CORDIC Accuracy Scaling Free

20160218

Scaling Constant K

K After initial phase

Scaling Constant K

$$\theta_0 \rightarrow \cdots \rightarrow \theta_i \rightarrow \theta_{i+1} \rightarrow \cdots \rightarrow \theta_n \to 0$$

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \rightarrow \cdots \rightarrow \begin{pmatrix} x_i \\ y_i \end{pmatrix} \rightarrow \begin{pmatrix} x_{i+1} \\ y_{i+1} \end{pmatrix} \rightarrow \cdots \rightarrow \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

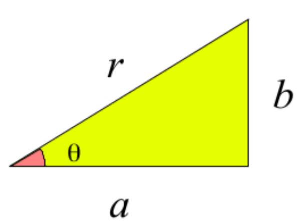
$$\begin{cases} x_{i+1} = x_i \cos\theta_i - y_i \sin\theta_i \\ y_{i+1} = x_i \sin\theta_i + y_i \cos\theta_i \end{cases}$$

$$\begin{cases} x_{i+1} = \cos\theta_i \,(x_i - y_i \tan\theta_i) \\ y_{i+1} = \cos\theta_i \,(x_i \tan\theta_i + y_i) \end{cases}$$

$$\begin{cases} x_{i+1} = \dfrac{1}{\sqrt{1 + \tan^2\theta_i}} (x_i - y_i \tan\theta_i) \\[2ex] y_{i+1} = \dfrac{1}{\sqrt{1 + \tan^2\theta_i}} (x_i \tan\theta_i + y_i) \end{cases}$$

Pseudo-rotation

$$\begin{cases} x'_{i+1} = (x_i - y_i \tan\theta_i) \\ y'_{i+1} = (x_i \tan\theta_i + y_i) \end{cases}$$



$$r = \sqrt{a^2 + b^2}$$

$$\cos\theta = \frac{a}{r} = \frac{a}{\sqrt{a^2 + b^2}}$$

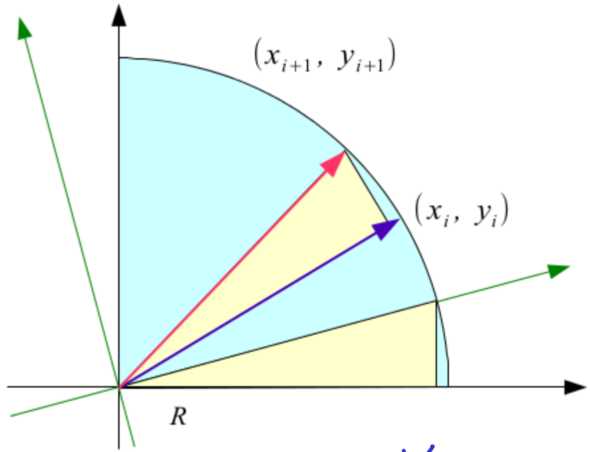
$$= \frac{1}{\sqrt{1 + (b/a)^2}}$$

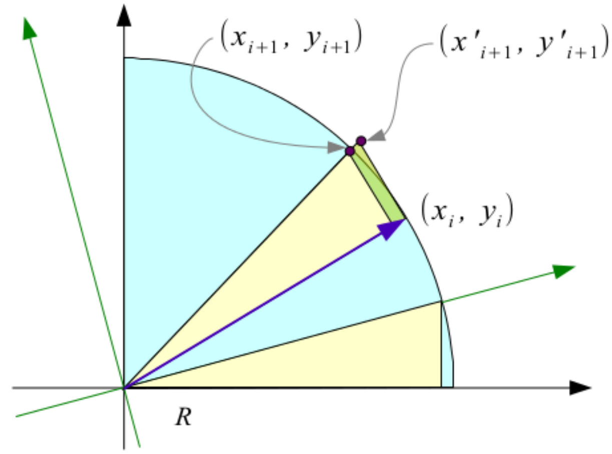
$$= \frac{1}{\sqrt{1 + \tan^2\theta}}$$

$$\cos\theta = \frac{a}{r}$$

$$\sin\theta = \frac{b}{r}$$

$$\tan\theta = \frac{b}{a}$$





$$\begin{cases} x_{i+1} = \cos\theta_i \left(x_i - y_i \tan\theta_i\right) \\ y_{i+1} = \cos\theta_i \left(x_i \tan\theta_i + y_i\right) \end{cases} \Rightarrow \begin{cases} x'_{i+1} = \left(x_i - y_i \tan\theta_i\right) \\ y'_{i+1} = \left(x_i \tan\theta_i + y_i\right) \end{cases}$$

$$\alpha_0 \rightarrow \cdots \rightarrow \alpha_i \rightarrow \alpha_{i+1} \rightarrow \cdots \rightarrow \alpha_n \to 0$$

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \rightarrow \cdots \rightarrow \begin{pmatrix} x_i \\ y_i \end{pmatrix} \rightarrow \begin{pmatrix} x_{i+1} \\ y_{i+1} \end{pmatrix} \rightarrow \cdots \rightarrow \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

$$\begin{aligned} x_{i+1} &= x_i \cos\theta_i - y_i \sin\theta_i &= \cos\theta_i (x_i - y_i \tan\theta_i) &= \left(1/\sqrt{1+\tan^2\theta_i}\right)(x_i - y_i \tan\theta_i) \\ y_{i+1} &= x_i \sin\theta_i + y_i \cos\theta_i &= \cos\theta_i (x_i \tan\theta_i + y_i) &= \left(1/\sqrt{1+\tan^2\theta_i}\right)(x_i \tan\theta_i + y_i) \end{aligned}$$

Pseudo-rotation

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \rightarrow \cdots \rightarrow \begin{pmatrix} x'_i \\ y'_i \end{pmatrix} \rightarrow \begin{pmatrix} x'_{i+1} \\ y'_{i+1} \end{pmatrix} \rightarrow \cdots \rightarrow \begin{pmatrix} x'_n \\ y'_n \end{pmatrix}$$

$$\begin{aligned} x'_{i+1} &= (x'_i - y'_i \tan\theta_i) &= (x'_i \cos\theta_i - y'_i \sin\theta_i)\sqrt{1+\tan^2\theta_i} \\ y'_{i+1} &= (x'_i \tan\theta_i + y'_i) &= (x'_i \sin\theta_i + y'_i \cos\theta_i)\sqrt{1+\tan^2\theta_i} \end{aligned}$$

