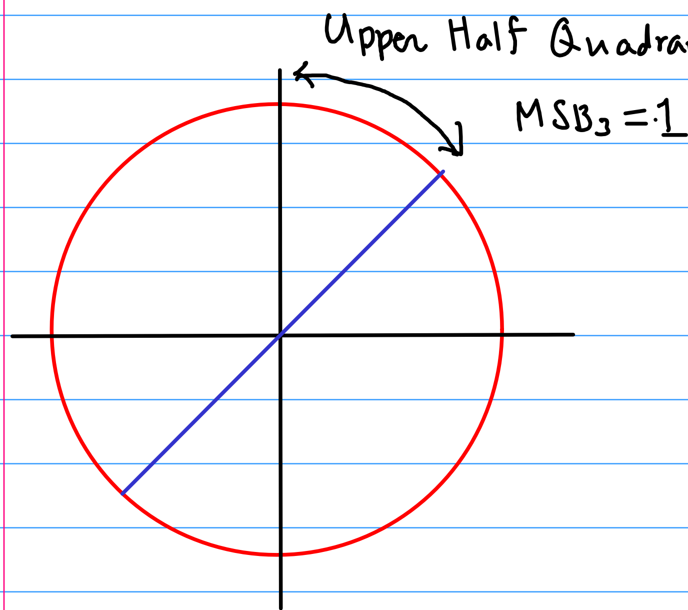
$$\theta \in [-\pi, +\pi] \Rightarrow \theta = [0, \tfrac{\pi}{2}]$$

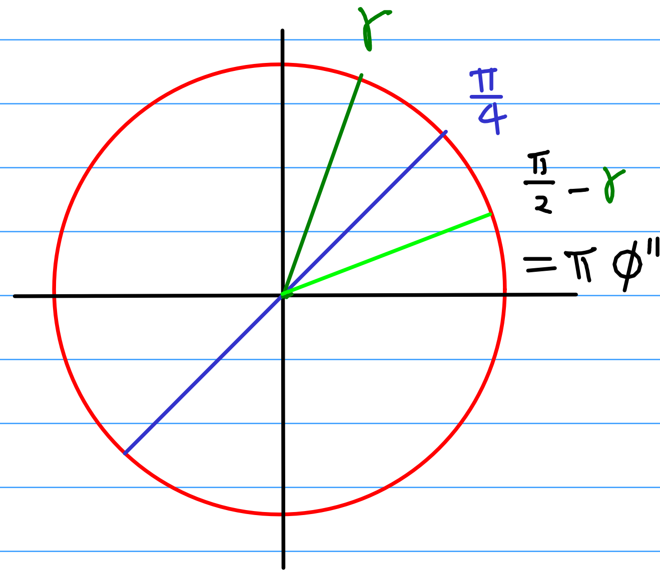
$$\phi \in [-1, +1] \Rightarrow \phi' = [0, 0.5]$$

ϕ'

0	0	MSB3					...	

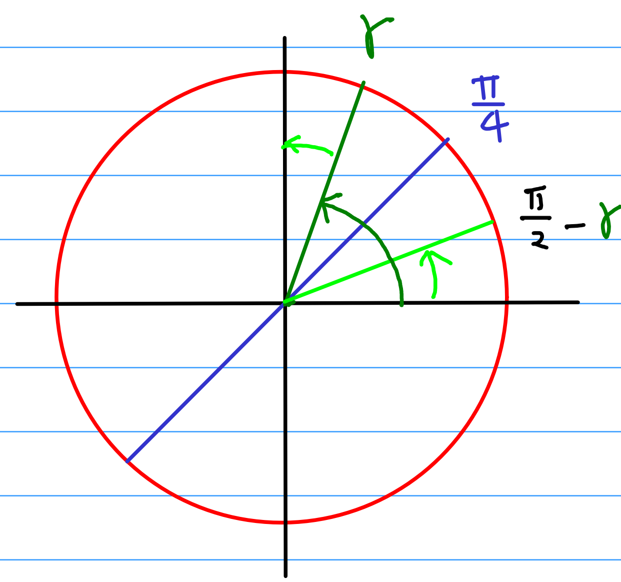
1st Quadrant





$\gamma > \frac{\pi}{4}$: Upper Half ($MSB_3 = 1$)

$\gamma < \frac{\pi}{4}$: Lower Half ($MSB_3 = 0$)

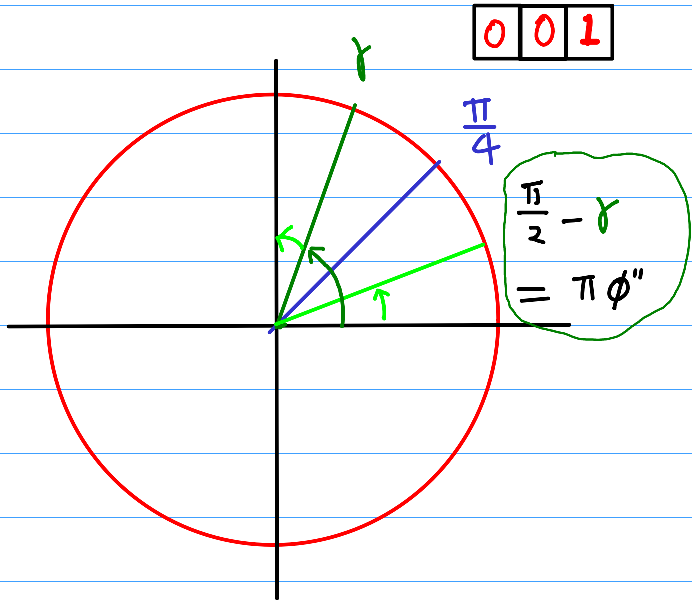


$$\cos\gamma = \sin\left(\frac{\pi}{2} - \gamma\right)$$

$$\sin\gamma = \cos\left(\frac{\pi}{2} - \gamma\right)$$

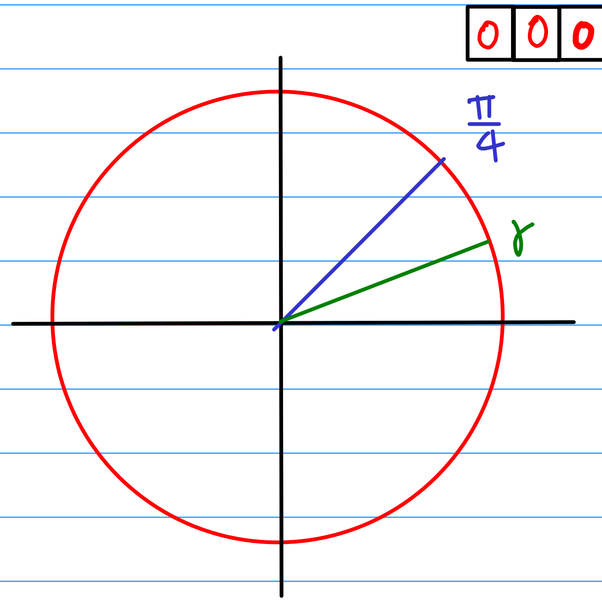
$\gamma > \frac{\pi}{4}$ $\quad$ $\frac{\pi}{2} - \gamma < \frac{\pi}{4}$.

π/4 mirror



$\frac{\pi}{4} < \gamma < \frac{\pi}{2}$

$\phi'' = 0.5 - \phi'$



$0 < \gamma < \frac{\pi}{4}$

$\phi'' = \phi'$

$$\theta = \pi\phi \rightarrow \theta' = \pi\phi' \rightarrow \theta'' = \pi\phi''$$

$$\theta \in [-\pi, +\pi] \rightarrow \theta' = [0, \pi/2] \rightarrow \theta'' = [0, \pi/4]$$

$$\phi \in [-1, +1] \rightarrow \phi' = [0, 0.5] \rightarrow \phi'' = [0, 0.25]$$

ϕ'									
	0	0	MSB3					...	

