Matrix (2A)

Please send corrections (or suggestions) to youngwlim@hotmail.com.

This document was produced by using LibreOffice and Octave.

Row Vector, Column Vector, Square Matrix

Name	Size	Example	Description
Row vector	$1 \times n$	$\begin{bmatrix} 3 & 7 & 2 \end{bmatrix}$	A matrix with one row, sometimes used to represent a vector
Column vector	$n \times 1$	$\begin{bmatrix} 4 \\ 1 \\ 8 \end{bmatrix}$	A matrix with one column, sometimes used to represent a vector
Square matrix	$n \times n$	$\begin{bmatrix} 9 & 13 & 5 \\ 1 & 11 & 7 \\ 2 & 6 & 3 \end{bmatrix}$	A matrix with the same number of rows and columns, sometimes used to represent a linear transformation from a vector space to itself, such as reflection, rotation, or shearing.

https://en.wikipedia.org/wiki/Matrix_(mathematics)

Matrix Notation

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = (a_{ij}) \in \mathbb{R}^{m \times n}.$$

$$\mathbf{A} = \begin{bmatrix} 0 & -1 & -2 & -3 \\ 1 & 0 & -1 & -2 \\ 2 & 1 & 0 & -1 \end{bmatrix}$$

$$a_{i,j} = f(i, j).$$

$$a_{ij} = i - j.$$

https://en.wikipedia.org/wiki/Matrix_(mathematics)

Matrix Addition

Addition	The *sum* **A**+**B** of two *m*-by-*n* matrices **A** and **B** is calculated entrywise: $(\mathbf{A} + \mathbf{B})_{i,j} = \mathbf{A}_{i,j} + \mathbf{B}_{i,j}$, where $1 \le i \le m$ and $1 \le j \le n$.

$$\begin{bmatrix} 1 & 3 & 1 \\ 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 5 \\ 7 & 5 & 0 \end{bmatrix} = \begin{bmatrix} 1+0 & 3+0 & 1+5 \\ 1+7 & 0+5 & 0+0 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 6 \\ 8 & 5 & 0 \end{bmatrix}$$

https://en.wikipedia.org/wiki/Matrix_(mathematics)

Scalar Multiplication

Scalar multiplication	The product $c\mathbf{A}$ of a number c (also called a scalar in the parlance of abstract algebra) and a matrix $\mathbf{A}$ is computed by multiplying every entry of $\mathbf{A}$ by c: $(c\mathbf{A})_{i,j} = c \cdot \mathbf{A}_{i,j}$. This operation is called *scalar multiplication*, but its result is not named "scalar product" to avoid confusion, since "scalar product" is sometimes used as a synonym for "inner product".

$$2 \cdot \begin{bmatrix} 1 & 8 & -3 \\ 4 & -2 & 5 \end{bmatrix} = \begin{bmatrix} 2 \cdot 1 & 2 \cdot 8 & 2 \cdot -3 \\ 2 \cdot 4 & 2 \cdot -2 & 2 \cdot 5 \end{bmatrix} = \begin{bmatrix} 2 & 16 & -6 \\ 8 & -4 & 10 \end{bmatrix}$$

https://en.wikipedia.org/wiki/Matrix_(mathematics)

Transposition

Transposition	The *transpose* of an *m*-by-*n* matrix **A** is the *n*-by-*m* matrix $\mathbf{A}^{\mathrm{T}}$ (also denoted $\mathbf{A}^{\mathrm{tr}}$ or ${}^{\mathrm{t}}\mathbf{A}$) formed by turning rows into columns and vice versa: $(\mathbf{A}^{\mathrm{T}})_{i,j} = \mathbf{A}_{j,i}$.