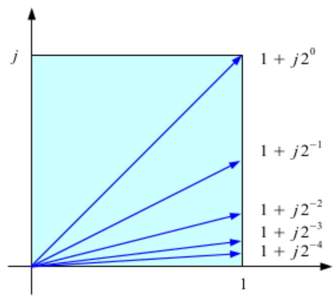
$$\begin{aligned} x'_n &= \{x_0 \cos(\Sigma\theta_i) - y_0 \sin(\Sigma\theta_i)\} \cdot \prod \left\{\sqrt{1+\tan^2\theta_i}\right\} \\ y'_n &= \{x_0 \sin(\Sigma\theta_i) + y_0 \cos(\Sigma\theta_i)\} \cdot \prod \left\{\sqrt{1+\tan^2\theta_i}\right\} \end{aligned}$$

$$\theta_n = \theta - \Sigma\theta_i$$

Choose θ_i such that $\tan\theta_i = \begin{cases} +2^{-i} \\ -2^{-i} \end{cases}$

$$\tan\theta_i = \sigma_i 2^{-i} \qquad \sigma_i \in \{+1, -1\}$$

$$\begin{aligned} x'_{i+1} &= (x'_i - y'_i \sigma_i 2^{-i}) \\ y'_{i+1} &= (x'_i \sigma_i 2^{-i} + y'_i) \\ \theta_{i+1} &= \theta_i - \tan^{-1}(\sigma_i 2^{-i}) \end{aligned}$$



Choose θ_i such that $\tan\theta_i = \begin{cases} +2^{-i} \\ -2^{-i} \end{cases}$

$$\tan\theta_i = \sigma_i 2^{-i} \qquad \sigma_i \in \{+1, -1\}$$

$$\begin{aligned} x'_{i+1} &= (x'_i - y'_i\,\sigma_i 2^{-i}) \\ y'_{i+1} &= (x'_i\,\sigma_i 2^{-i} + y'_i) \\ \theta_{i+1} &= \theta_i - \tan^{-1}(\sigma_i 2^{-i}) \end{aligned}$$

$$\begin{pmatrix} +\cos\theta & -\sin\theta \\ +\sin\theta & +\cos\theta \end{pmatrix}$$

$$= \boxed{\boxed{\frac{1}{\sqrt{1+2^{-2\cdot n}}}} \begin{pmatrix} +1 & \mp 2^{-n} \\ \pm 2^{-n} & +1 \end{pmatrix}} \cdots \boxed{\boxed{\frac{1}{\sqrt{1+2^{-2\cdot 1}}}} \begin{pmatrix} +1 & \mp 2^{-1} \\ \pm 2^{-1} & +1 \end{pmatrix}} \cdot \boxed{\boxed{\frac{1}{\sqrt{1+2^{0}}}} \begin{pmatrix} +1 & \mp 2^{0} \\ \pm 2^{0} & +1 \end{pmatrix}}$$

$$= \boxed{\boxed{\frac{1}{\sqrt{1+2^{-2\cdot n}}}} \cdots \boxed{\frac{1}{\sqrt{1+2^{-2\cdot 1}}}} \cdot \boxed{\frac{1}{\sqrt{1+2^{-2\cdot 0}}}}} \underline{\begin{pmatrix} +1 & \mp 2^{-n} \\ \pm 2^{-n} & +1 \end{pmatrix} \cdots \begin{pmatrix} +1 & \mp 2^{-1} \\ \pm 2^{-1} & +1 \end{pmatrix} \begin{pmatrix} +1 & \mp 2^{-0} \\ \pm 2^{-0} & +1 \end{pmatrix}}$$

$$\Rightarrow \boxed{K = \prod 1/\sqrt{1+\tan^2\theta_i} = 0.607} \qquad \Rightarrow \begin{pmatrix} +\cos(\Sigma\theta_i) & -\sin(\Sigma\theta_i) \\ +\sin(\Sigma\theta_i) & +\cos(\Sigma\theta_i) \end{pmatrix}$$

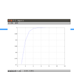
$$\begin{pmatrix} +\cos(\Sigma\theta_i) & -\sin(\Sigma\theta_i) \\ +\sin(\Sigma\theta_i) & +\cos(\Sigma\theta_i) \end{pmatrix} = \frac{1}{K} \cdot \begin{pmatrix} +\cos\theta & -\sin\theta \\ +\sin\theta & +\cos\theta \end{pmatrix}$$

$$\begin{aligned} 1/K &= \prod \sqrt{1+\tan^2\theta_i} = 1.647 \\ &= A = \mathit{CORDIC\ Gain} \end{aligned}$$

$$K_n = \frac{1}{\sqrt{1+2^{-2n}}}$$

$$K = \prod \frac{1}{\sqrt{1+2^{-2n}}}$$

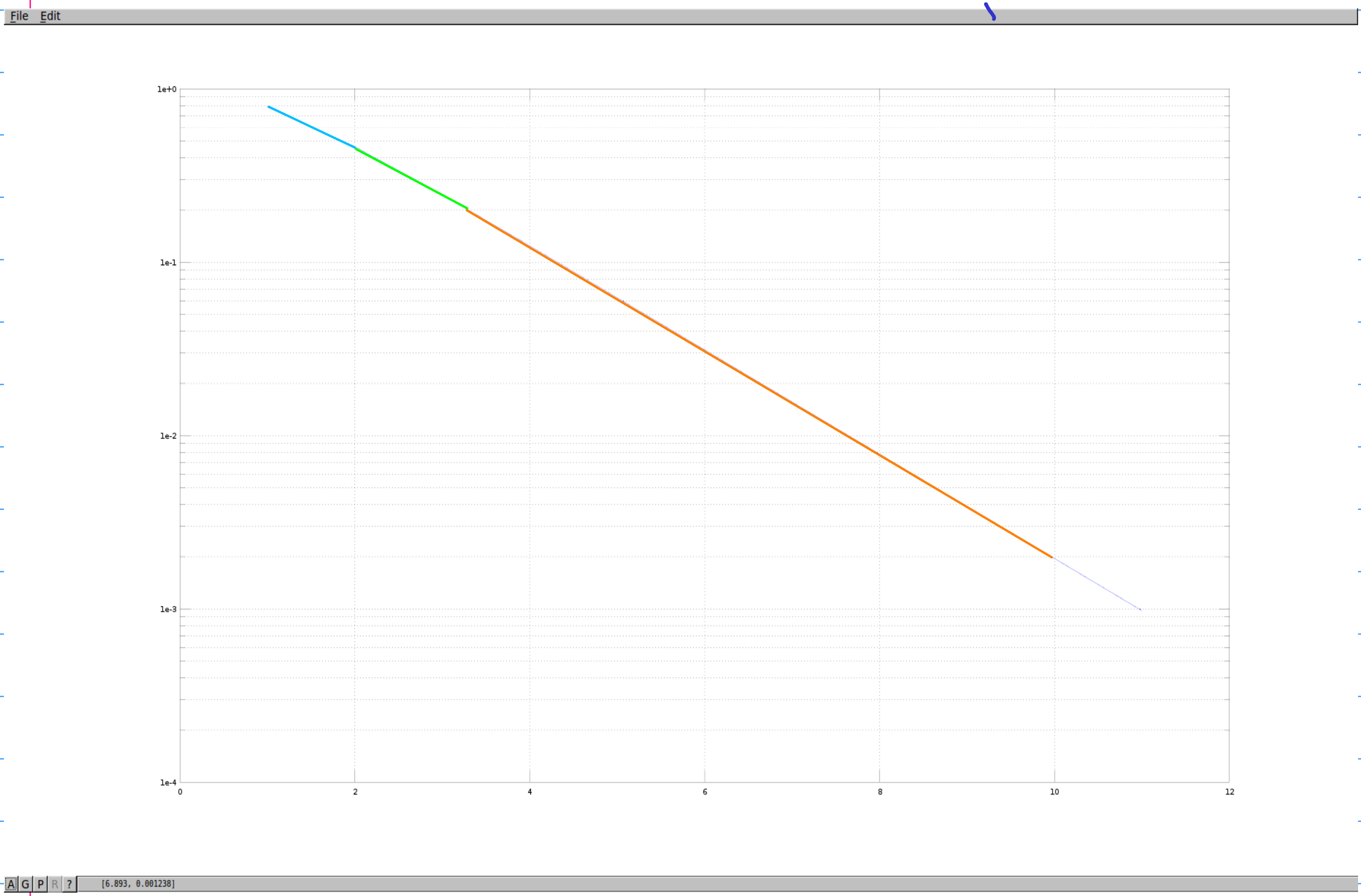
```
Kn = 1;
K = 1;
VKn = [];
VK = [];

for n = 0:10
  Kn = 1 / sqrt(1 + 2^(-2*n));
  K = K * Kn;

  VKn = [VKn, Kn];
  VK  = [VK,  K];
endfor

figure(1);
plot(VK);

figure(2)
plot(VKn);
```