MSB3 = 1 $\quad \phi' > \frac{\pi}{4}$

$$\phi'' = \frac{\pi}{2} - \phi'$$

0.5	1	0	0					...	
$\ominus$ ϕ'	0	0	MSB3					...	

$$\begin{cases} MSB3 = 0 & \phi'' = \phi' \\ MSB_3 = 1 & \phi'' = 0.5 - \phi' \end{cases}$$

$\theta = \pi \phi''$ (Hardwired Multiplier)

$0 < \theta < \frac{\pi}{4}$

$\phi \longrightarrow \phi' \longrightarrow \phi''$

1st Quad $\quad$ Lower Half

$$\phi \longrightarrow \phi' \longrightarrow \phi''$$

normalized angle ϕ

ϕ	MSB1	MSB2	MSB3				...		

1st Quadrant

ϕ'	0	0	MSB3				...		

$\boxed{\text{MSB3}} = 0 \qquad \phi'' = \phi'$

$\boxed{\text{MSB3}} = 1 \qquad \phi'' = 0.5 - \phi'$

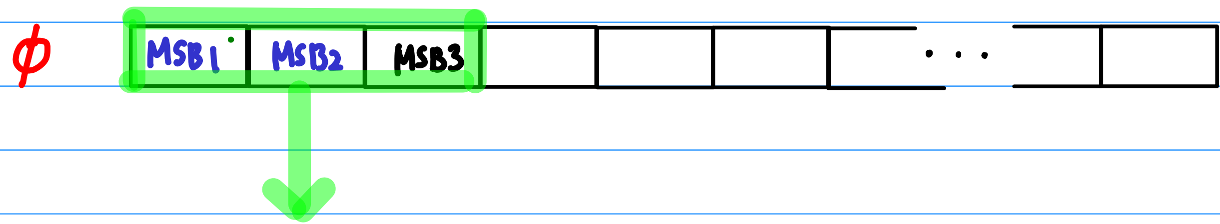
0.5	1	0	0	0	0	0	...	0
$-\ \phi'$	0	0	1	a_4	a_5	a_6	...	a_N

0.5	1	0	0	0	0	0	...	0
$\overline{\phi'}$	1	1	0	$\bar{a}_4$	$\bar{a}_5$	$\bar{a}_6$	...	$\bar{a}_N$

$0.5 - \phi'$	0	1	0	$\bar{a}_4$	$\bar{a}_5$	$\bar{a}_6$	...	$\bar{a}_N$
								$+1$
max	0	1	1	0	0	0	...	0

Control Signals : x invert, y invert, swap

normalized angle ϕ



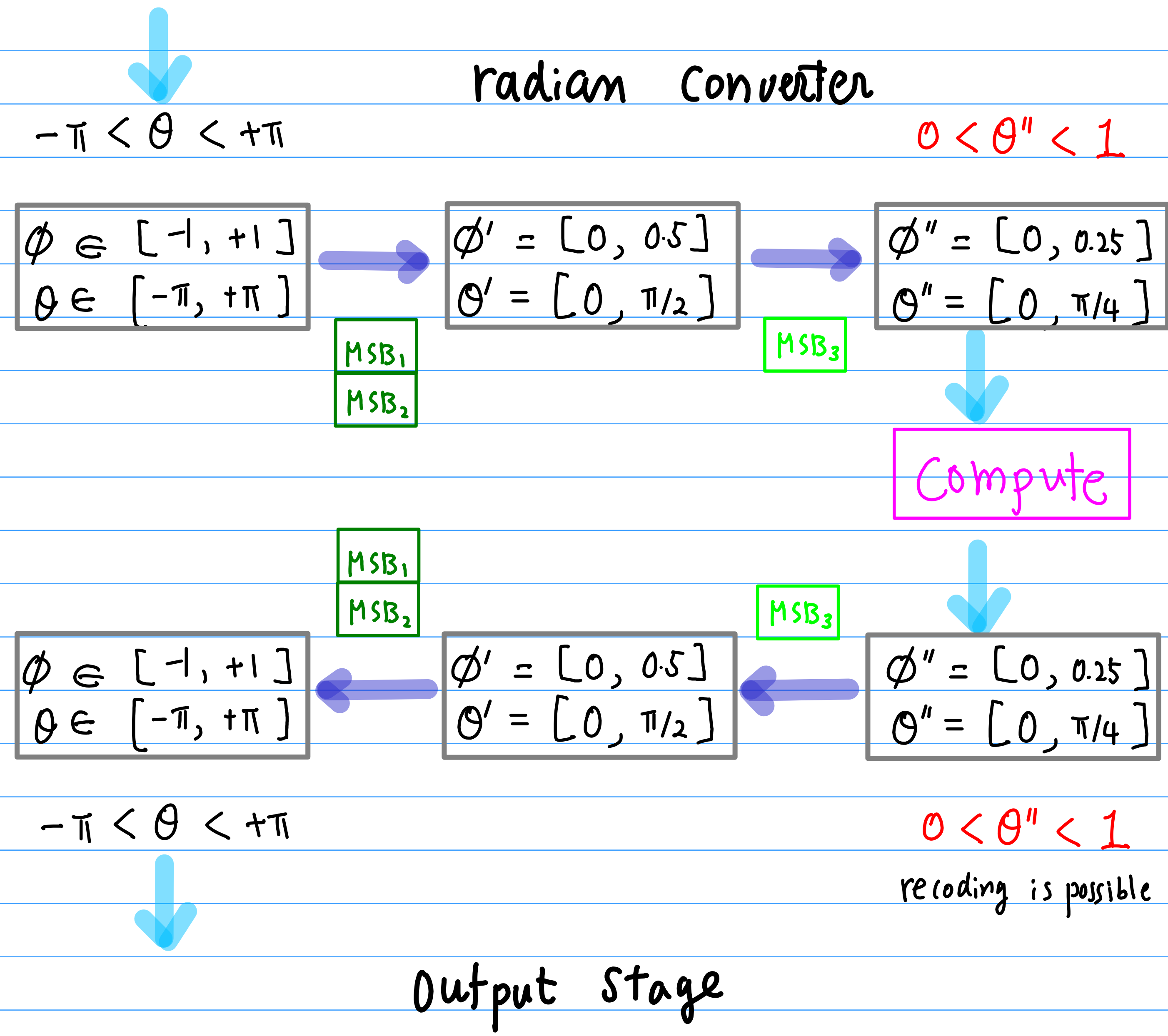
x - invert
y - invert
interchange

MSB1	MSB2	MSB3	$x$inv	$y$inv	swap
0	0	0	0	0	0
0	0	1	0	0	1
0	1	0	0	1	1
0	1	1	1	0	0
1	0	0	1	1	0
1	0	1	1	1	1
1	1	0	1	0	1
1	1	1	0	1	0

$$\theta = \pi\phi \longrightarrow \theta' = \pi\phi' \longrightarrow \theta'' = \pi\phi''$$

$$\theta \in [-\pi, +\pi] \longrightarrow \theta' = [0, \pi/2] \longrightarrow \theta'' = [0, \pi/4]$$

$$\phi \in [-1, +1] \longrightarrow \phi' = [0, 0.5] \longrightarrow \phi'' = [0, 0.25]$$



$$\boxed{\theta''} = \pi \boxed{\phi''}$$

radian angle

$0 < \theta'' < \pi/4$

normalized angle

$0 < \phi'' < 0.25$

$0 < \theta'' < 1$

The multiplication by π

→ could have used a handwired multiplier

→ but don't have to use a multiplier at all

① in table lookup DDFS architecture

→ here, the multiplication by π is implicit

② in CORDIC architecture

the elementary angle are divided by π

$$\theta_k = \tan^{-1}(2^{-k}) / 2\pi$$

the direction of subrotations are determined by the sign of angle difference

therefore the multiplication by π is <u>not</u> <u>necessary</u>

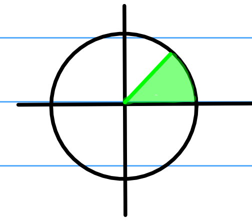
③ Sine / Cosine Generator

given angle θ (in radian)