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -6 & 7 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} 1 & 0 \\ 2 & -6 \\ 3 & 7 \end{bmatrix}$$

https://en.wikipedia.org/wiki/Matrix_(mathematics)

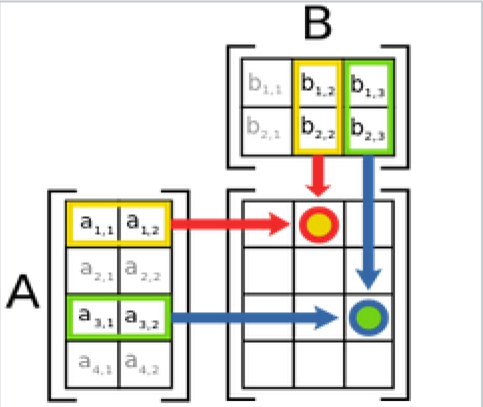
Matrix Multiplication

Multiplication of two matrices is defined if and only if the number of columns of the left matrix is the same as the number of rows of the right matrix. If **A** is an *m*-by-*n* matrix and **B** is an *n*-by-*p* matrix, then their *matrix product* **AB** is the *m*-by-*p* matrix whose entries are given by dot product of the corresponding row of **A** and the corresponding column of **B**:

$$[\mathbf{AB}]_{i,j} = A_{i,1}B_{1,j} + A_{i,2}B_{2,j} + \cdots + A_{i,n}B_{n,j} = \sum_{r=1}^{n} A_{i,r}B_{r,j}\,,$$

where $1 \le i \le m$ and $1 \le j \le p$.[13] For example, the underlined entry 2340 in the product is calculated as $(2 \times 1000) + (3 \times 100) + (4 \times 10) = 2340$:

$$\begin{bmatrix} \underline{2} & \underline{3} & \underline{4} \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & \underline{1000} \\ 1 & \underline{100} \\ 0 & \underline{10} \end{bmatrix} = \begin{bmatrix} 3 & \underline{2340} \\ 0 & 1000 \end{bmatrix}.$$



Schematic depiction of the matrix product **AB** of two matrices **A** and **B**.

https://en.wikipedia.org/wiki/Matrix_(mathematics)

Properties

Matrix multiplication satisfies the rules (**AB**)**C** = **A**(**BC**) (associativity), and (**A**+**B**)**C** = **AC**+**BC** as well as **C**(**A**+**B**) = **CA**+**CB** (left and right distributivity), whenever the size of the matrices is such that the various products are defined.[14] The product **AB** may be defined without **BA** being defined, namely if **A** and **B** are *m*-by-*n* and *n*-by-*k* matrices, respectively, and $m \neq k$. Even if both products are defined, they need not be equal, that is, generally

$$\mathbf{AB} \neq \mathbf{BA},$$

that is, matrix multiplication is not commutative, in marked contrast to (rational, real, or complex) numbers whose product is independent of the order of the factors. An example of two matrices not commuting with each other is:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 3 \end{bmatrix},$$

whereas

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 0 & 0 \end{bmatrix}.$$

https://en.wikipedia.org/wiki/Matrix_(mathematics)

Matrix Types

Name	Example with n = 3
Diagonal matrix	$\begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$
Lower triangular matrix	$\begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$
Upper triangular matrix	$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$

https://en.wikipedia.org/wiki/Matrix_(mathematics)

Identity Matrix

Identity matrix [edit]

Main article: Identity matrix

The *identity matrix* $\mathbf{I}_n$ of size n is the n-by-n matrix in which all the elements on the main diagonal are equal to 1 and all other elements are equal to 0, for example,

$$I_1 = [1],\ I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ \cdots,\ I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

It is a square matrix of order n, and also a special kind of diagonal matrix. It is called an identity matrix because multiplication with it leaves a matrix unchanged:

$\mathbf{AI}_n = \mathbf{I}_m\mathbf{A} = \mathbf{A}$ for any m-by-n matrix $\mathbf{A}$.

https://en.wikipedia.org/wiki/Matrix_(mathematics)

Inverse Matrix

In relation to its adjugate [edit]

The adjugate of a matrix A can be used to find the inverse of A as follows:

If A is an $n \times n$ invertible matrix, then

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A).$$

In relation to the identity matrix [edit]

