Difference Equation
First Order (H.1)

20161117

Based on
Complex Analysis for Mathematics and Engineering
J. Mathews

First Order Difference Equation

$y[n+1] - a\,y[n] = 0$

trial solution

$y[n] = c_1 r^n$

$c_1 r^{n+1} - a c_1 r^n = 0 \qquad (r-a) = 0$

$y[n] = c_1 a^n$

Linear Constant Coefficient Difference Equation

$y[n+1] - a\,y[n] = x[n]$ $\qquad$ $y[0] = y_0$ initial condition

Exponential growth or decay

$y[n+1] = (1+r)\,y[n]$ $\qquad$ $y[n] = y_0 (1+r)^n$

Newton's law of cooling

$y[n+1] = aL + (1-a)\,L\,y[n]$ $\qquad$ $y[n] = y_0 (1-a)^n + L(1-(1-a)^n)$

Repeated dosage drug level

$y[n+1] = a\,y[n] + b$ $\qquad$ $y[n] = y_0 a^n + \frac{a^n - 1}{a-1} b$

Value of an annuity due

$y[n+1] = (1+r)(y[n] + P)$ $\qquad$ $y[n] = \frac{(1+r)^{n+1} - 1}{r} P - P$

Linear Constant Coefficient Difference Equation

$y[n+1] - a\,y[n] = x[n]$ $\qquad$ $y[0] = y_0$ initial condition

Trial Solution

$y_h[n+1] - a\,y_h[n] = 0$ $\qquad$ $y_h[n] = C_1 a^n$ homogeneous solution

$y_p[n+1] - a\,y_p[n] = x[n]$ $\qquad$ $y_p[n]$ particular solution

$$y[n] = y_h[n] + y_p[n]$$

Z-transform method

$$\mathcal{Z}[y[n+1]] = z\,(Y(z) - y_0)$$

$$z\,(Y(z) - y_0) - a\,Y(z) = X(z)$$

$$y[n] = \mathcal{Z}^{-1}[Y(z)]$$

Residue Method

$$y[n] = \mathcal{Z}^{-1}[Y(z)] = \sum_{i=1}^{k} \mathrm{Res}\left[Y(z)\,z^{n-1}, z_i\right]$$

Convolution Method

ⓐ homogeneous solution

$$y_h[n+1] - a\, y_h[n] = 0 \qquad y_h[n] = C_1 a^n$$

transfer function

$$H(z) = \frac{1}{1-az^{-1}}$$

unit sample response

$$h[n] = Z^{-1}[1 \cdot H(z)] = a^n$$

ⓑ particular solution

$$y_p[n] = Z^{-1}[X(z)\,H(z)]$$

$$y_p[n] = h[n] * x[n] = \sum_{i=0}^{n} x[n-i]\,h[i] = \sum_{i=0}^{n} x[n-i]\,a^i$$

ⓒ general solution

$$y[n] = y_h[n] + y_p[n] = C_1 a^n + \sum_{i=0}^{n} x[n-i]\,a^i$$

constant $C_1 = y_0 - x[0]$

$$y[n] = (y_0 - x[0])\,a^n + \sum_{i=0}^{n} x[n-i]\,a^i$$

Solving First Order Difference Equations

① Trial Solution Method

$$y_h[n] = C_1 a^n$$

$$y[n] = y_h[n] + y_p[n]$$

② z-Transform Method

$$y[n] = \mathcal{Z}^{-1}[Y(z)]$$

③ Residue Method

$$y[n] = \mathcal{Z}^{-1}[Y(z)] = \sum_{i=1}^{k} \mathrm{Res}\left(Y(z)\cdot z^{n-1},\ z_i\right)$$

④ Convolution Method

$$h[n] = \mathcal{Z}^{-1}[H(z)]$$

$$y_p[n] = \mathcal{Z}^{-1}[H(z)X(z)]$$

$$y_p[n] = \sum_{i=0}^{n} x[n-i]\,h[i]$$

$$y_p[0] = \sum_{i=0}^{0} x[0-i]\,h[i] = x[0]\,h[0] = x[0]$$

Discrete-time Exponential

$$e^{\lambda t} = \gamma^t \qquad e^{\lambda} = \gamma \qquad \lambda = \ln \gamma$$

$$e^{-0.3t} \Rightarrow (0.7408)^t$$

$$e^{1.386t} \Leftarrow 4^t$$

$$e^{\lambda n} = \gamma^n \qquad e^{\lambda} = \gamma \qquad \lambda = \ln \gamma$$