$0 \leq \theta \leq \pi/4 < 1$

$\Downarrow$

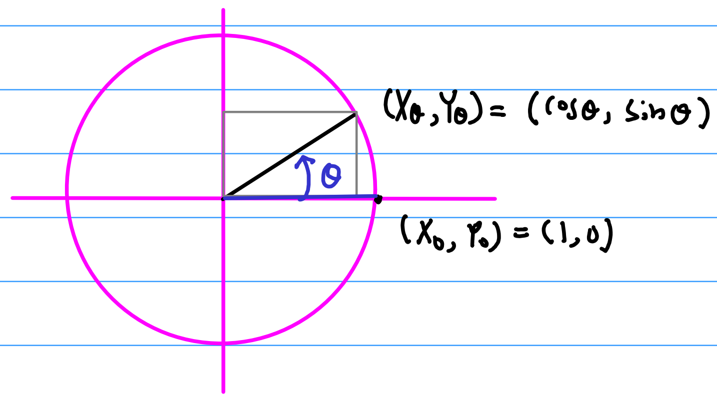
0.785398163



Compute $\boxed{\cos\theta}$, $\boxed{\sin\theta}$?

$$\begin{bmatrix} X_\theta \\ Y_\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} X_0 \\ Y_0 \end{bmatrix}$$

$$= \cos\theta \begin{bmatrix} 1 & -\tan\theta \\ \tan\theta & 1 \end{bmatrix} \begin{bmatrix} X_0 \\ Y_0 \end{bmatrix}$$



$$\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} X_\theta \\ Y_\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} X_0 \\ Y_0 \end{bmatrix}$$

$$= \cos\theta \begin{bmatrix} 1 & -\tan\theta \\ \tan\theta & 1 \end{bmatrix} \begin{bmatrix} X_0 \\ Y_0 \end{bmatrix}$$

$$= \cos\theta \begin{bmatrix} 1 & -\tan\theta \\ \tan\theta & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

a **sequence of subrotations** of the priori known angle

Suppose: θ as a sequence of sub-rotation

$\{\theta_k\}$ the subrotation angles are known a priori

then $\theta = \sigma_0 \theta_0 + \sigma_1 \theta_1 + \cdots + \sigma_N \theta_N$

$$\sigma_k = \{-1, 0, +1\}$$

① $\boxed{\theta_k = \tan^{-1} 2^{-k}}$ traditional CORDIC

② $\boxed{\theta_k = 2^{-k}}$ possible because $\theta'' < 1$

$$\theta = \sigma_0 \theta_0 + \sigma_1 \theta_1 + \cdots + \sigma_N \theta_N$$

$$\sigma_k = \{-1, 0, +1\}$$

$$\sigma_0 \theta_0 \rightarrow \cos(\sigma_0 \theta_0) \begin{bmatrix} 1 & -\tan(\sigma_0 \theta_0) \\ \tan(\sigma_0 \theta_0) & 1 \end{bmatrix}$$

$$\sigma_1 \theta_1 \rightarrow \cos(\sigma_1 \theta_1) \begin{bmatrix} 1 & -\tan(\sigma_1 \theta_1) \\ \tan(\sigma_1 \theta_1) & 1 \end{bmatrix}$$

$$\sigma_N \theta_N \rightarrow \cos(\sigma_N \theta_N) \begin{bmatrix} 1 & -\tan(\sigma_N \theta_N) \\ \tan(\sigma_N \theta_N) & 1 \end{bmatrix}$$

Sequence of sub-rotations

$$\theta = \sigma_0 \theta_0 + \sigma_1 \theta_1 + \cdots + \sigma_N \theta_N$$

$$\sigma_k = \{-1, 0, +1\}$$

$$\begin{bmatrix} X_\theta \\ Y_\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} X_0 \\ Y_0 \end{bmatrix}$$

$$\begin{bmatrix} X_\theta \\ Y_\theta \end{bmatrix} = \boxed{\cos(\sigma_0 \theta_0)} \begin{bmatrix} 1 & -\tan(\sigma_0 \theta_0) \\ \tan(\sigma_0 \theta_0) & 1 \end{bmatrix} \boxed{\cos(\sigma_1 \theta_1)} \begin{bmatrix} 1 & -\tan(\sigma_1 \theta_1) \\ \tan(\sigma_1 \theta_1) & 1 \end{bmatrix}$$
$$\cdots \boxed{\cos(\sigma_N \theta_N)} \begin{bmatrix} 1 & -\tan(\sigma_N \theta_N) \\ \tan(\sigma_N \theta_N) & 1 \end{bmatrix} \begin{bmatrix} X_0 \\ Y_0 \end{bmatrix}$$

$$\begin{bmatrix} X_\theta \\ Y_\theta \end{bmatrix} = K \begin{bmatrix} 1 & -\tan(\sigma_0 \theta_0) \\ \tan(\sigma_0 \theta_0) & 1 \end{bmatrix} \begin{bmatrix} 1 & -\tan(\sigma_1 \theta_1) \\ \tan(\sigma_1 \theta_1) & 1 \end{bmatrix} \cdots \begin{bmatrix} 1 & -\tan(\sigma_N \theta_N) \\ \tan(\sigma_N \theta_N) & 1 \end{bmatrix} \begin{bmatrix} X_0 \\ Y_0 \end{bmatrix}$$

$$K = \boxed{\cos(\sigma_0 \theta_0)} \boxed{\cos(\sigma_1 \theta_1)} \cdots \boxed{\cos(\sigma_N \theta_N)}$$ scale factor

$\sigma_k = +1$ positive angle rotation

$\sigma_k = -1$ negative angle rotation

A. CORDIC Algorithm

$$\theta = \sigma_0 \theta_0 + \sigma_1 \theta_1 + \cdots + \sigma_N \theta_N$$

$$\sigma_k = \{-1, 0, +1\}$$

$$\boxed{\boxed{\theta_k = \tan^{-1} 2^{-k}} \quad \tan \theta_k = 2^{-k} \quad \tan(\sigma_k \theta_k) = (\sigma_k 2^{-k})}$$

$$\sigma_k = \{-1, +1\}$$

$$K \begin{bmatrix} 1 & -\tan(\sigma_0\theta_0) \\ \tan(\sigma_0\theta_0) & 1 \end{bmatrix} \begin{bmatrix} 1 & -\tan(\sigma_1\theta_1) \\ \tan(\sigma_1\theta_1) & 1 \end{bmatrix} \cdots \begin{bmatrix} 1 & -\tan(\sigma_N\theta_N) \\ \tan(\sigma_N\theta_N) & 1 \end{bmatrix}$$

$$K = \cos(\sigma_0\theta_0)\cos(\sigma_1\theta_1)\cdots\cos(\sigma_N\theta_N)$$ scale factor

$$K \begin{bmatrix} 1 & -\sigma_0 2^{-0} \\ \sigma_0 2^{-0} & 1 \end{bmatrix} \begin{bmatrix} 1 & -\sigma_1 2^{-1} \\ \sigma_1 2^{-1} & 1 \end{bmatrix} \cdots \begin{bmatrix} 1 & -\sigma_N 2^{-N} \\ \sigma_N 2^{-N} & 1 \end{bmatrix}$$

