Ergodic Random Processes

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Based on
Probability, Random Variables and Random Signal Principles,
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Outline

Sample Average

The sample average

$$\hat{\hat{m}}_X(t) = \frac{1}{N}\sum_{i=1}^{N} X_i(t)$$

N independent sample realizations of the process
$X_i(t)$ for $i = 1, ..., N$
each realization is a time function
it is difficult to get many sample realizations

Time Average

The time average

$$A_T[\bullet] = \frac{1}{2T}\int_{-T}^{T}[\bullet]dt$$

average over time of a <u>single</u> <u>realization</u>

Time-Autocorrelation Function

The time average

$$\overline{x}_T = A_T[x(t)] = \frac{1}{2T}\int_{-T}^{T} x(t)dt$$

The time autocorrelation function

$$R_T(\tau) = A_T[x(t)x(t+\tau)] = \frac{1}{2T}\int_{-T}^{T} x(t)x(t+\tau)dt$$

Stationary Processes

first order stationary processes

$$m_X(t) = E[X(t)] = \overline{X} = constant$$

second order stationary processes

$$R_{XX}(t, t+\tau) = E[X(t)X(t+\tau)] = R_{XX}(\tau)$$

Stationary Processe Example

As a example of a stationary process
for which any single realisation has
an apparently noise-free structure,
let Y have a uniform distribution on $(0, 2\pi]$
and define the time series $X(t)$ by

$$X(t) = \cos(t + Y)$$

.
Then $X(t)$ is strictly stationary.

Expectation of Time-Autocorrelation Function

Convergence in square

the conditions under which random sequences of time average $A[\bullet]$ converge as $T \to \infty$

convergence in square

A random sequence X_n is said to **converge** to a random variable X in mean squre if

$$\lim_{n\to\infty} E\left[(X_n - X)^2\right] = 0$$

$$A[\bullet] = \lim_{n\to\infty} A_T[\bullet]$$

Example B.1: $X(t) = Y$

Let Y be any scalar **random variable**,
and define a time-series $\{X(t)\}$, by

$$X(t) = Y \qquad \text{for all } t.$$

Then $\{X(t)\}$ is a **stationary** time series

- **realisations** consist of a series of **constant** values,
- a different **constant** value for each **realisation**.

https://en.wikipedia.org/wiki/Stationary_process

Example B.1: $X(t) = Y$

A **law of large numbers** does not apply on this case,
as the limiting value of an average from a single realisation
takes the **random value** determined by Y,
rather than taking the **expected value** of Y.

The **time average** of $X(t)$ does not converge
since the process is not **ergodic**.

https://en.wikipedia.org/wiki/Stationary_process

Central Limit Theorem

Let $\{X_1, \ldots, X_n\}$ be a random sample of size n— that is, a sequence of independent and identically distributed (i.i.d.) random variables drawn from a distribution of expected value given by μ and finite variance given by σ^2. Suppose we are interested in the sample average

$$\bar{X}_n \equiv \frac{X_1 + \cdots + X_n}{n}$$

of these random variables. By the law of large numbers, the sample averages converge almost surely (and therefore also converge in probability) to the expected value μ as $n \to \infty$.

Central Limit Theorem

In probability theory, the central limit theorem (CLT) establishes that, in many situations, when independent random variables are summed up, their properly normalized sum tends toward a normal distribution (informally a bell curve) even if the original variables themselves are not normally distributed. The theorem is a key concept in probability theory because it implies that probabilistic and statistical methods that work for normal distributions can be applicable to many problems involving other types of distributions.