$e^{\lambda n}$ grows exponentially with n if $\mathrm{Re}\{\lambda\} > 0$

decays if $\mathrm{Re}\{\lambda\} < 0$

constant / oscillates if $\mathrm{Re}\{\lambda\} = 0$

$$\gamma = e^{\lambda} = e^{a+jb} = e^{a} e^{jb}$$

$$|\gamma| = |e^{a}|\,|e^{jb}| = e^{a}$$

$|\gamma| = e^{a}$

$\lvert\gamma\rvert < 1$	$a < 0$
$\lvert\gamma\rvert = 1$	$a = 0$
$\lvert\gamma\rvert > 1$	$a > 0$

First Order Difference Equations

$y[n] + 2y[n-1] = 3.5\,u[n]$ $\qquad$ $y[-1]=0 \Rightarrow$ zero state response

$$Y(z) + 2z^{-1}Y(z) = 3.5\,\frac{z}{z-1}$$

$$Y(z) = 3.5\,\frac{z}{z-1}\,\frac{1}{1+2z^{-1}} = 3.5\,\frac{z}{z-1}\cdot\frac{z}{z+2}$$

$$\frac{1}{z-1}\cdot\frac{1}{z+2} = \frac{a}{z-1} + \frac{b}{z+2}$$

$az + 2a + bz - b = 1$

$(a+b)z + (2a-b) = 1$

$(a+b) = 0 \quad (2a-b) = 1$

$a = \frac{1}{3},\ b = -\frac{1}{3}$

$$Y(z) = \frac{7}{2}\left(\frac{1}{3}\,\frac{z}{z-1} - \frac{1}{3}\,\frac{z}{z+2}\right)z$$

$$= \frac{7}{6}\,z^{-1}\left[z\left(\frac{-z}{z+2} + \frac{z}{z-1}\right)\right]$$

$$= \frac{7}{6}\left(-(-2)^{n+1} + u[n]\right)$$

$$= \frac{7}{3}(-2)^n + \frac{7}{6}u[n]$$

$X_1(z) = \frac{-z}{z+2}$	$X_2(z) = \frac{z}{z-1}$
$x_1[n] = -(-2)^n$	$x_2[n] = u[n]$
$x_1[0] = -1$	$x_2[0] = 1$
$zX_1(z)$ $= x_1[n+1] + x_1[0]\,z$ $= -(-2)^{n+1} \cancel{-z}$	$zX_2(z)$ $= x_2[n+1] + x_2[0]\,z$ $= u[n] \cancel{+z}$

$u[n] \leftrightarrow$	$\frac{z}{z-1}$
$b^n \leftrightarrow$	$\frac{z}{z-b}$

$$\frac{7}{6}\left(\frac{1}{1-\left(\frac{1}{z}\right)} - \frac{1}{1-\left(-\frac{2}{z}\right)}\right)z$$

$$= \frac{7}{6}u[n] - \frac{7}{6}(-2)^{n+1}$$

$$= \frac{7}{6}u[n] + \frac{7}{3}(-2)^n$$

$$y[n] + 2y[n-1] = 3.5\,u[n] \qquad y[-1] = 4$$

$$Y(z) + 2\left(z^{-1}Y(z) + y[-1]\right) = 3.5\,U(z)$$

x_{n+1}	$\Longleftrightarrow$	$zX(z) - x_0 z$
x_{n-1}	$\Longleftrightarrow$	$z^{-1}X(z) + x_{-1}$

$$(1+2z^{-1})\,Y(z) + 2y[-1] = 3.5\,U(z)$$

$$Y(z) = \frac{1}{1+2z^{-1}}\left(3.5\,U(z) - 2y[-1]\right)$$

$$= \boxed{\frac{3.5}{1+2z^{-1}}U(z)} - \boxed{\frac{2y[-1]}{1+2z^{-1}}}$$

Zero State Response Zero Input Response

$$y_{zI}[n] = \mathcal{Z}^{-1}\left[-\frac{2y[-1]}{1+2z^{-1}}\right]$$

$$= -2y[-1]\,\mathcal{Z}^{-1}\left[\frac{1}{1+2z^{-1}}\right] \quad = -2y[-1]\,\mathcal{Z}^{-1}\left[\frac{z}{z-(-2)}\right]$$

$$= -2y[-1](-2)^n$$

$$y_{zs}[n] = \mathcal{Z}^{-1}\left[\frac{3.5}{1+2z^{-1}}\,\frac{1}{1-z^{-1}}\right] = 3.5\,\mathcal{Z}^{-1}\left[\frac{z}{(z+2)}\,\frac{z}{(z-1)}\right]$$