↳ shift-and-add

$$K = \cos(\theta_0)\cos(\theta_1)\cdots\cos(\theta_N)$$

constant ← each rotation
+/− rotation
is actually performed

$$\begin{pmatrix} X_0 \\ Y_0 \end{pmatrix} = \begin{pmatrix} K \\ 0 \end{pmatrix}$$

$$\begin{bmatrix} X_\theta \\ Y_\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} X_0 \\ Y_0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -\sigma_0 2^{-0} \\ \sigma_0 2^{-0} & 1 \end{bmatrix} \begin{bmatrix} 1 & -\sigma_1 2^{-1} \\ \sigma_1 2^{-1} & 1 \end{bmatrix} \cdots \begin{bmatrix} 1 & -\sigma_N 2^{-N} \\ \sigma_N 2^{-N} & 1 \end{bmatrix} \begin{bmatrix} K \\ 0 \end{bmatrix}$$

$$K = \cos(\theta_0) \cos(\theta_1) \cdots \cos(\theta_N)$$

σ_k determines pos / neg subrotation by an angle θ_k

σ_k values are determined iteratively by the successive approximation

at the k-th iteration

- if the current appraximation $>$ the input angle θ then subtract θ_k
- if the current appraximation $<$ the input angle θ then add θ_k

CORDIC HW $\frac{1}{3}$ of the total HW

ⓐ computes σ_k
updates the current approximation by the angle θ_k

ⓑ performs the rotation by θ_k

$$\begin{pmatrix} \text{addition} \\ \text{comparison} \end{pmatrix}$$

redundant CSA $\boxed{\text{addition}}$ $\sum \theta_k$
eliminates the carry propagate delay
improves the throughput

the evaluation of each σ_k $\boxed{\text{comparison}}$
requires the knowledge of the sign difference
between two angles

the sign detection in redundant arithmetic
non-trivial, bottleneck

B. Recoding Algorithm

$$\theta = \sigma_0 \theta_0 + \sigma_1 \theta_1 + \cdots + \sigma_N \theta_N .$$

$$\sigma_k = \{-1, 0, +1\}$$

$$\boxed{\begin{gathered} \theta_k = 2^{-k} \\ \tan \theta_k = \tan 2^{-k} \\ \tan(\sigma_k \theta_k) = \tan(\sigma_k 2^{-k}) \end{gathered}}$$

$$\sigma_k = \{-1, +1\}$$

$$\begin{bmatrix} X_\theta \\ Y_\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} X_0 \\ Y_0 \end{bmatrix} = \cos\theta \begin{bmatrix} 1 & -\tan\theta \\ \tan\theta & 1 \end{bmatrix} \begin{bmatrix} X_0 \\ Y_0 \end{bmatrix}$$

$$K \begin{bmatrix} 1 & -\tan(\sigma_0\theta_0) \\ \tan(\sigma_0\theta_0) & 1 \end{bmatrix} \begin{bmatrix} 1 & -\tan(\sigma_1\theta_1) \\ \tan(\sigma_1\theta_1) & 1 \end{bmatrix} \cdots \begin{bmatrix} 1 & -\tan(\sigma_N\theta_N) \\ \tan(\sigma_N\theta_N) & 1 \end{bmatrix}$$

$$K = \cos(\sigma_0\theta_0)\cos(\sigma_1\theta_1)\cdots\cos(\sigma_N\theta_N)$$ scale factor

$$\theta = \sigma_0 \theta_0 + \sigma_1 \theta_1 + \cdots + \sigma_N \theta_N \qquad \sigma_k \in \{-1, 0, +1\}$$

$$\theta'' = \boxed{\sum_{k=1}^{N} b_k \cdot 2^{-k} = \phi_0 + \sum_{k=2}^{N+1} r_k \cdot 2^{-k}} \qquad \begin{array}{l} b_k \in \{0, 1\} \\ r_k \in \{-1, +1\} \end{array}$$

$$K \begin{bmatrix} 1 & -\tan(\sigma_0 \theta_0) \\ \tan(\sigma_0 \theta_0) & 1 \end{bmatrix} \begin{bmatrix} 1 & -\tan(\sigma_1 \theta_1) \\ \tan(\sigma_1 \theta_1) & 1 \end{bmatrix} \cdots \begin{bmatrix} 1 & -\tan(\sigma_N \theta_N) \\ \tan(\sigma_N \theta_N) & 1 \end{bmatrix}$$

$$K = \cos(\theta_0) \cos(\theta_1) \cdots \cos(\theta_N) \qquad \theta_k = \tan^{-1} 2^{-k}$$

$$K \begin{bmatrix} 1 & -\tan(r_2 \theta_2) \\ \tan(r_2 \theta_2) & 1 \end{bmatrix} \begin{bmatrix} 1 & -\tan(r_3 \theta_3) \\ \tan(r_3 \theta_3) & 1 \end{bmatrix} \cdots \begin{bmatrix} 1 & -\tan(r_N \theta_N) \\ \tan(r_N \theta_N) & 1 \end{bmatrix}$$

$$K = \cos(\theta_2) \cos(\theta_3) \cdots \cos(\theta_N) \qquad \theta_k = 2^{-k}$$

The recoding maintains a constant scale factor K

$$K \begin{bmatrix} 1 & -\tan(r_2\theta_2) \\ \tan(r_2\theta_2) & 1 \end{bmatrix} \begin{bmatrix} 1 & -\tan(r_3\theta_3) \\ \tan(r_3\theta_3) & 1 \end{bmatrix} \cdots \begin{bmatrix} 1 & -\tan(r_N\theta_N) \\ \tan(r_N\theta_N) & 1 \end{bmatrix}$$

$$K = \cos(\theta_2)\cos(\theta_3)\cdots\cos(\theta_N) \qquad \theta_k = 2^{-k}$$

$$\begin{bmatrix} X_{k+1} \\ Y_{k+1} \end{bmatrix} = \begin{bmatrix} 1 & -\tan(r_k\theta_k) \\ \tan(r_k\theta_k) & 1 \end{bmatrix} \begin{bmatrix} X_k \\ Y_k \end{bmatrix}.$$

$$= \begin{bmatrix} X_k - \tan(r_k\theta_k)Y_k \\ Y_k + \tan(r_k\theta_k)X_k \end{bmatrix}$$

Sub rotation

$$X_{k+1} = X_k - \tan(r_k\theta_k)Y_k$$

$$Y_{k+1} = Y_k + \tan(r_k\theta_k)X_k$$

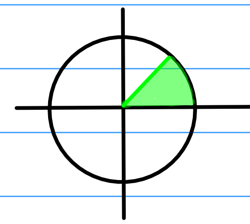
Madisetti & Willson, DDS Freq synthe

Angle Recoding

given angle θ (in radian)

$0 \leq \theta \leq \pi/4 < 1$

$\pi/4 \Rightarrow 0.785398163$



$$\theta = \sum_{k=1}^{N} b_k \theta_k$$ Binary Representation

$b_k \in \{0, 1\}$

$\theta_k = 2^{-k}$

Sign + N bit $\Rightarrow$ (N+1) bit fractional binary

S	b_1 b_2 $\cdots$ b_N

assume θ is positive $\boxed{b_0 = 0}$ $\boxed{S = 0}$

$$\boxed{\theta = \sum_{k=1}^{N} b_k 2^{-k} = \phi_0 + \sum_{k=2}^{N+1} r_k 2^{-k}}$$

$r_k \in \{-1, +1\}$ signed digits

ϕ_0 constant

⊕ subrotation by 2^{-k}

2 equal ⊕ half rotations by 2^{-k-1}

⊙ subrotation

2 equal opposite half rotations by $\pm 2^{-k-1}$

Binary Representation

$b_k = 1$: rotation by 2^{-k}

$b_k = 0$: zero rotation

k-th rotation

- fixed rotation by 2^{-k-1}
- pos rotation $\leftarrow b_k = 1$
- neg rotation $\leftarrow b_k = 0$