It follows from the theory of matrices that if

$$\mathbf{AB} = \mathbf{I}$$

for *finite square* matrices **A** and **B**, then also

$$\mathbf{BA} = \mathbf{I}$$ [1]

https://en.wikipedia.org/wiki/Invertible_matrix

Inverse Matrix Examples

Consider the following 2-by-2 matrix:

$$\mathbf{A} = \begin{pmatrix} -1 & \frac{3}{2} \\ 1 & -1 \end{pmatrix}.$$

The matrix $\mathbf{A}$ is invertible. To check this, one can compute that $\det \mathbf{A} = -1/2$, which is non-zero.

As an example of a non-invertible, or singular, matrix, consider the matrix

$$\mathbf{B} = \begin{pmatrix} -1 & \frac{3}{2} \\ \frac{2}{3} & -1 \end{pmatrix}.$$

The determinant of $\mathbf{B}$ is 0, which is a necessary and sufficient condition for a matrix to be non-invertible.

https://en.wikipedia.org/wiki/Matrix_(mathematics)

Solving Linear Equations

A set of linear equations

If the inverse matrix exists

$$\begin{aligned} a\,x + b\,y &= e \\ c\,x + d\,y &= f \end{aligned} \quad \Leftrightarrow \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1}\begin{bmatrix} e \\ f \end{bmatrix}$$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \neq 0 \quad \Rightarrow \quad \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} e \\ f \end{bmatrix}$$

$$\begin{vmatrix} e & b \\ f & d \end{vmatrix} = de - bf$$

$$\begin{vmatrix} a & e \\ c & f \end{vmatrix} = af - ce$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{ad - bc}\begin{bmatrix} de-bf \\ -ce+af \end{bmatrix}$$

Cramer's Rule

$$x = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$$

Rule of Sarrus (1)

Determinant of order 3 only

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix}$$

Copy and concatenate

Rule of Sarrus

$$\begin{array}{ccc} + & + & + \end{array} \qquad\qquad\qquad\qquad \begin{array}{ccc} - & - & - \end{array}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix} \qquad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix}$$

$$+ a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \qquad - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

Young Won Lim
03/22/2018

Linear Equations

(Eq 1) $a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n = b_1$

(Eq 2) $a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n = b_2$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

(Eq 3) $a_{n1} x_1 + a_{n2} x_2 + \cdots + a_{nn} x_n = b_n$

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$$

Cramer's Rule (1) – solutions

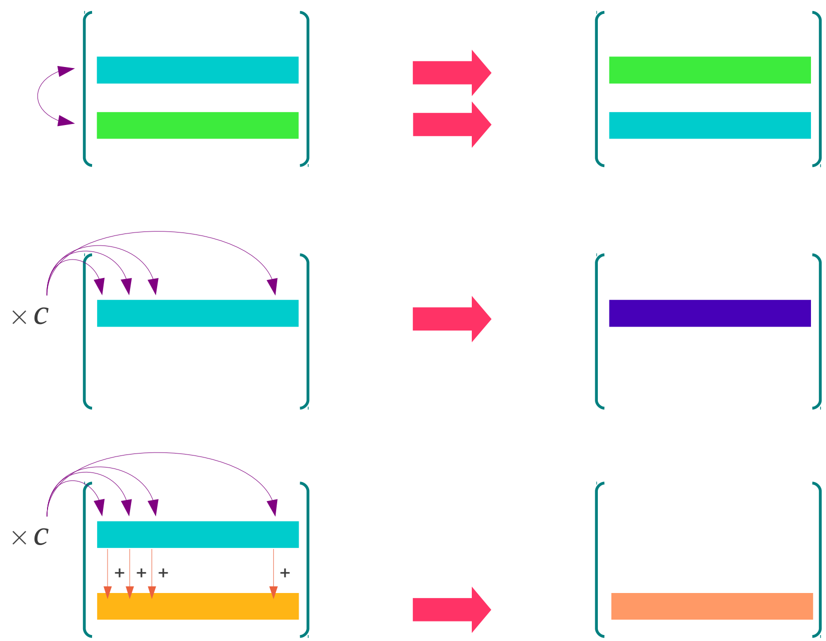
$$\overset{\boldsymbol{A}}{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}} \overset{\boldsymbol{x}}{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}} \begin{matrix} = \\ = \\ \\ = \end{matrix} \overset{\boldsymbol{b}}{\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}}$$