$$\frac{A}{z+2} + \frac{B}{z-1} \qquad Az - A + Bz + 2B = 1$$

$$A + B = 0 \qquad A = -B$$

$$-A + 2B = 1 \qquad 3B = 1 \qquad B = \frac{1}{3},\ A = -\frac{1}{3}$$

$$= \frac{7}{6}\,\mathcal{Z}^{-1}\left[z\left(\frac{-z}{z+2} + \frac{z}{z-1}\right)\right]$$

$$= \frac{7}{6}\left(-(-2)^{n+1} + u[n]\right)$$

$$= \frac{7}{3}(-2)^n + \frac{7}{6}u[n]$$

$X_1(z) = \frac{-z}{z+2}$	$X_2(z) = \frac{z}{z-1}$
$x_1[n] = -(-2)^n$	$x_2[n] = u[n]$
$x_1[0] = -1$	$x_2[0] = 1$
$zX_1(z)$ $= x_1[n+1] + x_1[0]\,z$ $= -(-2)^{n+1} - \cancel{z}$	$zX_2(z)$ $= x_2[n+1] + x_2[0]\,z$ $= u[n] + \cancel{z}$

$$y[n] + 2y[n-1] = 3.5\,u[n] \qquad y[-1] = 4$$

$$y[n+1] + 2y[n] = 3.5\,u[n+1] \qquad y[0] = 3.5 - 2\cdot 4 = -4.5$$

$$zY(z) - y[0]z + 2Y(z) = 3.5\cdot\left(zU(z) - u[0]z\right)$$

$$(z+2)Y(z) = -4.5z - 3.5z + 3.5zU(z) = z + 3.5zU(z)$$

$$Y(z) = \frac{-8z}{z+2} + 3.5\frac{z}{z+2}\,\frac{z}{z+1}$$

$$y[n] + 2y[n-1] = 3.5\,u[n] \qquad y[-1] = 4$$

$$Y(z) + 2\left(z^{-1}Y(z) + y[-1]\right) = 3.5\,U(z)$$

$$Y(z) = \frac{1}{1+2z^{-1}}\left(3.5\,U(z) - 2y[-1]\right) = \frac{3.5z}{z+2}\,\frac{z}{z+1} - \frac{8z}{z+1}$$

$$= \underbrace{\boxed{\frac{3.5}{1+2z^{-1}}U(z)}}_{\text{ZSR}} - \underbrace{\boxed{\frac{2y[-1]}{1+2z^{-1}}}}_{\text{ZIR}}$$

$$y[n+1] - 2y[n] = 3^n \qquad y[0] = 2$$

(I) Z - Transform method

$$\boxed{\begin{aligned} &z(Y(z) - 2) - 2Y(z) = \frac{z}{z-3} \\ &(z-2)Y(z) = 2z + \frac{z}{z-3} = \frac{2z}{(z-2)} + \frac{z}{(z-2)(z-3)} \\ &Y(z) = \frac{2z(z-3)+z}{(z-2)(z-3)} = \frac{2z^2 - 5z}{(z-2)(z-3)} \end{aligned}}$$

$2(z^2-5z+6) + 10z - 12 - 5z$

$$= 2 + \frac{5z-12}{(z-2)(z-3)}$$

$$= 2 + \frac{2}{(z-2)} + \frac{3}{(z-3)}$$

$$y[n] = Z^{-1}[2] + Z^{-1}\left[\frac{2}{(z-2)}\right] + Z^{-1}\left[\frac{3}{(z-3)}\right]$$

$$= 2u[n] + 2^n u[n-1] + 3^n u[n-1]$$

ⓑ residue

$$\boxed{\begin{aligned} & z(Y(z)-2) - 2Y(z) = \frac{z}{z-3} \\ & (z-2)Y(z) = 2z + \frac{z}{z-3} = \frac{2z}{(z-2)} + \frac{z}{(z-2)(z-3)} \\ & Y(z) = \frac{2z(z-3)+z}{(z-2)(z-3)} = \frac{2z^2-5z}{(z-2)(z-3)} \end{aligned}}$$

$$\mathrm{Res}[Y(z)z^{n-1}, 2] = \lim_{z\to 2} \frac{2z^2-5z}{(z-3)} z^{n-1} = \frac{-2}{-1} 2^{n-1}$$

$$\mathrm{Res}[Y(z)z^{n-1}, 3] = \lim_{z\to 3} \frac{2z^2-5z}{(z-2)} z^{n-1} = \frac{+3}{+1} 3^{n-1}$$

$$\begin{aligned} y[n] &= \mathrm{Res}[Y(z)z^{n-1}, 2] + \mathrm{Res}[Y(z)z^{n-1}, 3] \\ &= 2^n + 3^n \end{aligned}$$