Combining all the fixed rotations

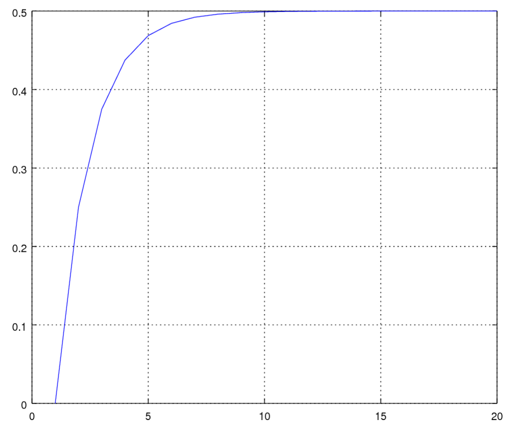
$\rightarrow$ initial fixed rotation ϕ_0

	b_1	b_2	b_3		b_N
	2^{-1}	2^{-2}	2^{-3}		2^{-N}
fixed $\Rightarrow$	$+2^{-2}$	$+2^{-3}$	$+2^{-4}$		$+2^{-N-1}$
	$(b_1=1)$ $+2^{-2}$	$(b_2=1)$ $+2^{-3}$	$(b_3=1)$ $+2^{-4}$		$(b_N=1)$ $+2^{-N-1}$
	$(b_1=0)$ -2^{-2}	$(b_2=0)$ -2^{-3}	$(b_3=0)$ -2^{-4}		$(b_N=0)$ -2^{-N-1}

initial fixed rotation

$$\phi_0 = \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{N+1}}$$

$$= \frac{\frac{1}{2^2}\left(1-\frac{1}{2^N}\right)}{\left(1-\frac{1}{2}\right)} = \frac{1}{2}\left(1-\frac{1}{2^N}\right) = \frac{1}{2} - \frac{1}{2^{N+1}}$$



Signed Digit Recoding

the rotation after recoding

— a fixed initial rotation ϕ_0

a sequence of $\oplus / \ominus$ rotations

$b_k = 1$ $\quad + 2^{-k-1}$ rotation

$b_k = 0$ $\quad - 2^{-k-1}$ rotation

$r_k = (2b_{k-1} - 1)$

$2 \cdot 1 - 1 = +1$ $\qquad b_{k-1} = 1 \longrightarrow r_k = +1$

$2 \cdot 0 - 1 = -1$ $\qquad b_{k-1} = 0 \longrightarrow r_k = -1$

The recoding need not be explicitly performed

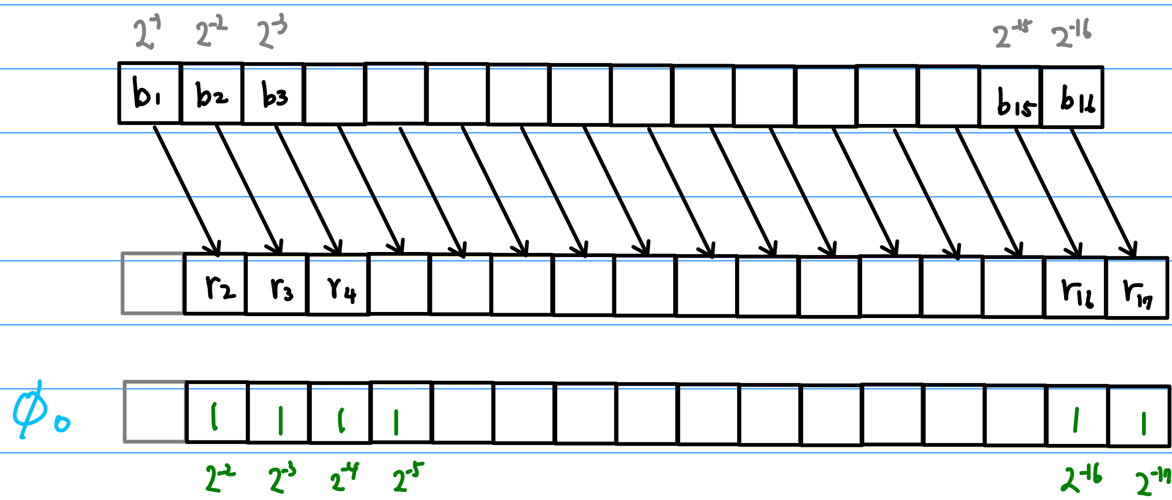
Simply replacing $b_k = 0$ with -1

This recoding maintains

a constant scaling factor K

$$\theta = \sum_{k=1}^{N} b_k 2^{-k} = \phi_0 + \sum_{k=2}^{N+1} r_k 2^{-k}$$

Binary Representation $\{b_k\}$



Signed Digit Recoding $\{r_k\}$

$\boxed{MSB_1}\boxed{MSB_2}\boxed{MSB_3} \longrightarrow 0 < \theta'' < 1 \longrightarrow$ recoding $\{r_k\}$

$$\boxed{\sum_{k=1}^{N} b_k \cdot 2^{-k} = \phi_0 + \sum_{k=2}^{N+1} r_k \cdot 2^{-k}}$$

$b_k \in \{0, 1\}$ $\qquad$ $r_k \in \{-1, +1\}$

$$\begin{cases} b_k = 1 \longrightarrow r_{k+1} = +1 \\ b_k = 0 \longrightarrow r_{k+1} = -1 \end{cases} \qquad \boxed{r_k = (2b_{k-1} - 1)}$$

ϕ_0 depends only on bit width N

for fixed N, $\boxed{\phi_0 = \frac{1}{2} - \frac{1}{2^{N+1}}}$ is a constant

The Scaling K.

The initial rotation $\phi_0 = \frac{1}{2} - \frac{1}{2^{N+1}}$

rotation Starting point

$$(X_0, Y_0) = (K\cos\phi_0,\ K\sin\phi_0)$$

rotation always starts from this fixed point.

Cascade of feed forward rotational stages

$$\theta \rightarrow \boxed{MSB_1 \mid MSB_2 \mid MSB_3} \rightarrow \theta'' \rightarrow b_k 2^{-k} \rightarrow r_k 2^{-k}$$

binary representation

recoding

{ no comparison
no error build up

① $\boxed{\theta_k = \tan^{-1} 2^{-k}}$ traditional CORDIC

☆ ② $\boxed{\theta_k = 2^{-k}}$ possible because $\theta'' < 1$

$$\begin{bmatrix} 1 & -\sigma_k \tan(2^{-k}) \\ \sigma_k \tan(2^{-k}) & 1 \end{bmatrix}$$

④ Output stage

$$-\pi < \theta = \pi\phi < \pi$$

$$0 < \theta'' = \pi\phi'' < \frac{\pi}{4}$$

$$X_{N+1} = \cos\theta'' \longrightarrow \cos\theta$$

$$Y_{N+1} = \sin\theta'' \longrightarrow \sin\theta$$

$\theta'' \in [0, \frac{\pi}{4}]$ $\qquad$ $\theta \in [-\pi, +\pi]$

Output stage

$\sin\theta \longrightarrow \sin\pi\phi$
$\cos\theta \longrightarrow \cos\pi\phi$

$[-\pi, +\pi]$

$\sin\theta'' \longrightarrow \sin\theta$
$\cos\theta'' \longrightarrow \cos\theta$