$$\overset{\boldsymbol{A}_1}{\begin{pmatrix} b_1 & a_{12} & \cdots & a_{1n} \\ b_2 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ b_n & a_{n2} & \cdots & a_{nn} \end{pmatrix}} \quad \overset{\boldsymbol{A}_2}{\begin{pmatrix} a_{11} & b_1 & \cdots & a_{1n} \\ a_{21} & b_2 & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & b_n & \cdots & a_{nn} \end{pmatrix}} \quad \overset{\boldsymbol{A}_n}{\begin{pmatrix} a_{11} & a_{12} & \cdots & b_1 \\ a_{21} & a_{22} & \cdots & b_2 \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & b_n \end{pmatrix}}$$

$$x_1 = \frac{det(\boldsymbol{A}_1)}{det(\boldsymbol{A})} \qquad x_2 = \frac{det(\boldsymbol{A}_2)}{det(\boldsymbol{A})} \qquad x_n = \frac{det(\boldsymbol{A}_n)}{det(\boldsymbol{A})}$$

Gauss-Jordan Elimination

$$\begin{pmatrix} +2 & +1 & -1 \\ -3 & -1 & +2 \\ -2 & +1 & +2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} +8 \\ -11 \\ -3 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} * \\ * \\ * \end{pmatrix}$$



https://en.wikiversity.org/wiki/Linear_Algebra_in_plain_view

Gauss-Jordan Elimination – Step 1

$$+2x_1 + x_2 - x_3 = 8 \qquad (L_1)$$
$$-3x_1 - x_2 + 2x_3 = -11 \qquad (L_2)$$
$$-2x_1 + x_2 + 2x_3 = -3 \qquad (L_3)$$

$$\left[\begin{array}{ccc|c} +2 & +1 & -1 & +8 \\ -3 & -1 & +2 & -11 \\ -2 & +1 & +2 & -3 \end{array}\right]$$

$$+1x_1 + \tfrac{1}{2}x_2 - \tfrac{1}{2}x_3 = 4 \qquad (\tfrac{1}{2} \times L_1)$$

+2/2 +1/2 –1/2 +8/2

$$+1x_1 + \tfrac{1}{2}x_2 - \tfrac{1}{2}x_3 = 4 \qquad (\tfrac{1}{2} \times L_1)$$
$$-3x_1 - x_2 + 2x_3 = -11 \qquad (L_2)$$
$$-2x_1 + x_2 + 2x_3 = -3 \qquad (L_3)$$

$$\left[\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ -3 & -1 & +2 & -11 \\ -2 & +1 & +2 & -3 \end{array}\right]$$

https://en.wikiversity.org/wiki/Linear_Algebra_in_plain_view

Gauss-Jordan Elimination – Step 2

$$+1x_1 + \tfrac{1}{2}x_2 - \tfrac{1}{2}x_3 = +4 \qquad (L_1)$$
$$-3x_1 - x_2 + 2x_3 = -11 \qquad (L_2)$$
$$-2x_1 + x_2 + 2x_3 = -3 \qquad (L_3)$$

$$\left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ \boxed{-3} & -1 & +2 & -11 \\ \boxed{-2} & +1 & +2 & -3 \end{array}\right)$$

$$+3x_1 + \tfrac{3}{2}x_2 - \tfrac{3}{2}x_3 = +12 \qquad (\boxed{3 \times L_1})$$
$$-3x_1 - x_2 + 2x_3 = -11 \qquad (L_2)$$