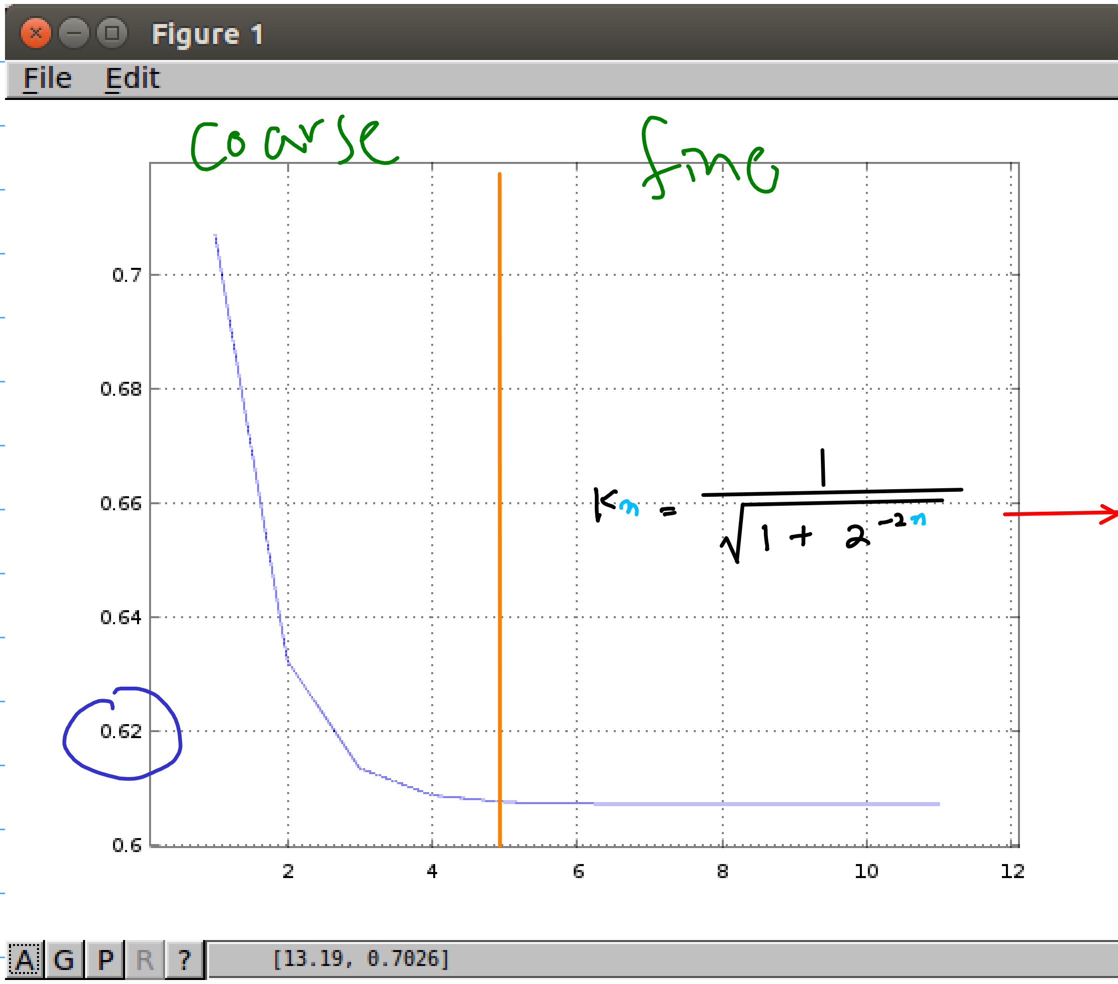


different slope

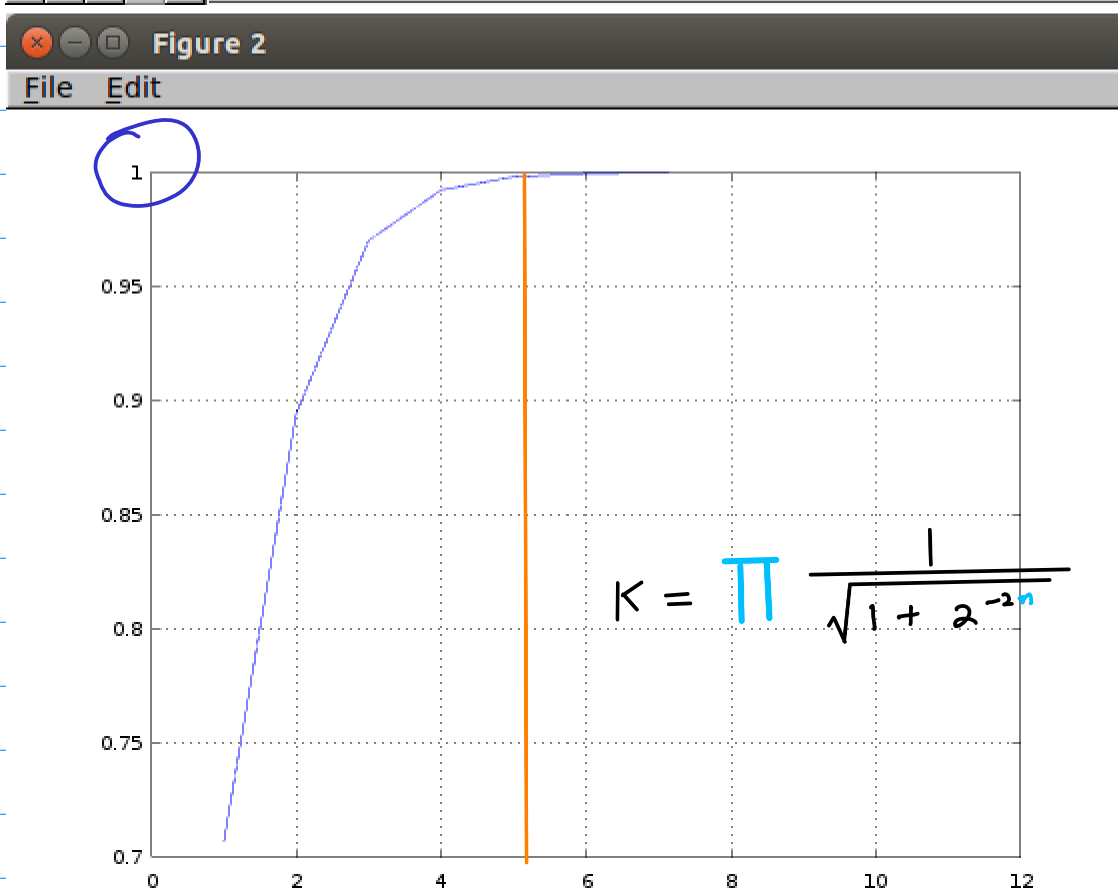
Write the reason of different slope

$$\log(\theta_i) = \log(\operatorname{atan}(\tfrac{1}{2^i}))$$



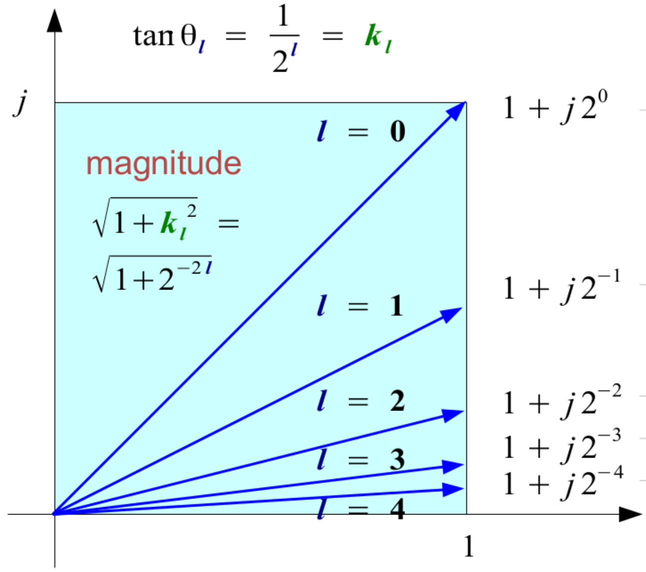