$[0, \frac{\pi}{4}]$ $[-\pi, +\pi]$

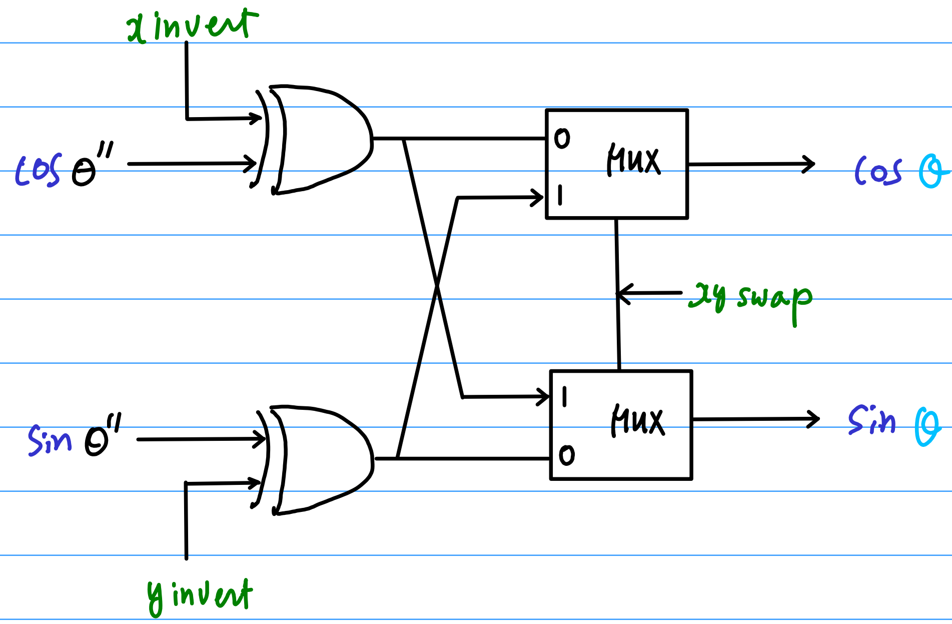
negation / interchange

MSB1, MSB2, MSB3 $\rightarrow$ x invert, y invert, xy swap

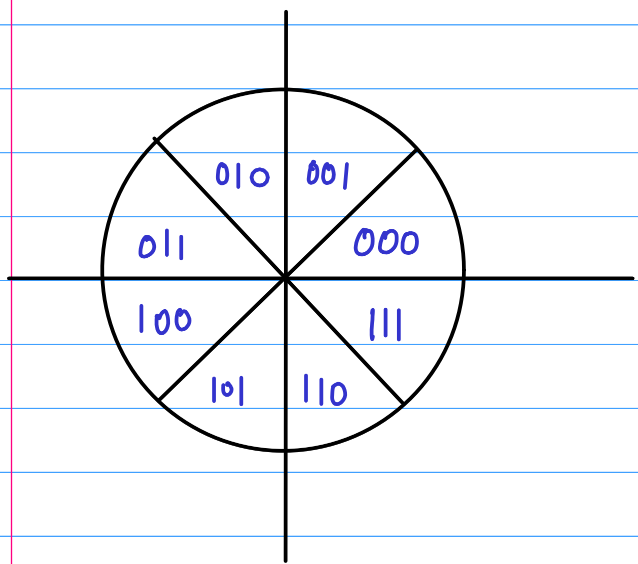
the negation of $\boxed{\begin{array}{l}\cos\theta'' = X_{N+1} \\ \sin\theta'' = Y_{N+1}\end{array}}$

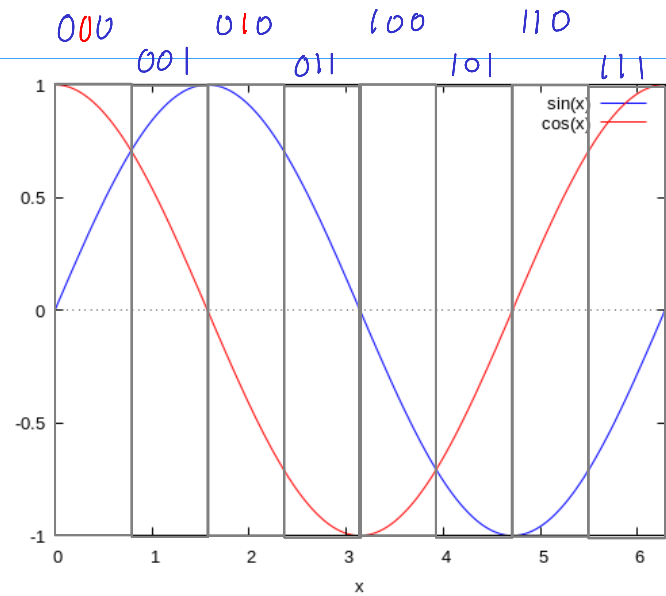
interchange $\boxed{\begin{array}{l}\cos\theta'' = X_{N+1} \\ \sin\theta'' = Y_{N+1}\end{array}}$

negate before swap



$\theta'' \in [0, \frac{\pi}{4}]$ $\theta \in [-\pi, +\pi]$



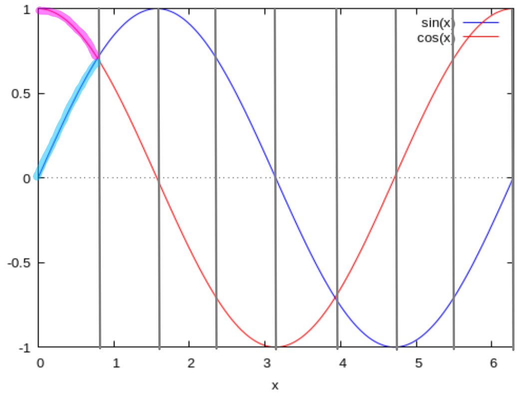


	cos	sin.			
	xinv	yinv	swap	$\cos \pi \theta$	$\sin \pi \phi$
000	0	0	0	$\cos\theta$	$\sin\theta$
001	0	0	1	$\sin\theta$	$\cos\theta$
010	0	1	1	$-\sin\theta$	$\cos\theta$
011	1	0	0	$-\cos\theta$	$\sin\theta$
100	1	1	0	$-\cos\theta$	$-\sin\theta$
101	1	1	1	$-\sin\theta$	$-\cos\theta$
110	1	0	1	$\sin\theta$	$-\cos\theta$
111	0	1	0	$\cos\theta$	$-\sin\theta$