$$\begin{array}{cccc} +3 & +3/2 & -3/2 & +12 \\ -3 & -1 & +2 & -11 \end{array}$$

$$+2x_1 + \tfrac{2}{2}x_2 - \tfrac{2}{2}x_3 = +8 \qquad (\boxed{2 \times L_1})$$
$$-2x_1 + x_2 + 2x_3 = -3 \qquad (L_3)$$

$$\begin{array}{cccc} +2 & +2/2 & -2/2 & +8 \\ -2 & +1 & +2 & -3 \end{array}$$

$$+1x_1 + \tfrac{1}{2}x_2 - \tfrac{1}{2}x_3 = +4 \qquad (L_1)$$
$$0x_1 + \tfrac{1}{2}x_2 + \tfrac{1}{2}x_3 = +1 \qquad (\boxed{3 \times L_1} + L_2)$$
$$0x_1 + 2x_2 + 1x_3 = +5 \qquad (\boxed{2 \times L_1} + L_3)$$

$$\left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ \boxed{0} & +1/2 & +1/2 & +1 \\ \boxed{0} & +2 & +1 & +5 \end{array}\right)$$

https://en.wikiversity.org/wiki/Linear_Algebra_in_plain_view

Gauss-Jordan Elimination – Step 3

$$+1x_1 + \tfrac{1}{2}x_2 - \tfrac{1}{2}x_3 = +4 \qquad (L_1)$$
$$0x_1 + \tfrac{1}{2}x_2 + \tfrac{1}{2}x_3 = +1 \qquad (L_2)$$
$$0x_1 + 2x_2 + 1x_3 = +5 \qquad (L_3)$$

$$\left[\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1/2 & +1/2 & +1 \\ 0 & +2 & +1 & +5 \end{array}\right]$$

$$0x_1 + 1x_2 + 1x_3 = +2 \qquad (2 \times L_2)$$

$$\begin{array}{cccc} 0 & +1 & +1 & +2 \end{array}$$

$$+1x_1 + \tfrac{1}{2}x_2 - \tfrac{1}{2}x_3 = +4 \qquad (L_1)$$
$$0x_1 + 1x_2 + 1x_3 = +2 \qquad (2 \times L_2)$$
$$0x_1 + 2x_2 + 1x_3 = +5 \qquad (L_3)$$

$$\left[\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1 & +1 & +2 \\ 0 & +2 & +1 & +5 \end{array}\right]$$

https://en.wikiversity.org/wiki/Linear_Algebra_in_plain_view

Gauss-Jordan Elimination – Step 4

$$+1x_1 + \tfrac{1}{2}x_2 - \tfrac{1}{2}x_3 = +4 \qquad (L_1)$$
$$0x_1 + 1x_2 + 1x_3 = +2 \qquad (L_2)$$
$$0x_1 + 2x_2 + 1x_3 = +5 \qquad (L_3)$$

$$\left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1 & +1 & +2 \\ 0 & \boxed{+2} & +1 & +5 \end{array}\right)$$

$$0x_1 - 2x_2 - 2x_3 = -4 \qquad \boxed{(-2 \times L_2)}$$
$$0x_1 + 2x_2 + 1x_3 = +5 \qquad (L_3)$$

$$\begin{array}{cccc} 0 & -2 & -2 & -4 \\ 0 & +2 & +1 & +5 \end{array}$$

$$+1x_1 + \tfrac{1}{2}x_2 - \tfrac{1}{2}x_3 = +4 \qquad (L_1)$$
$$0x_1 + 1x_2 + 1x_3 = +2 \qquad (L_2)$$
$$0x_1 + 0x_2 - 1x_3 = +1 \qquad (\boxed{-2 \times L_2} + L_3)$$

$$\left(\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1 & +1 & +2 \\ 0 & \boxed{0} & -1 & +1 \end{array}\right)$$

https://en.wikiversity.org/wiki/Linear_Algebra_in_plain_view

Gauss-Jordan Elimination – Step 5

$$+1x_1 + \tfrac{1}{2}x_2 - \tfrac{1}{2}x_3 = +4 \qquad (L_1)$$
$$0x_1 + 1x_2 + 1x_3 = +2 \qquad (L_2)$$
$$0x_1 + 0x_2 - 1x_3 = +1 \qquad (L_3)$$

