

# Temporal Characteristics of Random Processes

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Nov 30, 2022

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Based on  
Probability, Random Variables and Random Signal Principles,  
P.Z. Peebles,Jr. and B. Shi

# Outline

- 1 Random Variables
- 2 Random Processes
- 3 Stochastic Process

# Random Variable Definition

## A random variable

a **function** over a **sample space**  $S = \{s_1, s_2, s_3, \dots, s_n\}$

$$s \rightarrow X(s)$$

$$x = X(s)$$

a **function** of a possible **outcome**  $s$  of an **experiment**

# Random Variable Definition

## A random variable

- a **random variable** : a capital letter  $X$
- a particular value : a lowercase letter  $x$
- a **sample space**  $S = \{s_1, s_2, s_3, \dots, s_n\}$
- an **outcome** (an element of  $S$ ) :  $s$

$$s \rightarrow X(s)$$

$$x = X(s)$$

$$s \rightarrow x$$

# Understanding Random Variables (1)

- **random variables** are used to quantify **outcomes** of a random occurrence, and therefore, can take on many **values**.
- **random variables** are required to be **measurable** and are typically real numbers.

for example, the letter **X** may be designated to represent the *sum* of the resulting numbers after *three dice* are rolled.

therefore, X could be 3 ( $1 + 1 + 1$ ), 18 ( $6 + 6 + 6$ ), or somewhere between 3 and 18

<https://www.investopedia.com/terms/r/random-variable.asp>

## Understanding Random Variables (2)

- A **random variable** is different from an algebraic variable.
  - The variable in an algebraic equation is an unknown value that can be calculated.
  - The equation  $10 + x = 13$  shows that we can calculate the specific value for  $x$  which is 3.
  - a **random variable** has a set of values, and any of those values *could be* the resulting **outcome** as seen in the example of the dice above.

<https://www.investopedia.com/terms/r/random-variable.asp>

## Understanding Random Variables (3)

- A typical example of a random variable is the **outcome** of a **coin toss**.
  - Consider a **probability distribution** in which the outcomes of a random event are not **equally likely** to happen.
  - If the **random variable**  $Y$  is the **number** of **heads** we get from **tossing two coins**, then  $Y$  could be 0, 1, or 2.  
(no heads, one head, or both heads)

<https://www.investopedia.com/terms/r/random-variable.asp>



## Understanding Random Variables (4)

- Let the **random variable**  $Y$  be the **number of heads of tossing two coins**
  - the two coins land in four different ways: TT, HT, TH, and HH.
  - the  $P(Y = 0) = 1/4$  since we have one chance of getting no heads (i.e., two tails [TT] when the coins are tossed).
  - the  $P(Y = 2) = 1/4$  : the probability of getting two heads (HH)
  - getting one head has a likelihood of occurring twice: in HT and TH. In this case,  $P(Y = 1) = 2/4 = 1/2$ .

<https://www.investopedia.com/terms/r/random-variable.asp>

## Formal definition of a random variable

A **random variable**  $X$  is a **measurable function**  $X: \Omega \rightarrow E$  from a set of possible **outcomes**  $\Omega$  to a **measurable space**  $E$ .

The technical axiomatic definition requires  $\Omega$  to be a **sample space** of a **probability triple**  $(\Omega, \mathcal{F}, P)$

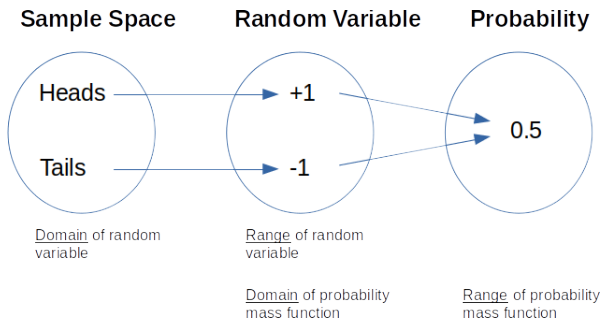
A **random variable** is often denoted by capital roman letters such as  $X, Y, Z, T$ .

The **probability** that  $X$  takes on a value in a **measurable set**  $S \subseteq E$  is written as

$$P(X \in S) = P(\{\omega \in \Omega \mid X(\omega) \in S\})$$

[https://en.wikipedia.org/wiki/Random\\_variable](https://en.wikipedia.org/wiki/Random_variable)

# Random variable example



This graph shows how random variable is a function from all possible outcomes to real values. It also shows how random variable is used for defining probability mass functions.

[https://en.wikipedia.org/wiki/Random\\_variable](https://en.wikipedia.org/wiki/Random_variable)

# Probability Space (1)

a **probability space** or a **probability triple**  $(\Omega, \mathcal{F}, P)$  is a mathematical construct that provides a formal model of a **random process** or "**experiment**".