⓪

$\cos\theta$

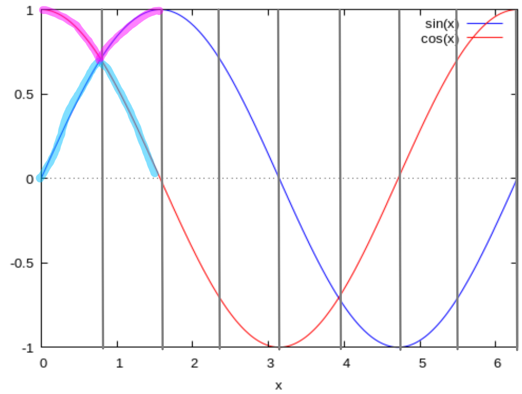
$\sin\theta$



①

$\sin\theta$

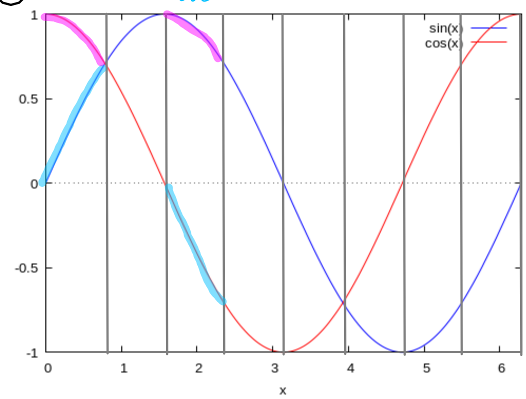
$\cos\theta$



②

$-\sin\theta$

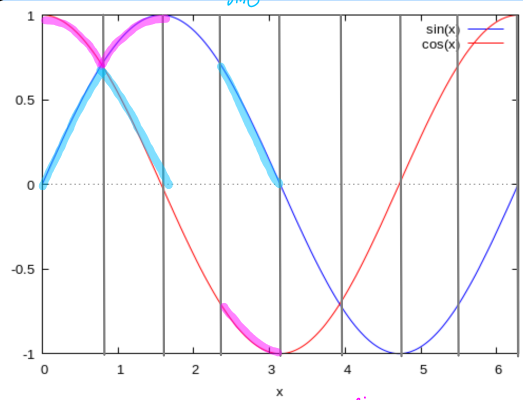
$\cos\theta$



③

$-\cos\theta$

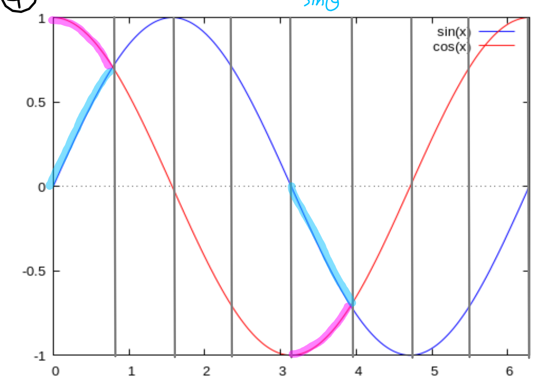
$\sin\theta$



④

$-\cos\theta$

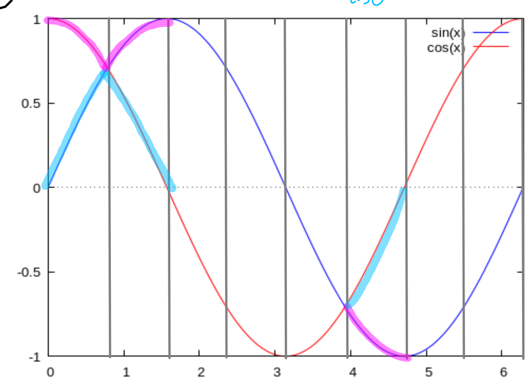
$-\sin\theta$



⑤

$-\sin\theta$

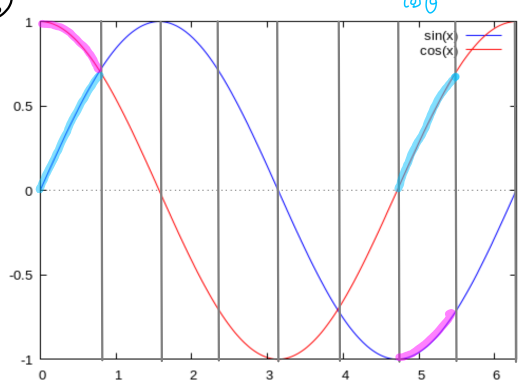
$-\cos\theta$



⑥

$\sin\theta$

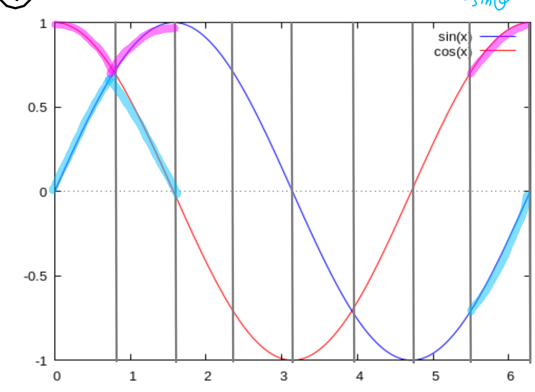
$-\cos\theta$



⑦

$\cos\theta$

$-\sin\theta$



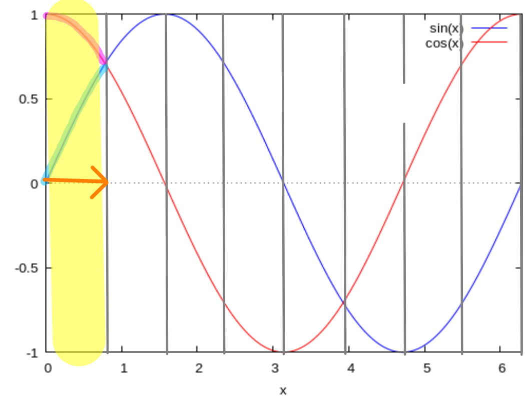
Madisetti & Willson, DDS Freq synthesizer

