# Power Density Spectrum - Continuous Time

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Based on
Probability, Random Variables and Random Signal Principles,
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# Outline

## Energy, Average Power – deterministic, time domain

a deterministic signal $x(t)$

$$x_T(t) = \begin{cases} x(t) & -T < t < T \\ 0 & otherwise \end{cases}$$

the **energy**

$$E(T) = \int_{-T}^{+T} x^2(t)dt = \int_{-\infty}^{+\infty} x_T^2(t)dt$$

the **average power**

$$P(T) = \frac{1}{2T}\int_{-T}^{+T} x^2(t)dt = \frac{1}{2T}\int_{-\infty}^{+\infty} x_T^2(t)dt$$

## Fourier Transform Pair $x(t) \Longleftrightarrow X(\omega)$

Fourier transform

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

a deterministic signal $x(t)$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} x(\omega) e^{j\omega t} d\omega$$

# Fourier transform of $x_T(t)$
for continuous time signals

## bounded duration, bounded variation

for a finite $T$, $x_T(t)$ is assumed to have bounded variation

$$\int_{-T}^{+T} |x(t)| dt < \infty$$

the Fourier transform of $x_T(t)$

$$X_T(\omega) = \int_{-\infty}^{+\infty} x_T(t) e^{-j\omega t} dt$$

$$= \int_{-T}^{+T} x(t) e^{-j\omega t} dt$$

**deterministic $X_T(\omega)$ v.s. random $X_T(\omega)$**

a **deterministic** sample signal $x_T(t)$

$$x_T(t) \Longleftrightarrow X_T(\omega)$$

a **random process** signal $X_T(t)$

$$X_T(t) \Longleftrightarrow X_T(\omega)$$

# Parseval's theorem (I)

## power density spectrum for continuous time signals

### for a deterministic $x_T(t)$

a **deterministic** sample signal $x_T(t)$

$$\int_{-\infty}^{+\infty} x_T(\tau) x_T^*(\tau) d\tau = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X_T(\omega) X_T^*(\omega) d\omega$$

$$\int_{-\infty}^{+\infty} |x_T(\tau)|^2 d\tau = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |X_T(\omega)|^2 d\omega$$

# Parseval's theorem (II)

## power density spectrum for continuous time signals

### for a deterministic $x_T(t)$ v.s. a random $X_T(t)$

- a **deterministic** signal $x_T(t) \Longleftrightarrow X_T(\omega)$

$$\int_{-\infty}^{+\infty} |x_T(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |X_T(\omega)|^2 d\omega$$

- a **random** signal $X_T(t) \Longleftrightarrow X_T(\omega)$

$$\int_{-\infty}^{+\infty} E\left[|X_T(t)|^2\right] dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} E\left[|X_T(\omega)|^2\right] d\omega$$

## Energy, Average Power – Parseval's theorem applied

a deterministic signal $x_T(t)$

$$x_T(t) = \begin{cases} x(t) & -T < t < T \\ 0 & otherwise \end{cases} \qquad x_T(t) \Longleftrightarrow X_T(\omega)$$

the **energy** by Parseval's theorem

$$E(T) = \int_{-T}^{+T} x^2(t)dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |X_T(\omega)|^2 d\omega$$

the **average power** by Parseval's theorem

$$P(T) = \frac{1}{2T} \int_{-T}^{+T} x^2(t)dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{|X_T(\omega)|^2}{2T} d\omega$$

## deterministic $x_T(t) \Longleftrightarrow X_T(\omega)$

the **energy** for the **deterministic** $X_T(\omega)$ in $x_T(t) \Longleftrightarrow X_T(\omega)$

$$E(T) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |X_T(\omega)|^2 d\omega$$

the **average power** for the **deterministic** $X_T(\omega)$

$$P(T) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{|X_T(\omega)|^2}{2T} d\omega$$

the **power density spectrum** for the **deterministic** $X_T(\omega)$

$$\lim_{T \to \infty} \frac{|X_T(\omega)|^2}{2T}$$

**random $X_T(t) \Longleftrightarrow X_T(\omega)$**

the **energy** for the **random** $X_T(\omega)$ in $X_T(t) \Longleftrightarrow X_T(\omega)$

$$E(T) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} E\left[|X_T(\omega)|^2\right] d\omega$$

the **average power** for the **random** $X_T(\omega)$

$$P(T) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{E\left[|X_T(\omega)|^2\right]}{2T} d\omega$$

the **power density spectrum** for the **random** $X_T(\omega)$

$$\lim_{T\to\infty} \frac{E\left[|X_T(\omega)|^2\right]}{2T}$$

# Average power $P(T)$ – bounded duraton $(-T, +T)$

## power density spectrum for continuous time signals

$$P(T) = \frac{1}{2T} \int_{-T}^{+T} x^2(t) dt$$

- not the average power in a **random process**
  only the power in one **sample function**
  - to obtain the **average power** over all possible **realizations**,
    replace $x(t)$ by $X(t)$
    take the **expected value** of $x^2(t)$, that is $TE\left[X^2(t)\right]$
  - then, the **average power** is a **random variable**
    with respect to the **random process** $X(t)$
- not the average power in an entire **sample function**
  - take $T \to \infty$ to include all power in the **ensemble** member