$$K_n = \frac{1}{\sqrt{1 + 2^{-2n}}} \rightarrow \boxed{0.607}$$



$$K = \prod \frac{1}{\sqrt{1 + 2^{-2n}}} \Rightarrow 1$$

45
26.5
14.0
7.1
3.6



$$\theta_l = \tan^{-1}(2^{-l}) = \tan^{-1}(k_l)$$

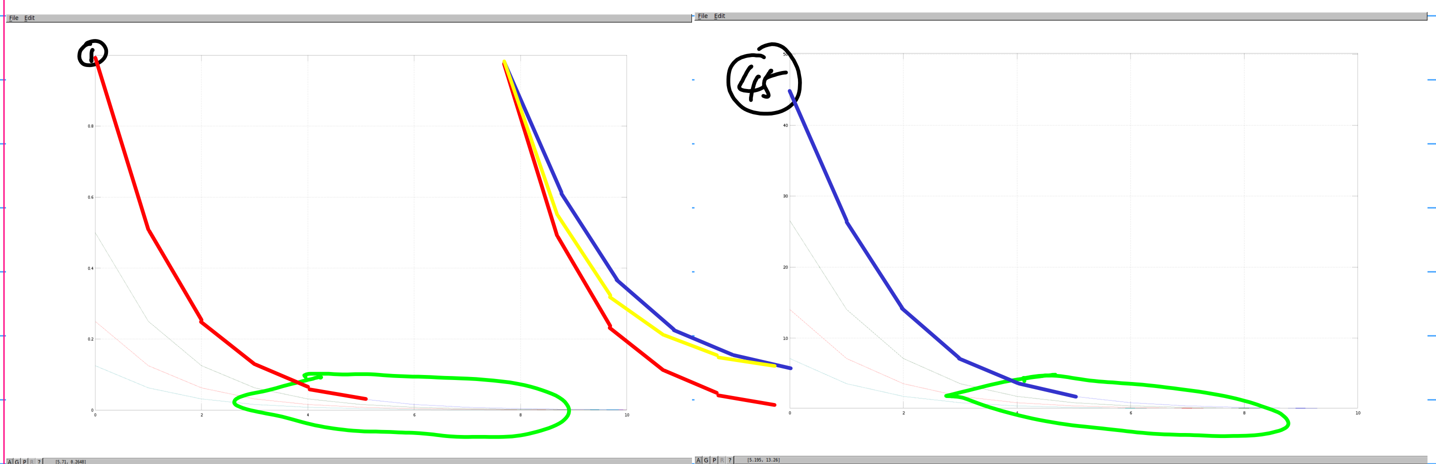
$$\theta_0 = \tan^{-1}(2^{0}) = 45.00000^\circ$$