$$\begin{cases} \cos\phi \\ \sin\phi \end{cases}$$

0

$\cos\theta$

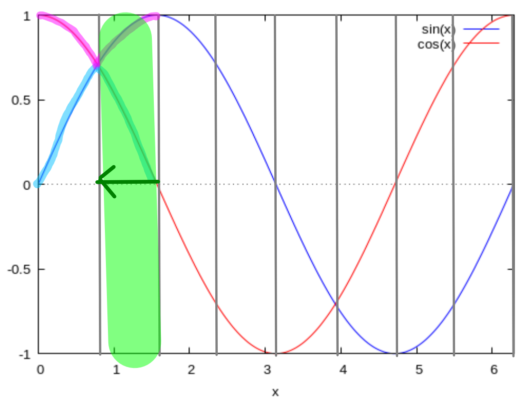
$\sin\theta$



1

$\sin\theta$

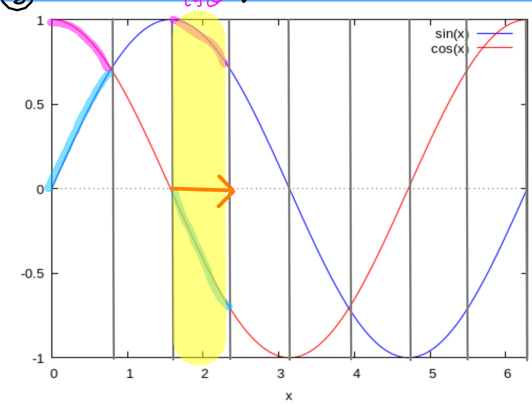
$\cos\theta$



2

$-\sin\theta$

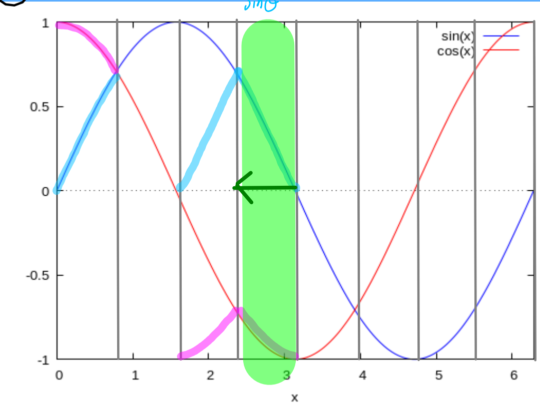
$\cos\theta$



3

$-\cos\theta$

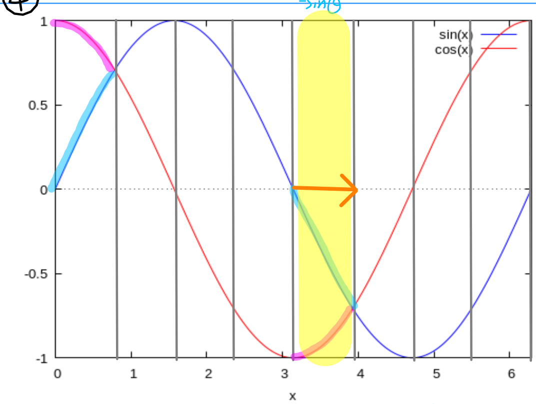
$\sin\theta$



4

$-\cos\theta$

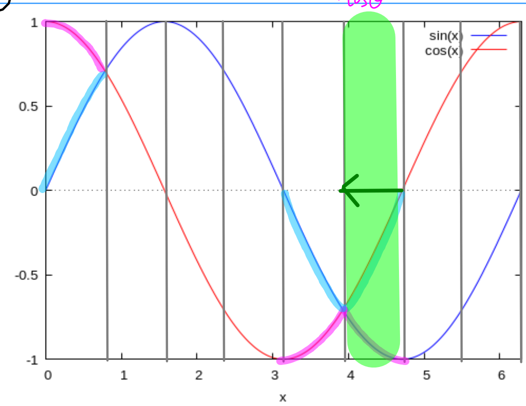
$-\sin\theta$



5

$-\sin\theta$

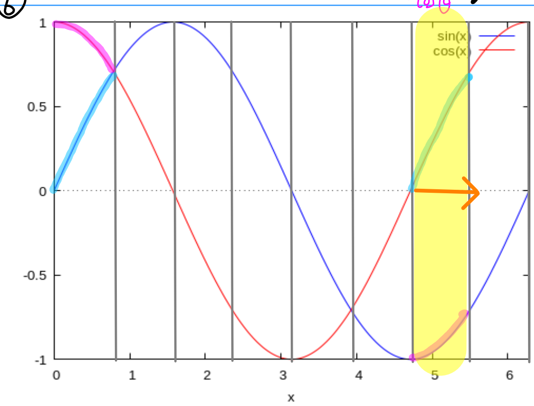
$-\cos\theta$



6

$\sin\theta$

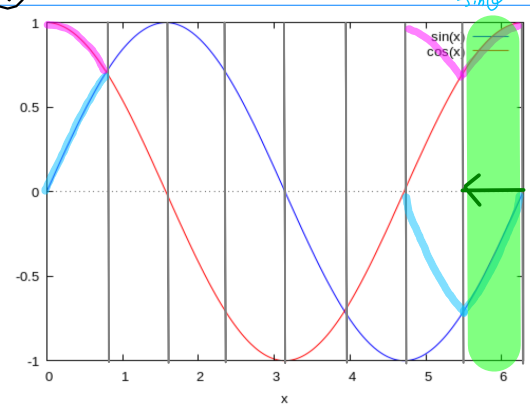
$-\cos\theta$



7

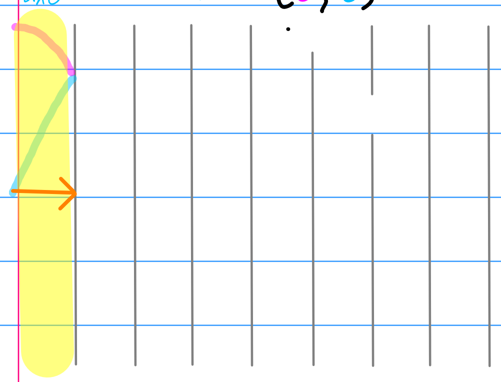
$\cos\theta$

$-\sin\theta$

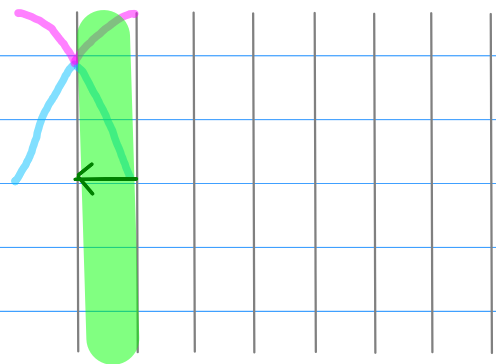


sin Φ

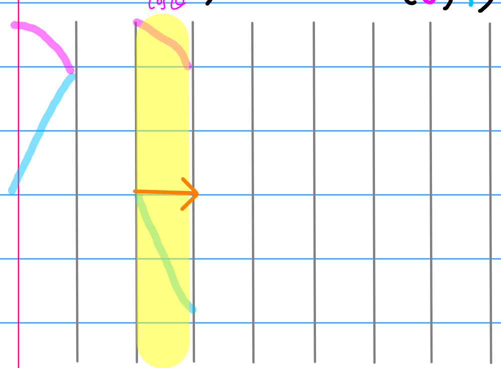
⓪ cos θ sin θ + + (0, 0)



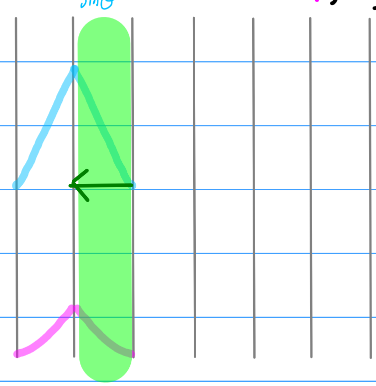
① sin θ cos θ) + + (0, 0)



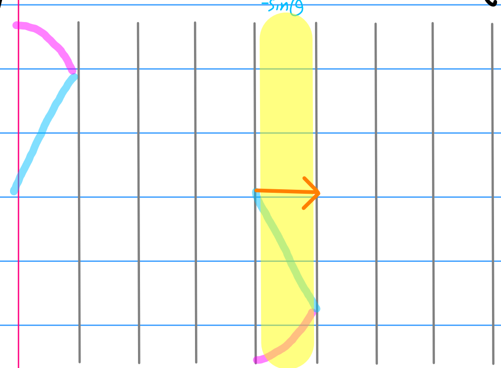
② -sin θ cos θ) + - (0, 1)



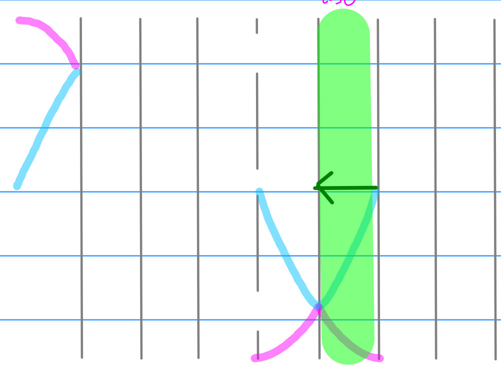
③ -cos θ sin θ - + (1, 0)



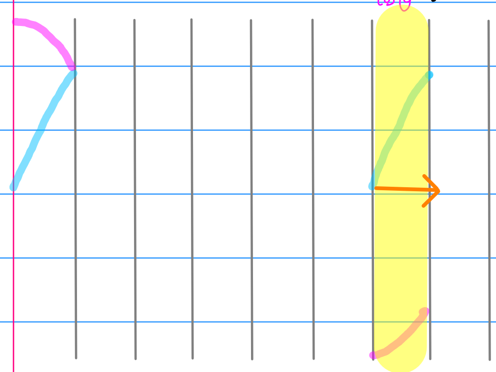
④ -cos θ -sin θ - - (1, 1)



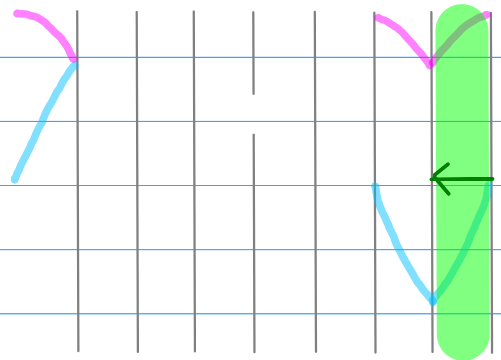
⑤ -sin θ -cos θ) - - (1, 1)



⑥ sin θ -cos θ) - + (1, 0)



⑦ cos θ -sin θ + - (0, 1)



	Xinv	Yinv	swap	$\cos \pi \phi$	$\sin \pi \phi$
0 0 0	0	0	0	$\cos\theta$	$\sin\theta$
0 0 1	0	0	1	$\sin\theta$	$\cos\theta$
0 1 0	0	1	1	$-\sin\theta$	$\cos\theta$
0 1 1	1	0	0	$-\cos\theta$	$\sin\theta$
1 0 0	1	1	0	$-\cos\theta$	$-\sin\theta$
1 0 1	1	1	1	$-\sin\theta$	$-\cos\theta$
1 1 0	1	0	1	$\sin\theta$	$-\cos\theta$
1 1 1	0	1	0	$\cos\theta$	$-\sin\theta$

0 0	0 0
0 1	1 0
1 1	1 1
1 0	0 1

0 0 0 0

0 1 1 0

1 1 1 1

1 0 0 1