$$P(T) = \frac{1}{2T} \int_{-T}^{+T} x^2(t) dt$$

- replace $x(t)$ by the **random variable** $X(t)$
- take the **expected value** of $x^2(t)$, that is $E\left[X^2(t)\right]$

$$P(T) = \frac{1}{2T} \int_{-T}^{+T} E\left[X^2(t)\right] dt$$

- take $T \to \infty$ to include all power

$$\boxed{P_{XX} = \lim_{T\to\infty} P(T)} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{+T} E\left[X^2(t)\right] dt$$

**The time average**

$$A_T[\bullet] = \frac{1}{2T}\int_{-T}^{T}[\bullet]dt \qquad A[\bullet] = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}[\bullet]dt$$

**time average and sample average operations**

$$\boxed{P_{XX} = \lim_{T\to\infty} P(T)} = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{+T} E\left[X^2(t)\right]dt$$

$$= \lim_{T\to\infty} A_T\left[E\left[X^2(t)\right]\right]$$

$$= A\left[E\left[X^2(t)\right]\right]$$

## for deterministic and random signals

the **average power** $P(T)$ for a **deterministic** signal $x(t)$

$$P(T) = \frac{1}{2T} \int_{-T}^{+T} x^2(t) dt$$

the **average power** $P_{XX}$ for a **random process** $X(t)$

$$\begin{aligned} P_{XX} &= \lim_{T \to \infty} P(T) \\ &= \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{+T} E\left[X^2(t)\right] dt \\ &= A\left[E\left[X^2(t)\right]\right] \end{aligned}$$

## the average power via power density

the average power $P_{XX}$ for the **random process** $X_T(\omega)$

$$P_{XX} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \boxed{\lim_{T\to\infty} \frac{E\left[|X_T(\omega)|^2\right]}{2T}} d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \boxed{S_{XX}(\omega)} d\omega$$

the **power density spectrum** $S_{XX}(\omega)$

$$\boxed{S_{XX}(\omega)} = \boxed{\lim_{T\to\infty} \frac{E\left[|X_T(\omega)|^2\right]}{2T}}$$

- $S_{XX}(\omega) \geq 0$
- $S_{XX}(-\omega) = S_{XX}(\omega)$ $\quad\quad X(t)$ real
- $S_{XX}(\omega)$ real
- $\frac{1}{2\pi}\int_{-\infty}^{+\infty} S_{XX}(\omega) d\omega = A\left[E\left[X^2(t)\right]\right]$
- $S_{\dot{X}\dot{X}}(\omega) = \omega^2 S_{XX}(\omega)$
- $\frac{1}{2\pi}\int_{-\infty}^{+\infty} S_{XX}(\omega) e^{j\omega t} d\omega = A[R_{XX}(t, t+\tau)]$
- $S_{XX}(\omega) = \int_{-\infty}^{+\infty} A[R_{XX}(t, t+\tau)]\, e^{-j\omega\tau} d\tau$

# Equations involving $S_{XX}(\omega)$

## power density spectrum for continuous time signals

**the average power $P_{xx}$ and the inverse Fourier transform of $S_{XX}(\omega)$**

the **average power** related equation

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} S_{XX}(\omega)d\omega = A\left[E\left[X^2(t)\right]\right]$$

the **autocorrelation** related equation

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} S_{XX}(\omega)e^{j\omega t}d\omega = A[R_{XX}(t, t+\tau)]$$

## the average power $P_{xx}$

the average power related equation

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} S_{XX}(\omega)d\omega = A\left[E\left[X^2(t)\right]\right]$$

- a **random process** $X(t)$ in time domain
- a **random process** $X(\omega)$ in frequency domain

$$X(t) = \lim_{T\to\infty} X_T(t) \qquad X(\omega) = \lim_{T\to\infty} X_T(\omega)$$

- Parseval's theorem over $X_T(t) \Longleftrightarrow X_T(\omega)$

## Average power $P_{XX}$ using $X_T(t)$ and $X_T(\omega)$

- Using a **random process** $X_T(t)$ in time domain

$$P_{XX} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{+T} E\left[X^2(t)\right] dt$$
$$= \lim_{T\to\infty} \frac{1}{2T} \int_{-\infty}^{+\infty} E\left[X_T^2(t)\right] dt$$
$$= \lim_{T\to\infty} A_T\left[E\left[X^2(t)\right]\right] = \boxed{A\left[E\left[X^2(t)\right]\right]}$$

- Using a **random process** $X_T(\omega)$ in frequency domain

$$P_{XX} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \boxed{\lim_{T\to\infty} \frac{E\left[|X_T(\omega)|^2\right]}{2T}} d\omega$$
$$= \boxed{\frac{1}{2\pi} \int_{-\infty}^{+\infty} \boxed{S_{XX}(\omega)} d\omega}$$