$$\theta_1 = \tan^{-1}(2^{-1}) = 26.56505^\circ$$

$$\theta_2 = \tan^{-1}(2^{-2}) = 14.03624^\circ$$

$$\theta_3 = \tan^{-1}(2^{-3}) = 7.12502^\circ$$

$$\theta_4 = \tan^{-1}(2^{-4}) = 3.57633^\circ$$



$\tan \theta_i = 2^{-i}$

$\theta_i \approx \operatorname{atan}(2^{-i})$

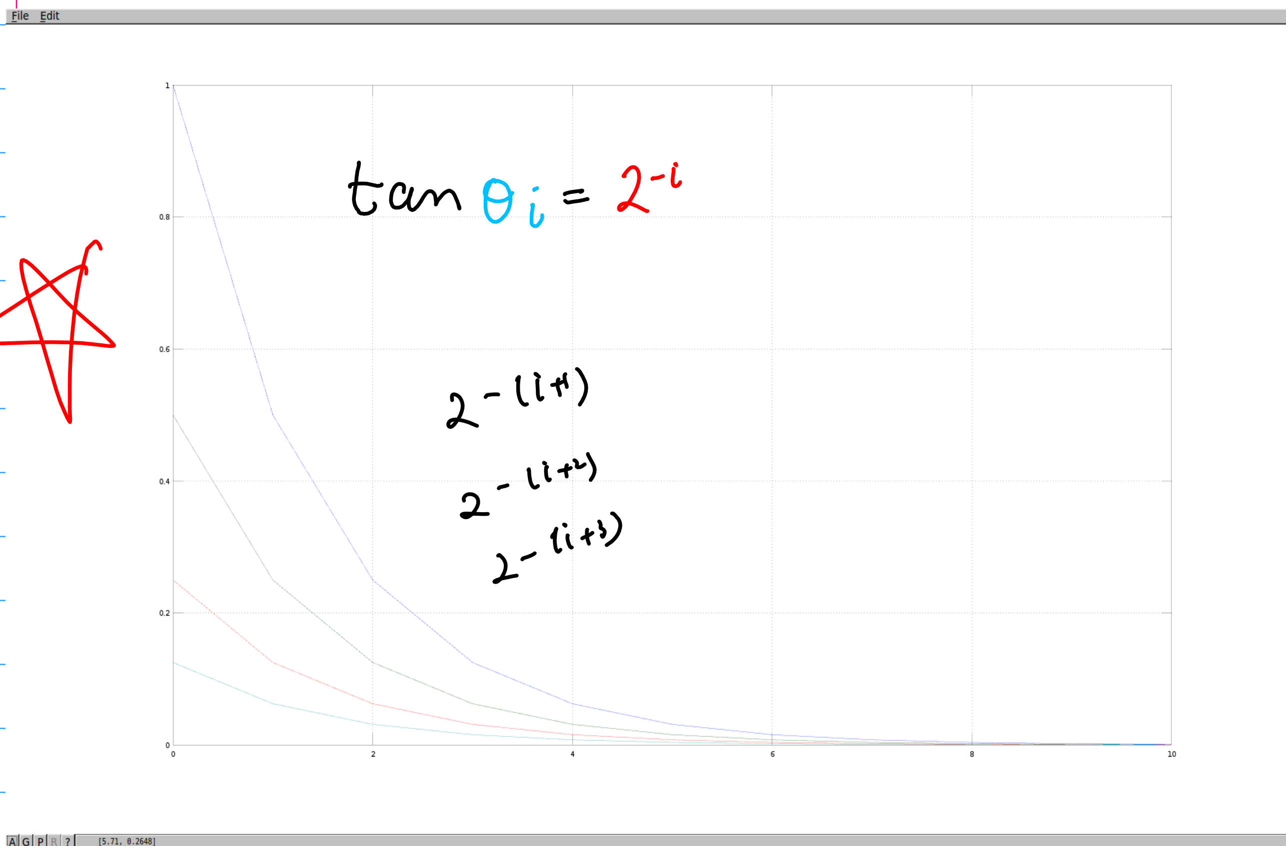
angle points

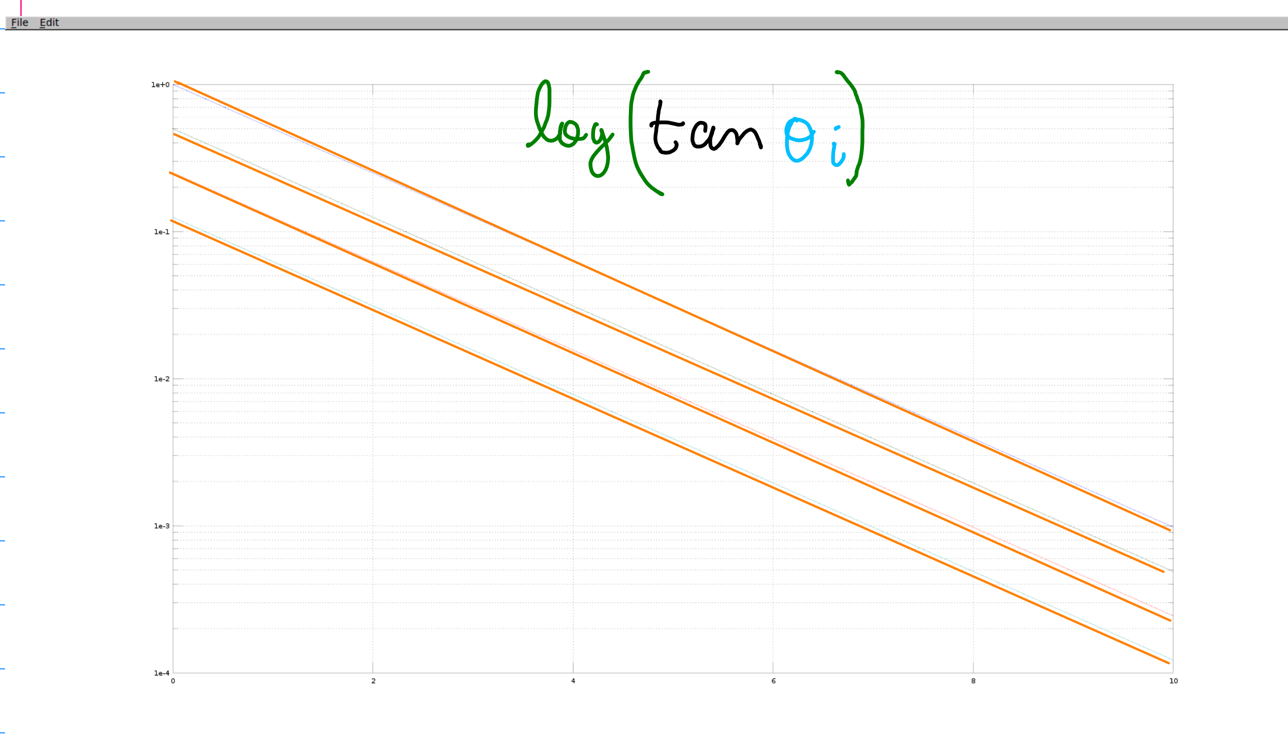
$$\tan \theta_i = \frac{1}{2^i} = 2^{-i}$$

$$\tan \theta_{i+1} = \frac{1}{2^{i+1}} = 2^{-(i+1)}$$

$$\tan \theta_{i+2} = \frac{1}{2^{i+2}} = 2^{-(i+1)}$$

angle point


File Edit
$\tan \theta_i = 2^{-i}$
$2^{-(i+1)}$
$2^{-(i+2)}$
$2^{-(i+3)}$
1
0.8
0.6
0.4
0.2
0
0
2
4
6
8
10
A G P R ?
[5.71, 0.2648]


File Edit
$\log(\tan \theta_i)$
1e+0
1e-1
1e-2
1e-3
1e-4
0
2
4
6
8
10
A G P R ?
[5.585, 0.001252]

```
(%i3) taylor(cos(x), x, 0, 8);
```

(%o3)/T/ $1-\frac{x^2}{2}+\frac{x^4}{24}-\frac{x^6}{720}+\frac{x^8}{40320}+...$

```
(%i4) taylor(sin(x), x, 0, 8);
```