$$\left[\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1 & +1 & +2 \\ 0 & 0 & -1 & +1 \end{array}\right]$$

$$0x_1 - 0x_2 + 1x_3 = -1 \qquad (-1 \times L_3)$$

$$\begin{array}{cccc} 0 & 0 & +1 & -1 \end{array}$$

$$+1x_1 + \tfrac{1}{2}x_2 - \tfrac{1}{2}x_3 = +4 \qquad (L_1)$$
$$0x_1 + 1x_2 + 1x_3 = +2 \qquad (L_2)$$
$$0x_1 + 0x_2 + 1x_3 = -1 \qquad (-1 \times L_3)$$

$$\left[\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1 & +1 & +2 \\ 0 & 0 & +1 & -1 \end{array}\right]$$

https://en.wikiversity.org/wiki/Linear_Algebra_in_plain_view

Forward Phase

$$\left[\begin{array}{ccc|c} +2 & +1 & -1 & +8 \\ -3 & -1 & +2 & -11 \\ -2 & +1 & +2 & -3 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ -3 & -1 & +2 & -11 \\ -2 & +1 & +2 & -3 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ \boxed{0} & +1/2 & +1/2 & +1 \\ \boxed{0} & +2 & +1 & +5 \end{array}\right] \Rightarrow$$

$$\left[\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1 & +1 & +2 \\ 0 & +2 & +1 & +5 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1 & +1 & +2 \\ 0 & \boxed{0} & -1 & +1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1 & +1 & +2 \\ 0 & 0 & +1 & -1 \end{array}\right] \Rightarrow$$

Forward Phase - Gaussian Elimination

https://en.wikiversity.org/wiki/Linear_Algebra_in_plain_view

Gauss-Jordan Elimination – Step 6

$$+1x_1 + \tfrac{1}{2}x_2 - \tfrac{1}{2}x_3 = +4 \qquad (L_1)$$
$$0x_1 + 1x_2 + 1x_3 = +2 \qquad (L_2)$$
$$0x_1 + 0x_2 + 1x_3 = -1 \qquad (L_3)$$

$$\left(\begin{array}{ccc|c} +1 & +1/2 & \boxed{-1/2} & +4 \\ 0 & +1 & \boxed{+1} & +2 \\ 0 & 0 & +1 & -1 \end{array}\right)$$

$$0x_1 + 0x_2 + \tfrac{1}{2}x_3 = -\tfrac{1}{2} \qquad \boxed{\left(+\tfrac{1}{2} \times L_3\right)}$$
$$+1x_1 + \tfrac{1}{2}x_2 - \tfrac{1}{2}x_3 = +4 \qquad (L_1)$$

0	0	+1/2	–1/2
+1	+1/2	–1/2	+4

$$0x_1 + 0x_2 - 1x_3 = +1 \qquad \boxed{\left(-1 \times L_3\right)}$$
$$0x_1 + 1x_2 + 1x_3 = +2 \qquad (L_2)$$

0	0	–1	+1
0	+1	+1	+2

$$+1x_1 + \tfrac{1}{2}x_2 + 0x_3 = +\tfrac{7}{2} \qquad \left(+\tfrac{1}{2} \times L_3 + L_1\right)$$
$$0x_1 + 1x_2 + 0x_3 = +3 \qquad \left(-1 \times L_3 + L_2\right)$$
$$0x_1 + 0x_2 + 1x_3 = -1 \qquad (L_3)$$

$$\left(\begin{array}{ccc|c} +1 & +1/2 & \boxed{0} & +7/2 \\ 0 & +1 & \boxed{0} & +3 \\ 0 & 0 & +1 & -1 \end{array}\right)$$