- For example, one can define a **probability space** which models the **throwing of a die**

[https://en.wikipedia.org/wiki/Probability\\_space](https://en.wikipedia.org/wiki/Probability_space)

## Probability Space (2)

A probability space consists of three elements  $(\Omega, \mathcal{F}, P)$

- A **sample space**,  $\Omega$   
the set of all possible outcomes.
- An **event space**  
a set of events  $\mathcal{F}$ ,  
an event is a set of outcomes in the sample space
- A **probability function**,  $P$   
assigns each event in the event space a probability,  
which is a number between 0 and 1.

[https://en.wikipedia.org/wiki/Probability\\_space](https://en.wikipedia.org/wiki/Probability_space)

## Probability Space (3)

In the example of the throw of a standard die, we would take the sample space to be  $\{1, 2, 3, 4, 5, 6\}$ . For the event space, we could simply use the set of all subsets of the sample space, which would then contain simple events such as  $\{5\}$  ("the die lands on 5"), as well as complex events such as  $\{2, 4, 6\}$  ("the die lands on an even number"). Finally, for the probability function, we would map each event to the number of outcomes in that event divided by 6 — so for example,  $\{5\}$  would be mapped to  $1 / 6$   $1/6$ , and  $\{2, 4, 6\}$  would be mapped to  $3/6 = 1/2$ .

[https://en.wikipedia.org/wiki/Probability\\_space](https://en.wikipedia.org/wiki/Probability_space)

# Random Process (1)

## A random process

a function of both **outcome**  $s$  and **time**  $t$

$$X(t, s)$$

assigning a **time function** to every **outcome**  $s_i$

$$s_i \rightarrow x(t, s_i)$$

## Random Process (2)

## A random process

the family of such **time functions** is called a **random process**

$$x(t, s_i) = X(t, s_i)$$

$$x(t, s) = X(t, s)$$



## Random Process (3)

We have seen that a random variable  $X$  is a rule which assigns a number to every outcome  $e$  of an experiment.

The random variable is a function  $X(e)$  that maps the set of experiment outcomes to the set of numbers.

A random process is a rule that maps every outcome  $e$  of an experiment to a function  $X(t, e)$ .

A random process is usually conceived of as a function of time, but there is no reason to not consider random processes that are functions of other independent variables, such as spatial coordinates.

The function  $X(u, v, e)$  would be a function whose value depended on the location  $(u, v)$  and the outcome  $e$ .

# Ensemble of time functions

## Time functions

A random process  $X(t, s)$  represents a family or ensemble of **time functions**

$X(t, s)$  represents

- a **single time function**  $x(t, s)$
- when  $t$  is a variable and  $s$  is fixed at an outcome

$x(t, s)$  represents

- a **sample function**,
- an ensemble member,
- a realization of the process

# Short-form notation for time functions

## The short-form notation $x(t)$

to represent a specific waveform of a **random process**  $X(t)$   
for a given **outcome**  $s_j$

$$x(t) = x(t, s)$$

$$X(t) = X(t, s)$$

## Random Process Example

## Example

$$X(t, s_1) = x_1(t)$$

$$s_1 \rightarrow x_1(t)$$

$$X(t, s_2) = x_2(t)$$

$$s_2 \rightarrow x_2(t)$$

...

...

$$X(t, s_n) = x_n(t)$$

$$s_n \rightarrow x_n(t)$$

$S = \{s_1, s_2, s_3, \dots, s_n\}$  a sample space

$X(t) = \{x_1(t), x_2(t), x_3(t), \dots, x_n(t)\}$  a random process

# Random variables with time

a **random process**  $X(t, s)$  represents a **single time function** when  $t$  is a variable and  $s$  is fixed at an outcome

a random process  $X(t, s)$  represents a **single random variable** when both  $t$  and  $s$  are fixed at a time and an outcome, respectively

$$X_i = X(t_i, s) = X(t_i) \quad \text{random variable}$$

$$X(t, s) = X(t) \quad \text{random process}$$

# An alphabet

the **alphabet** of  $X(t)$

the set of its possible values

- the values of **time**  $t$  for which a **random process** is defined
- the **alphabet** of the random variable  $X = X(t)$  at time  $t$

# Classification of Random Processes

## (1) Types of time and alphabet

- the values of **time**  $t$  for which a **random process** is defined
  - continuous time
  - discrete time
- the **alphabet** of the random variable  $X = X(t)$  at time  $t$ 
  - continuous alphabet
  - discrete alphabet

# Classification of Random Processes

(2) types of the random variable  $X(t)$  and the time  $t$

- a continuous **alphabet** continuous **time** random process
  - $X(t)$  has continuous values and  $t$  has continuous values
- a discrete **alphabet** continuous **time** random process
  - $X(t)$  has discrete values and  $t$  has continuous values
- a continuous **alphabet** discrete **time** random process
  - $X(t)$  has continuous values and  $t$  has discrete values
- a discrete **alphabet** discrete **time** random process
  - $X(t)$  has discrete values and  $t$  has discrete values