(%o4)/T/ $x-\frac{x^3}{6}+\frac{x^5}{120}-\frac{x^7}{5040}+...$

```
(%i11) taylor(tan(x), x, 0, 8);
```

(%o11)/T/ $x+\frac{x^3}{3}+\frac{2x^5}{15}+\frac{17x^7}{315}+...$

```
(%i21) taylor(atan(x), x, 0, 16);
```

(%o21)/T/ $x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\frac{x^9}{9}-\frac{x^{11}}{11}+\frac{x^{13}}{13}-\frac{x^{15}}{15}+...$

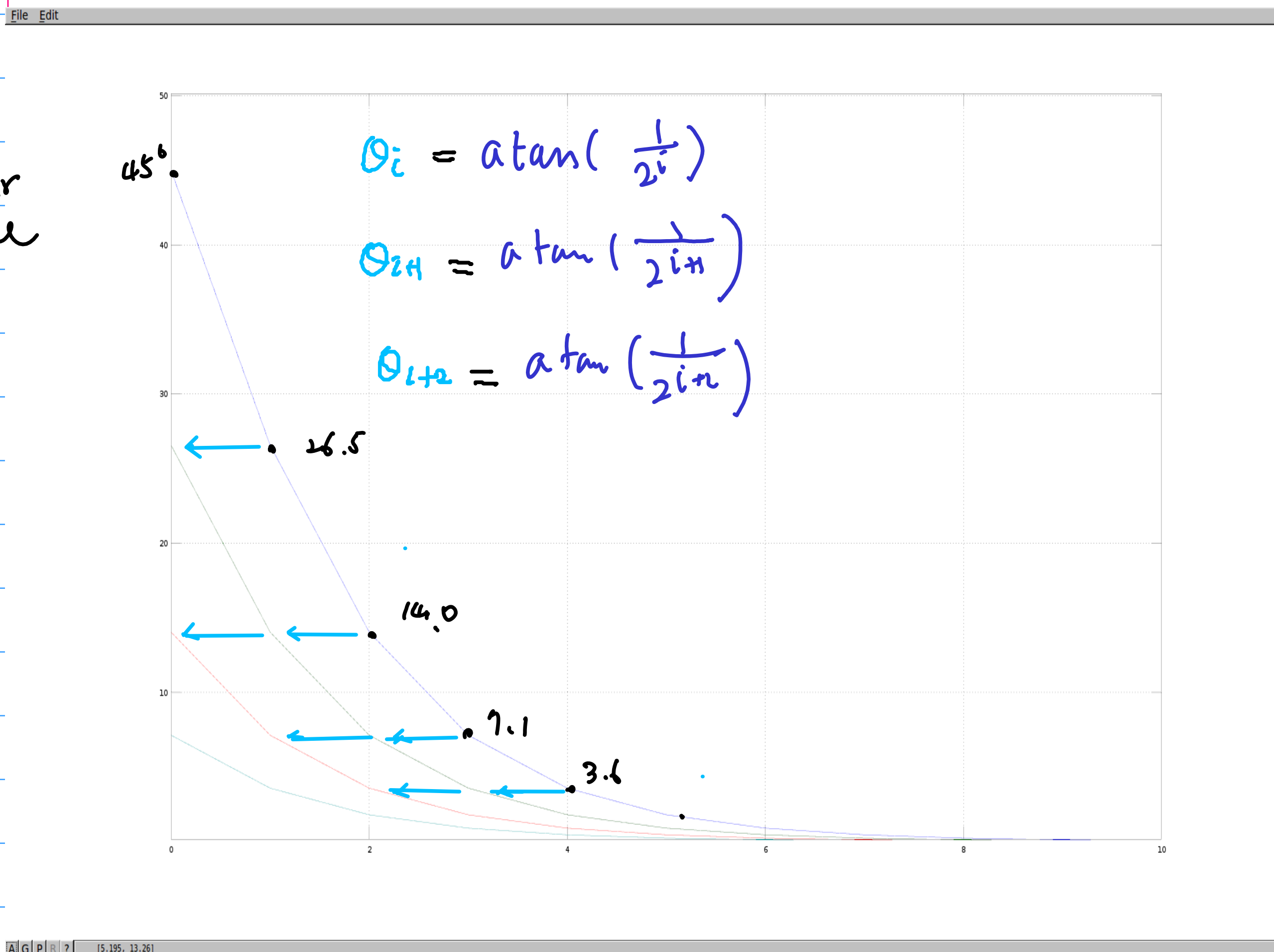
```
(%i23) f(x) := x-x^3/3+x^5/5-x^7/7+x^9/9-x^11/11+x^13/13-x^15/15;
```

(%o23) $f(x):=x-\frac{x^3}{3}+\frac{x^5}{5}+\frac{-x^7}{7}+\frac{x^9}{9}+\frac{-x^{11}}{11}+\frac{x^{13}}{13}+\frac{-x^{15}}{15}$

```
(%i25) float(f(1/2));
(%o25) 0.46364724210793
(%i26) float(f(1/2^2));
(%o26) 0.24497866312362
(%i27) float(f(1/2^3));
(%o27) 0.12435499454676
(%i28) float(f(1/2^4));
(%o28) 0.062418809995957
```

```
octave:5> atan(1 ./ 2.^n)'
ans =

   0.7853982
   0.4636476
   0.2449787
   0.1243550
   0.0624188
   0.0312398
   0.0156237
   0.0078123
   0.0039062
```