Conditions

1. $X(t)$ has a finite constant mean $\overline{X}$ for all t
2. $X(t)$ is bounded $x(t) < \infty$ for all t and all $x(t)$
3. Bounded time average of E[|X(t)|]

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} E[|X(t)|]dt < \infty$$

4. $X(t)$ is a regular process

$$E\left[|X(t)|^2\right] = R_{XX}(t,t) < \infty$$

Regular process

$X(t)$ is a regular process

$$E\left[|X(t)|^2\right] = R_{XX}(t,t) < \infty$$

for a real WSS process $X(t)$

$$E\left[|X(t)|^2\right] = R_{XX}(0) < \infty$$

since $\overline{X}$ is finit by assumption

$$C_{XX}(0) = R_{XX}(0) - \overline{X}^2 < \infty$$

Mean Erogodic

A wide snese stationary (WSS) process $X(t)$
with a constant **mean value** $\overline{X}$
is called **mean-ergodic**
if the **time average** $\overline{x}_T = A_T[x(t)]$ converges to $\overline{X}$ as $T \to \infty$

$$\lim_{T\to\infty} E\left[(\overline{x}_T - \overline{X})^2\right] = 0$$

$$\lim_{T\to\infty} \sigma^2_{\overline{x}_T} = 0$$

Covariance Functions

$$\begin{aligned} C_{XX}(t, t+\tau) &= E\left[\{X(t) - m_X(t)\}\{X(t+\tau) - m_X(t+\tau)\}\right] \\ &= R_{XX}(t, t+\tau) - m_X(t)m_X(t+\tau) \end{aligned}$$

for a WSS process $X(t)$

$$\begin{aligned} C_{XX}(\tau) &= E\left[\left\{X(t) - \overline{X}\right\}\left\{X(t+\tau) - \overline{X}\right\}\right] \\ &= R_{XX}(\tau) - \overline{X}^2 \end{aligned}$$

Variance of $\overline{x}_T$ (1)

N Gaussian random variables

$$\sigma^2_{\overline{x}_T} = E\left[\left\{\frac{1}{2T}\int_{-T}^{T}\left(X(t) - \overline{X}\right)dt\right\}^2\right]$$

$$= E\left[\left(\frac{1}{2T}\right)^2\left\{\int_{-T}^{T}\left(X(t) - \overline{X}\right)dt\right\}\left\{\int_{-T}^{T}\left(X(t_1) - \overline{X}\right)dt_1\right\}\right]$$

$$= E\left[\left(\frac{1}{2T}\right)^2\int_{-T}^{T}\left(X(t) - \overline{X}\right)\left(X(t_1) - \overline{X}\right)dt\,dt_1\right]$$

Variance of $\overline{x}_T$ (2)

$$\sigma^2_{\overline{x}_T} = E\left[(\overline{x}_T - \overline{X})^2\right]$$

$$= \left(\frac{1}{2T}\right)^2 \int_{-T}^{T} E\left[(X(t) - \overline{X})\,(X(t_1) - \overline{X})\right] dt dt_1$$

$$= \left(\frac{1}{2T}\right)^2 \int_{-T}^{T} C_{XX}(t, t_1) dt dt_1$$

for the WSS $X(t)$, let $C_{XX}(t, t_1) = C_{XX}(\tau)$, $\tau = t_1 - t$, and $d\tau = dt_1$

$$= \left(\frac{1}{2T}\right)^2 \int_{t=-T}^{T} \int_{\tau=-T-t}^{T-t} C_{XX}(\tau) d\tau dt$$

Variance of $\overline{x}_T$ (3)

$$\sigma^2_{\overline{x}_T} = \left(\frac{1}{2T}\right)^2 \int_{t=-T}^{T} \int_{\tau=-T-t}^{T-t} C_{XX}(\tau) d\tau dt$$

the Riemann strips in $\tau - t$ plane

1. using horizontal Rieman strips
2. using vertical Rieman strips
   using the symmetry $C_{XX}(-\tau) = C_{XX}(-\tau)$

$$\sigma^2_{\overline{x}_T} = \frac{1}{2T} \int_{-2T}^{2T} \left(1 - \frac{|\tau|}{2T}\right) C_{XX}(\tau) d\tau$$