# Deterministic and Non-deterministic Random Processes

- A process is **non-deterministic** if **future values** of any sample function cannot be predicted exactly from **observed past values**
- A process is **deterministic** if **future values** of any sample function can be predicted from **observed past values**

## Deterministic Random Process Example (1)

$$X(t) = A \cos(\omega_0 t + \Theta)$$

$A$ ,  $\Theta$ , or  $\omega_0$  (or all) can be random variables.

a sample function corresponds to the above equation with particular values of these random variables.

$$x_i(t) = A_i \cos(\omega_{0,i} t + \Theta_i)$$

## Deterministic Random Process Example (2)

$$x_i(t) = A_i \cos(\omega_{0,i}t + \Theta_i)$$

the knowledge of the sample function  
prior to any time instance fully allows  
the prediction of the sample function's future values  
because all the necessary information is known

$$x_i(t) \quad t \leq 0 \quad \implies \quad x_i(t) \quad t > 0$$

# Functions and variables of a random process $X(t, \theta)$ (1)

$X(t, \theta)$	a family of functions, an ensemble
$X(t, \theta_k)$	a single time function selected by the outcome $\theta_k$
$X(t_1, \theta)$	a random variable at the time $t = t_1$
$X(t_1, \theta_k)$	a number at the time $t = t_1$ , of the outcome $\theta_k$

<https://www.cis.rit.edu/class/simg713/notes/chap7-random-process.pdf>

Functions and variables of a random process  $X(t, \theta)$  (2)

- $X(t, \theta)$  is a **family of functions**. Imagine a giant strip chart recording in which each pen is identified with a different  $\theta$ . This family of functions is traditionally called an **ensemble**.
- A **single function**  $X(t, \theta_k)$  is selected by the **outcome**  $\theta_k$ . This is just a **time function** that we could call  $X_k(t)$ . Different **outcomes** give us different **time functions**.
- If  $t$  is fixed, say  $t = t_1$ , then  $X(t_1, \theta)$  is a **random variable**. Its value depends on the **outcome**  $\theta$ .
- If both  $t_1$  and  $\theta_k$  are given then  $X(t_1, \theta_k)$  is just a **number**.

<https://www.cis.rit.edu/class/simg713/notes/chap7-random-process.pdf>

# Stochastic Process (1)

In probability theory and related fields, a **stochastic** (/stou'kæstɪk/) or **random** process is a mathematical object usually defined as a family of **random variables**.

The word stochastic in English was originally used as an adjective with the definition "pertaining to **conjecturing**", and stemming from a Greek word meaning "to aim at a mark, guess", and the Oxford English Dictionary gives the year 1662 as its earliest occurrence.

From Ancient Greek στοχαστικός (stokhastikós), from στοχάζομαι (stokhá-zomai, "aim at a target, guess"), from στόχος (stókhos, "an aim, a guess").

<https://en.wikipedia.org/wiki/Stochastic>  
<https://en.wiktionary.org/wiki/stochastic>

## Stochastic Process (2)

The definition of a **stochastic process** varies, but a **stochastic process** is *traditionally* defined as a collection of **random variables** indexed by some set.

The terms **random process** and **stochastic process** are considered synonyms and are used interchangeably, without the **index set** being precisely specified.

Both "**collection**", or "**family**" are used while instead of "**index set**", sometimes the terms "**parameter set**" or "**parameter space**" are used.

[https://en.wikipedia.org/wiki/Stochastic\\_process](https://en.wikipedia.org/wiki/Stochastic_process)

## Stochastic Process (3)

The term **random function** is also used to refer to a **stochastic** or **random process**, though sometimes it is only used when the stochastic process takes real values.

This term is also used when the **index sets** are **mathematical spaces** other than the **real line**,

while the terms **stochastic process** and **random process** are usually used when the **index set** is interpreted as time,

and other terms are used such as **random field** when the **index set** is  $n$ -dimensional **Euclidean space**  $\mathbb{R}^n$  or a manifold

[https://en.wikipedia.org/wiki/Stochastic\\_process](https://en.wikipedia.org/wiki/Stochastic_process)



## Stochastic Process (4)

A **stochastic process** can be denoted, by  $\{X(t)\}_{t \in T}$ ,  $\{X_t\}_{t \in T}$ ,  $\{X(t)\}$ ,  $\{X_t\}$  or simply as  $X$  or  $X(t)$ , although  $X(t)$  is regarded as an abuse of function notation.

For example,  $X(t)$  or  $X_t$  are used to refer to the **random variable** with the **index**  $t$ , and not the entire **stochastic process**.