Linear Scale

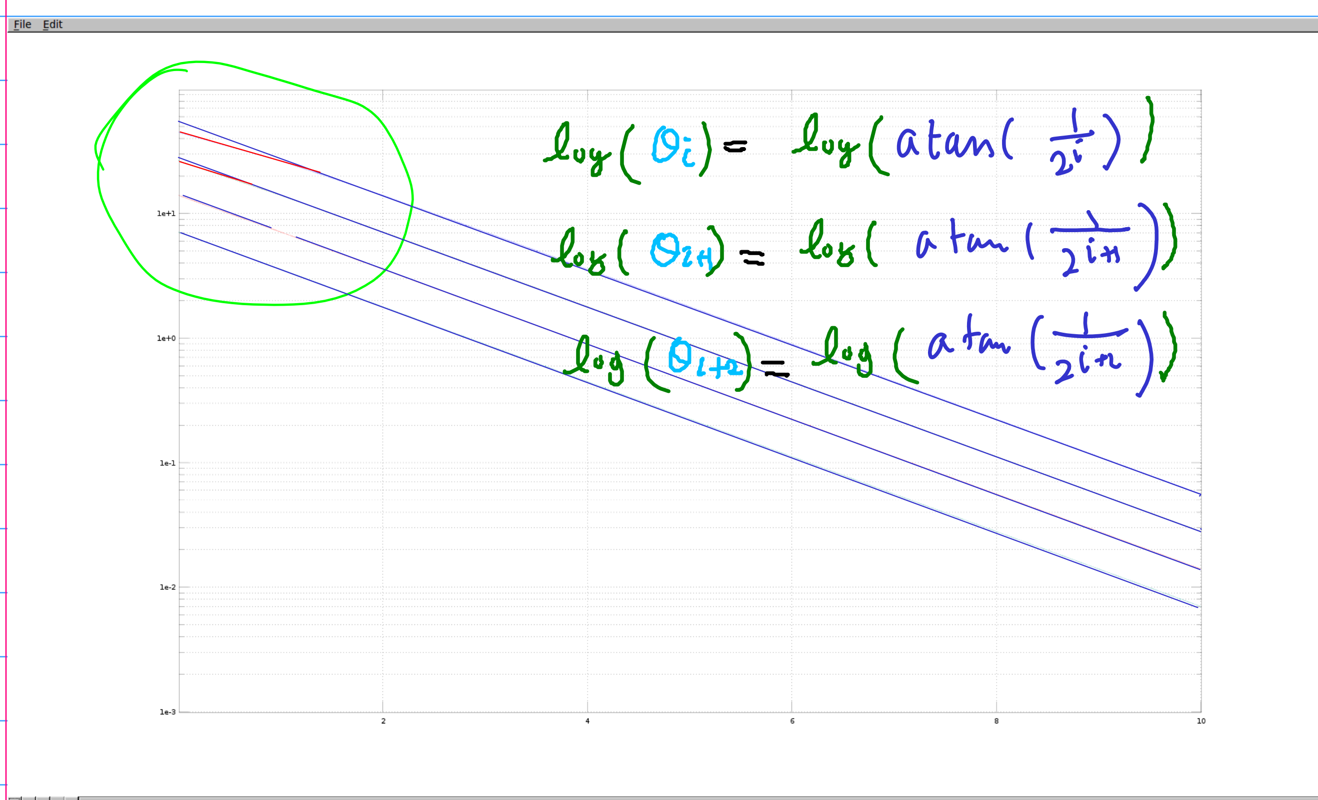


$$\theta_i = \operatorname{atan}\left(\frac{1}{2^i}\right)$$

$$\theta_{i+1} = \operatorname{atan}\left(\frac{1}{2^{i+1}}\right)$$

$$\theta_{i+2} = \operatorname{atan}\left(\frac{1}{2^{i+2}}\right)$$

log scale



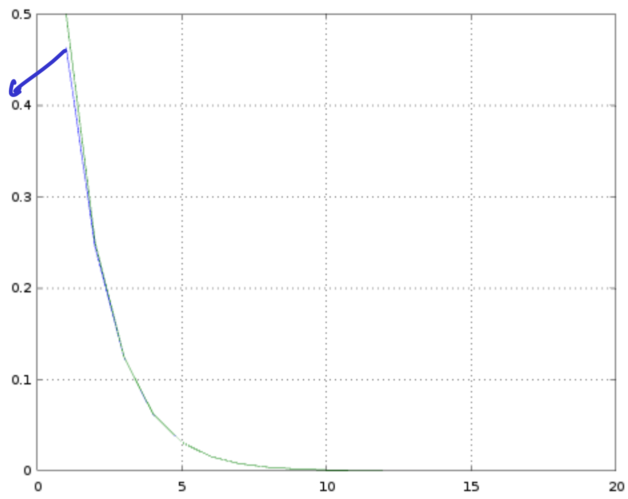
$$\log(\theta_i) = \log\left(\operatorname{atan}\left(\frac{1}{2^i}\right)\right)$$

$$\log(\theta_{i+1}) = \log\left(\operatorname{atan}\left(\frac{1}{2^{i+1}}\right)\right)$$

$$\log(\theta_{i+2}) = \log\left(\operatorname{atan}\left(\frac{1}{2^{i+2}}\right)\right)$$

After initial iterations

$$\tan^{-1}\left(\frac{1}{2^i}\right) \approx \frac{1}{2^i}$$

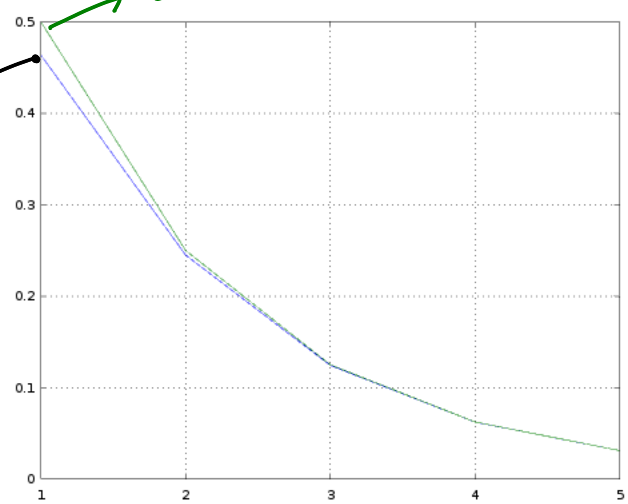


$\tan^{-1}\left(\frac{1}{2^i}\right)$

```
octave:1> i = 1:20;
octave:2> theta = atan( 1 ./ 2.^(i));
octave:3> p2 = 1./ 2.^i;
octave:4> plot(i, theta, i, p2)
```

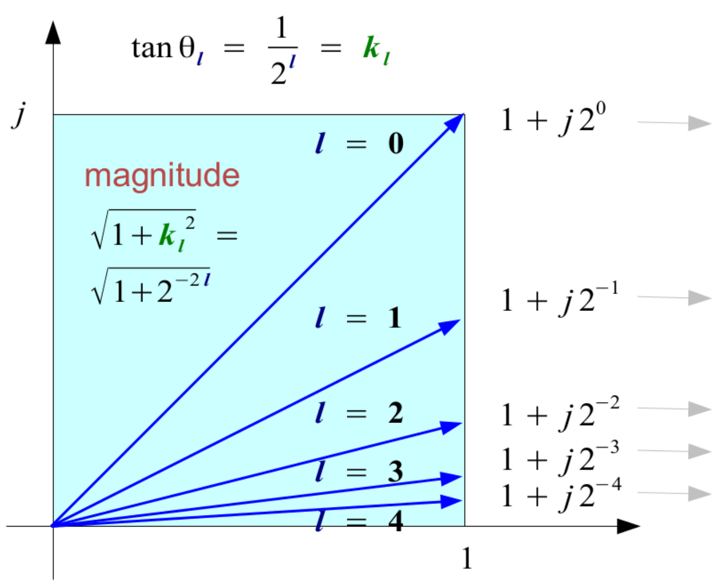
$$x'_{i+1} = (x'_i - y'_i\,\sigma_i 2^{-i})$$
$$y'_{i+1} = (x'_i\,\sigma_i 2^{-i} + y'_i)$$
$$\theta_{i+1} = \theta_i - \tan^{-1}(\sigma_i 2^{-i})$$
$$\Rightarrow \theta_i - \sigma_i\, 2^{-i}$$

after 10 iterations



$\frac{1}{2^i}$

$0.46365 = \tan^{-1}\left(\frac{1}{2}\right)$



$$\theta_l = \tan^{-1}(2^{-l}) = \tan^{-1}(k_l)$$

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$$\theta_i = \tan^{-1}\left(\frac{1}{2^i}\right) \approx \frac{1}{2^i}$$