Variance of $\overline{x}_T$ (4)

the necessary and sufficient condition
for a WSS process X(t) to be mean ergodic

$$\lim_{T\to\infty} \left\{ \frac{1}{2T} \int_{-2T}^{2T} \left(1 - \frac{|\tau|}{2T}\right) C_{XX}(\tau) d\tau \right\} = 0$$

$$\frac{1}{2T} \int_{-2T}^{2T} \left(1 - \frac{|\tau|}{2T}\right) C_{XX}(\tau) d\tau < \frac{1}{2T} \int_{-2T}^{2T} |C_{XX}(\tau)| \, d\tau$$

1. $C_{XX}(0) < \infty$ and $C_{XX}(\tau) \to 0$ as $|\tau| \to \infty$
2. $\int_{-\infty}^{\infty} |C_{XX}(\tau)| \, d\tau < \infty$

Mean Ergodic Condition

a necessary and sufficient condition
for a WSS process $X(t)$ to be **mean ergodic**

$$\lim_{T\to\infty}\left\{\frac{1}{2T}\int_{-2T}^{2T}\left(1-\frac{|\tau|}{2T}\right)C_{XX}(\tau)d\tau\right\}=0$$

Mean Ergodic Process - continuous time

$X(t)$ is a **mean ergodic** if

1. $C_{XX}(0) < \infty$ and $C_{XX}(\tau) \to 0$ as $|\tau| \to \infty$
2. $\int_{-\infty}^{\infty} |C_{XX}(\tau)| \, d\tau < \infty$

Mean Ergodic Process - discrete time

$X[n]$ is a mean ergodic if

$$\lim_{N\to\infty}\left\{\frac{1}{2N+1}\sum_{n=-N}^{+N}X[n]\right\}=\overline{X}$$

$$\lim_{T\to\infty}\left\{\frac{1}{2N+1}\sum_{n=-2N}^{+2N}\left(1-\frac{|n|}{2N+1}\right)C_{XX}[n]\right\}=0$$

Auto-correlation Ergodic Process - Continuous Time

A stationary continuous process $X(t)$
with autocorrelation function $R_{XX}(\tau)$
is called **autocorrelation ergodic**
if for all τ, $R_T(\tau) = A_T[x(t)x(t+\tau)]$
converges to $R_{XX}(\tau)$ as $T \to \infty$

Auto-correlation Ergodic Process - Discrete Time

A stationary sequence $X[n]$
with autocorrelation function $R_N[k]$
is called **autocorrelation ergodic**
if for all k, $R_N[k] = \frac{1}{2N+1} \sum_{n=-N}^{+N} x[n]x[n+k]$
converges to $R_{XX}[k]$ as $N \to \infty$

A necessary and sufficient condition

$$W(t) = X(t)X(t+\tau)$$
$$E[W(t)] = E[X(t)X(t+\tau)] = R_{XX}(\tau)$$

$$R_{WW}(\lambda) = E[W(t)W(t+\lambda)]$$
$$= E[X(t)X(t+\tau)X(t+\lambda)X(t+\tau+\lambda)]$$

$$C_{WW}(\lambda) = R_{WW}(\lambda) - \{E[W(t)]\}^2$$
$$= R_{WW}(\lambda) - R_{XX}^2(\tau)$$

Auto-correlation Ergodic Condition

a necessary and sufficient condition
for a WSS process $X(t)$ to be **auto-correlation ergodic**

$$\lim_{T\to\infty}\left\{\frac{1}{2T}\int_{-2T}^{2T}\left(1-\frac{|\tau|}{2T}\right)C_{WW}(\tau)d\tau\right\}=0$$

Auto-correlation Ergodic

- auto-correlation ergodicity requires that the 4-th order moments of X(t)
  $R_{WW}(\lambda) = E[X(t)X(t+\tau)X(t+\lambda)X(t+\tau+\lambda)]$
- for Gaussian processes, 4-th order moments are known via 2nd and 1st order moments