If the **index set** is  $T = [0, \infty)$ , then one can write, for example,  $(X_t, t \geq 0)$  to denote the **stochastic process**.

[https://en.wikipedia.org/wiki/Stochastic\\_process](https://en.wikipedia.org/wiki/Stochastic_process)

# Stochastic Process Definition (1)

A **stochastic process** is defined as a collection of **random variables** defined on a common **probability space**  $(\Omega, \mathcal{F}, P)$ ,

- $\Omega$  is a **sample space**,
- $\mathcal{F}$  is a  $\sigma$ -**algebra**,
- $P$  is a **probability measure**;
- the **random variables**, indexed by some set  $T$ ,
- all take values in the same **mathematical space**  $S$ , which must be **measurable** with respect to some  $\sigma$ -algebra  $\Sigma$

[https://en.wikipedia.org/wiki/Stochastic\\_process](https://en.wikipedia.org/wiki/Stochastic_process)

## Stochastic Process Definition (2)

In other words, for a given **probability space**  $(\Omega, \mathcal{F}, P)$  and a **measurable space**  $(S, \Sigma)$ , a **stochastic process** is a **collection** of  $S$ -valued **random variables**, which can be written as:

$$\{X(t) : t \in T\}.$$

[https://en.wikipedia.org/wiki/Stochastic\\_process](https://en.wikipedia.org/wiki/Stochastic_process)

# Stochastic Process Definition (3)

Historically, in many problems from the natural sciences a point  $t \in T$  had the meaning of time, so  $X(t)$  is a **random variable** representing a value observed at time  $t$ .

A **stochastic process** can also be written as  $\{X(t, \omega) : t \in T\}$  to reflect that it is actually a function of two variables,  $t \in T$  and  $\omega \in \Omega$ .

[https://en.wikipedia.org/wiki/Stochastic\\_process](https://en.wikipedia.org/wiki/Stochastic_process)

## Stochastic Process Definition (4)

There are other ways to consider a stochastic process, with the above definition being considered the traditional one.

For example, a stochastic process can be interpreted or defined as a  $S^T$ -valued **random variable**, where  $S^T$  is the space of all the possible functions from the set  $T$  into the space  $S$ .

However this alternative definition as a "**function-valued random variable**" in general requires additional regularity assumptions to be **well-defined**.

[https://en.wikipedia.org/wiki/Stochastic\\_process](https://en.wikipedia.org/wiki/Stochastic_process)

# Index set (1)

The set  $T$  is called the **index set** or **parameter set** of the **stochastic process**.

Often this set is some subset of the real line, such as the natural numbers or an interval, giving the set  $T$  the interpretation of time.

[https://en.wikipedia.org/wiki/Stochastic\\_process](https://en.wikipedia.org/wiki/Stochastic_process)

## Index set (2)

In addition to these sets, the index set  $T$  can be another set with a **total order** or a more general set, such as the Cartesian plane  $R^2$  or  $n$ -dimensional **Euclidean space**, where an element  $t \in T$  can represent a point in space.

That said, many results and theorems are only possible for **stochastic processes** with a **totally ordered index set**.

[https://en.wikipedia.org/wiki/Stochastic\\_process](https://en.wikipedia.org/wiki/Stochastic_process)

# State space

The **mathematical space**  $S$  of a **stochastic process** is called its **state space**.

This mathematical space can be defined using integers, real lines,  $n$ -dimensional Euclidean spaces, complex planes, or more abstract mathematical spaces.

The **state space** is defined using elements that reflect the different values that the **stochastic process** can take.

[https://en.wikipedia.org/wiki/Stochastic\\_process](https://en.wikipedia.org/wiki/Stochastic_process)



# Sample function (1)

A **sample function** is a single outcome of a **stochastic process**, so it is formed by taking a single possible value of each **random variable** of the **stochastic process**.

More precisely, if  $\{X(t, \omega) : t \in T\}$  is a **stochastic process**, then for any point  $\omega \in \Omega$ , the mapping  $X(\cdot, \omega) : T \rightarrow S$ , is called a **sample function**, a **realization**, or, particularly when  $T$  is interpreted as time, a **sample path** of the **stochastic process**  $\{X(t, \omega) : t \in T\}$ .

[https://en.wikipedia.org/wiki/Stochastic\\_process](https://en.wikipedia.org/wiki/Stochastic_process)

## Sample function (2)

This means that for a fixed  $\omega \in \Omega$  ,  
there exists a **sample function**  
that maps the **index set**  $T$  to the **state space**  $S$ .

Other names for a **sample function** of a **stochastic process**  
include **trajectory**, **path function** or **path**

[https://en.wikipedia.org/wiki/Stochastic\\_process](https://en.wikipedia.org/wiki/Stochastic_process)