After initial iterations

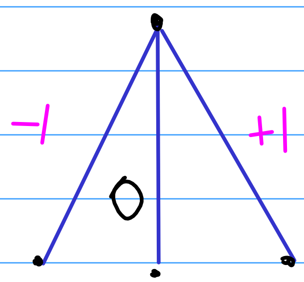
$K_i = 1$

scaling constant K doesn't change much after initial 5 ~ 10 iterations

during initial iterations, iteration angle   should be
either substracted or added

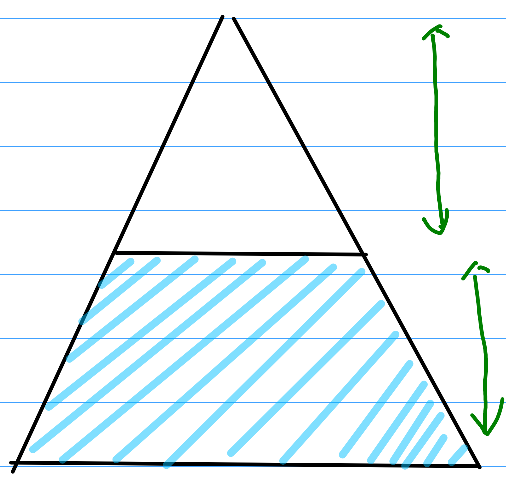
But after initial iterations, the iteration angles
don't have to be added or substracted,
in other words, zero can be added
this may cause fast convergence and improved resolution

fine stage — after initial iterations



ternary tree

$K_i \rightarrow \underline{1}$



$$K = \cos(\theta_0) \cdot \cos(\theta_1) \cdot \cos(\theta_2) \cdot \cdots$$

$$= \cos(\tan^{-1}(1)) \cdot \cos(\tan^{-1}(\tfrac{1}{2})) \cdot \cos(\tan^{-1}(\tfrac{1}{4})) \cdot \cdots$$

$$= 0.707 \cdot 0.894 \cdot 0.970 \cdot 0.992 \cdot 0.998 \cdot \cdots$$

$$\cong 0.60725$$

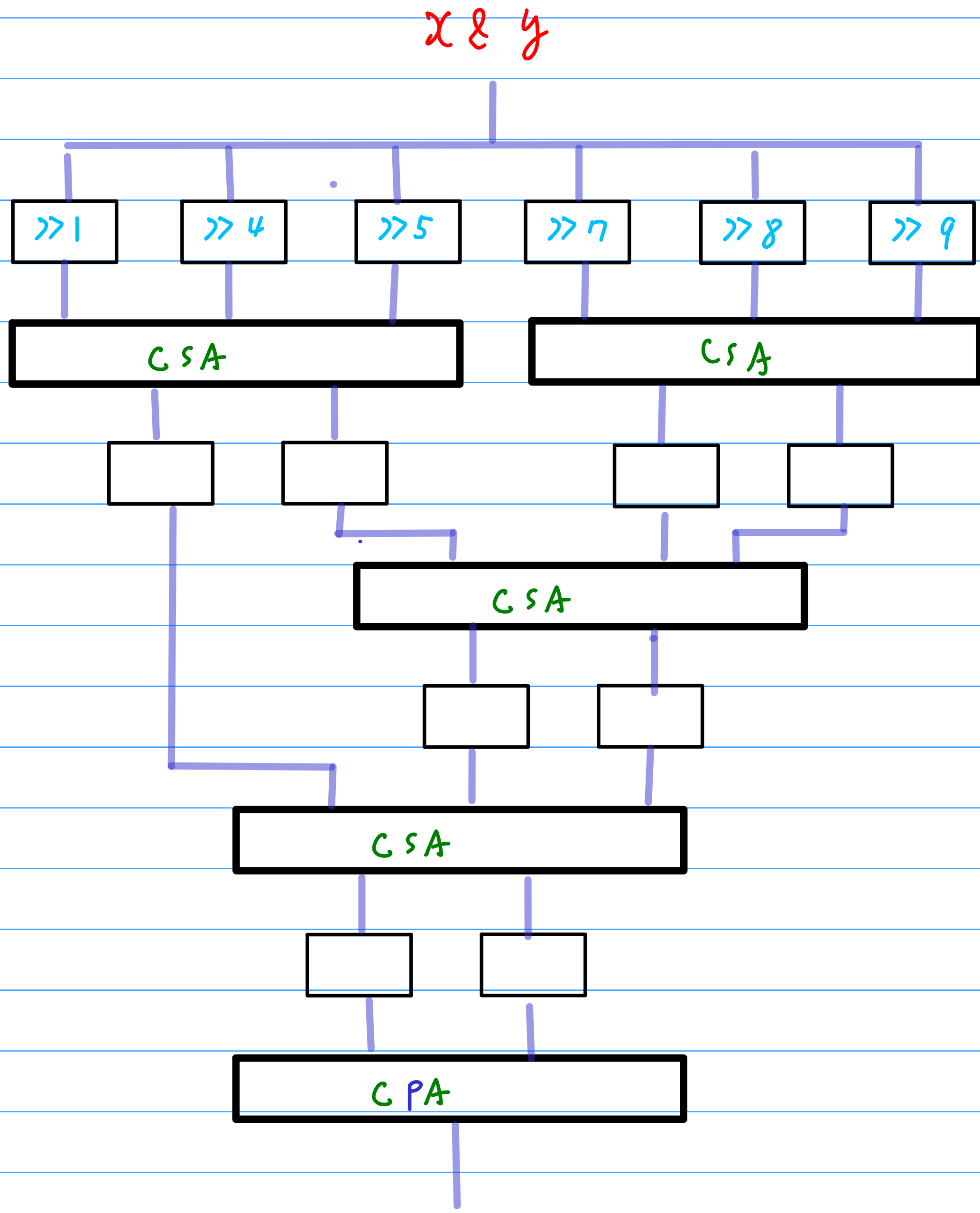
2^{-1} 2^{-2} 2^{-3} 2^{-4} 2^{-5} 2^{-6} 2^{-7} 2^{-8} 2^{-9}

0.100110111

$$\cos\theta_i = \frac{1}{\sqrt{1+\tan^2\theta_i}}$$

Fast Scaling Factor

compensation



x & y
>>1
>>4
>>5
>>7
>>8
>>9
CSA
CSA
CSA
CSA
CPA