## the Inverse Fourier transform of $S_{XX}(\omega)$

the average power related equation

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} S_{XX}(\omega)e^{j\omega t}d\omega = A[R_{XX}(t, t+\tau)]$$

- auto-correlation function

$$R_{XX}(t, t+\tau) = E[X(t)X(t+\tau)] \qquad \Rightarrow R_{XX}(\tau)$$

- a **random process** $X(t)$ in time domain
- a **random process** $X(\omega)$ in frequency domain

## Fourier transform pairs

- $A[R_{XX}(t, t+\tau)] \Longleftrightarrow S_{XX}(\omega)$

$$S_{XX}(\omega) = \int_{-\infty}^{+\infty} A[R_{XX}(t, t+\tau)]\, e^{-j\omega\tau} d\tau$$

$$A[R_{XX}(t, t+\tau)] = \frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{XX}(\omega) e^{j\omega t} d\omega$$

- $R_{XX}(\tau) \Longleftrightarrow S_{XX}(\omega)$

$$S_{XX}(\omega) = \int_{-\infty}^{+\infty} R_{XX}(\tau) e^{-j\omega\tau} d\tau$$

$$R_{XX}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{XX}(\omega) e^{+j\omega\tau} d\omega$$

for a WSS $X(t)$, $A[R_{XX}(t, t+\tau)] = R_{XX}(\tau)$

### $S_{XX}(\omega)$ and $R_{XX}(\tau)$

the power spectrum

$$S_{XX}(\omega) = \int_{-\infty}^{+\infty} R_{XX}(\tau) e^{-j\omega\tau} d\tau$$

the auto-correlation function

$$R_{XX}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{XX}(\omega) e^{+j\omega\tau} d\omega$$

### Fourier transform of $\frac{d^n}{dt^n}x(t)$

$$x(t) \Longleftrightarrow X(\omega)$$

$$\frac{d^n}{dt^n}x(t) \Longleftrightarrow (j\omega)^n X(\omega)$$

# Power Spectrum and Auto-Correlation of a Derivative Function

power density spectrum for continuous time signals

## $S_{\dot{X}\dot{X}}(\omega)$ and $R_{\dot{X}\dot{X}}(\tau)$

the power spectrum

$$S_{\dot{X}\dot{X}}(\omega) = \int_{-\infty}^{+\infty} R_{\dot{X}\dot{X}}(\tau) e^{-j\omega\tau} d\tau$$

$$\begin{aligned} S_{\dot{X}\dot{X}}(\omega) &= \omega^2 S_{XX}(\omega) \\ &= \omega^2 \int_{-\infty}^{+\infty} R_{XX}(\tau) e^{-j\omega\tau} d\tau \end{aligned}$$

the auto-correlation function

$$R_{\dot{X}\dot{X}}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{\dot{X}\dot{X}}(\omega) e^{+j\omega\tau} d\omega$$

## Definition

Fourier transform of an autocorrelation functions

$$S_{XX}(\omega) = \int_{-\infty}^{+\infty} R_{XX}(\tau) e^{-j\omega\tau} d\tau$$

$$S_{\dot{X}\dot{X}}(\omega) = \int_{-\infty}^{+\infty} R_{\dot{X}\dot{X}}(\tau) e^{-j\omega\tau} d\tau$$

- auto-correlation function

$$R_{XX}(t, t+\tau) = E\left[X(t)X(t+\tau)\right] \Rightarrow R_{XX}(\tau)$$

$$R_{\dot{X}\dot{X}}(t, t+\tau) = E\left[\dot{X}(t)\dot{X}(t+\tau)\right] \Rightarrow R_{\dot{X}\dot{X}}(\tau)$$

- a **random process** $X(t)$ in time domain
- $\dot{X}(t) = \frac{d}{dt}X(t)$ : the derivative of $X(t)$

## Definition

the standard deviation is
a measure of the spread in a density function.
the analogous quantity for the normalized power spectrum is
a measure of its spread that we call the rms bandwidth
(root-mean-square)

$$W_{rms}^2 = \frac{\int_{-\infty}^{+\infty} \omega^2 S_{XX}(\omega) d\omega}{\int_{-\infty}^{+\infty} S_{XX}(\omega) d\omega}$$

## Definition

the mean frequence $\bar{\omega}_0$

$$\bar{\omega}_0 = \frac{\int_{-\infty}^{+\infty} \omega S_{XX}(\omega) d\omega}{\int_{-\infty}^{+\infty} S_{XX}(\omega) d\omega}$$

the rms bandwidth

$$W_{rms}^2 = \frac{4 \int_{-\infty}^{+\infty} (\omega - \bar{\omega}_0)^2 S_{XX}(\omega) d\omega}{\int_{-\infty}^{+\infty} S_{XX}(\omega) d\omega}$$