https://en.wikiversity.org/wiki/Linear_Algebra_in_plain_view

Gauss-Jordan Elimination – Step 7

$$+1\,x_1 + \tfrac{1}{2}x_2 + 0\,x_3 = +\tfrac{7}{2} \qquad (L_1)$$
$$0\,x_1 + 1\,x_2 + 0\,x_3 = +3 \qquad (L_2)$$
$$0\,x_1 + 0\,x_2 + 1\,x_3 = -1 \qquad (L_3)$$

$$\left(\begin{array}{ccc|c} +1 & \boxed{+1/2} & 0 & +7/2 \\ 0 & +1 & 0 & +3 \\ 0 & 0 & +1 & -1 \end{array}\right)$$

$$0\,x_1 - \tfrac{1}{2}x_2 + 0\,x_3 = -\tfrac{3}{2} \qquad \boxed{\left(-\tfrac{1}{2} \times L_2\right)}$$
$$+1\,x_1 + 0\,x_2 - 0\,x_3 = +2 \qquad (L_1)$$

$$\begin{array}{cccc} 0 & -1/2 & 0 & -3/2 \\ +1 & +1/2 & 0 & +7/2 \end{array}$$

$$+1\,x_1 + 0\,x_2 - 0\,x_3 = +2 \qquad \left(-\tfrac{1}{2} \times L_2 + L_1\right)$$
$$0\,x_1 + 1\,x_2 + 0\,x_3 = +3 \qquad (L_2)$$
$$0\,x_1 + 0\,x_2 + 1\,x_3 = -1 \qquad (L_3)$$

$$\left(\begin{array}{ccc|c} +1 & \boxed{0} & 0 & +2 \\ 0 & +1 & 0 & +3 \\ 0 & 0 & +1 & -1 \end{array}\right)$$

https://en.wikiversity.org/wiki/Linear_Algebra_in_plain_view

Backward Phase

$$\left(\begin{array}{ccc|c} +1 & +1/2 & \boxed{-1/2} & +4 \\ 0 & +1 & \boxed{+1} & +2 \\ 0 & 0 & +1 & -1 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} +1 & +1/2 & \boxed{0} & +7/2 \\ 0 & +1 & \boxed{0} & +3 \\ 0 & 0 & +1 & -1 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} +1 & \boxed{0} & 0 & +2 \\ 0 & +1 & 0 & +3 \\ 0 & 0 & +1 & -1 \end{array}\right) \Rightarrow$$

https://en.wikiversity.org/wiki/Linear_Algebra_in_plain_view

Gauss-Jordan Elimination

Forward Phase – Gaussian Elimination

$$\left[\begin{array}{ccc|c} +2 & +1 & -1 & +8 \\ -3 & -1 & +2 & -11 \\ -2 & +1 & +2 & -3 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ -3 & -1 & +2 & -11 \\ -2 & +1 & +2 & -3 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ \boxed{0} & +1/2 & +1/2 & +1 \\ \boxed{0} & +2 & +1 & +5 \end{array}\right] \Rightarrow$$

$$\left[\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1 & +1 & +2 \\ 0 & +2 & +1 & +5 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1 & +1 & +2 \\ 0 & \boxed{0} & -1 & +1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} +1 & +1/2 & -1/2 & +4 \\ 0 & +1 & +1 & +2 \\ 0 & 0 & +1 & -1 \end{array}\right] \Rightarrow$$

Backward Phase – Guass-Jordan Elimination

$$\left[\begin{array}{ccc|c} +1 & +1/2 & \boxed{-1/2} & +4 \\ 0 & +1 & \boxed{+1} & +2 \\ 0 & 0 & +1 & -1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} +1 & +1/2 & \boxed{0} & +7/2 \\ 0 & +1 & \boxed{0} & +3 \\ 0 & 0 & +1 & -1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} +1 & \boxed{0} & 0 & +2 \\ 0 & +1 & 0 & +3 \\ 0 & 0 & +1 & -1 \end{array}\right] \Rightarrow$$

https://en.wikiversity.org/wiki/Linear_Algebra_in_plain_view

References

[1] http://en.wikipedia.org/
[2]