Repeated Dosage Drug Level

① Trial and Error Method

$$y[n+1] = a\,y[n] + b \qquad\qquad y[n] = y_0 a^n + \frac{a^n-1}{a-1} b$$

ⓐ homogeneous equation — homogeneous solution

$$y[n+1] - a\,y[n] = 0$$

assume $y_h[n] = r^n$

$$r^{n+1} - a r^n = 0$$

$r - a = 0$ Characteristic equation

$$y_h[n] = C_1 a^n$$

ⓑ non-homogeneous equation — particular solution

$$y[n+1] - a\,y[n] = b$$

$y_p[n] = c$ constant

$$c - ac = b$$

$$c = \frac{b}{1-a}$$

general solution

$$y[n] = y_h[n] + y_p[n]$$

$$y[n] = c a^n + \frac{b}{1-a}$$

initial condition

$$y[0] = c a^0 + \frac{b}{1-a} = y_0$$

$$c = y_0 + \frac{b}{a-1}$$

$$y[n] = \left(y_0 + \frac{b}{a-1}\right) a^n + \frac{b}{1-a}$$

$$= y_0 a^n + \left(\frac{b}{a-1}\right) a^n - \frac{b}{a-1}$$

$$y[n] = y_0 a^n + \frac{a^n - 1}{a-1} b$$

Repeated Dosage Drug Level

II z-Transform Method – Partial Fraction

$$y[n+1] = a\,y[n] + b \qquad\qquad y[n] = y_0 a^n + \frac{a^n-1}{a-1} b$$

$$z\left(Y(z) - y_0\right) = a\,Y(z) + b \boxed{\frac{z}{z-1}}$$

$$(z-a)\,Y(z) = y_0 z + b\frac{z}{z-1}$$

$$Y(z) = y_0 \frac{z}{(z-a)} + b\frac{z}{(z-a)(z-1)}$$

$$\frac{z}{(z-a)(z-1)} = \frac{A}{z-a} + \frac{B}{z-1}$$

$$A = (z-a)\frac{z}{(z-a)(z-1)}\Big|_{z=a} = \frac{a}{a-1}$$

$$B = (z-1)\frac{z}{(z-a)(z-1)}\Big|_{z=1} = \frac{1}{1-a} = \frac{-1}{a-1}$$

$$Y(z) = y_0 \frac{z}{(z-a)} + \left(\frac{ab}{a-1}\right)\boxed{\frac{1}{z-a}} - \left(\frac{b}{a-1}\right)\boxed{\frac{1}{z-1}}$$

$$\Updownarrow \qquad\qquad \Updownarrow$$

$$y[n] = y_0\, a^n + \left(\frac{ab}{a-1}\right)\boxed{a^{n-1} u[n-1]} - \left(\frac{b}{a-1}\right)\boxed{u[n-1]}$$

$$\begin{cases} n=0 & y[0] = y_0 \\ n\geq 1 & y[n] = y_0 a^n + \left(\frac{ab}{a-1}\right) a^{n-1} - \left(\frac{b}{a-1}\right) \end{cases}$$

$$n \geq 0 \quad y[n] = y_0 a^n + \frac{a^n-1}{a-1} b$$

Repeated Dosage Drug Level

III Z-Transform Method – Residue Method

$$y[n+1] = a\,y[n] + b \qquad\qquad y[n] = y_0 a^n + \frac{a^n - 1}{a-1} b$$

$$z\left(Y(z) - y_0\right) = a\,Y(z) + b \boxed{\frac{z}{z-1}}$$

$$(z-a)\,Y(z) = y_0 z + b\frac{z}{z-1}$$

$$Y(z) = y_0 \frac{z}{(z-a)} + b\frac{z}{(z-a)(z-1)}$$

$$y[n] = \mathcal{Z}^{-1}[Y(z)] = \mathrm{Res}\left(Y(z)z^{n-1}, 1\right) + \mathrm{Res}\left(Y(z)z^{n-1}, a\right)$$

$$\mathrm{Res}\left(Y(z)z^{n-1}, 1\right) = \lim_{z\to 1} (z-1)\,Y(z)\,z^{n-1}$$

$$= \lim_{z\to 1} y_0 \frac{z^n(z-1)}{z-a} + b\frac{z^n}{z-a} = \frac{b}{1-a}$$

$$\mathrm{Res}\left(Y(z)z^{n-1}, a\right) = \lim_{z\to a} (z-a)\,Y(z)\,z^{n-1}$$

$$= \lim_{z\to a} y_0 z^n + \frac{b z^n}{z-1} = y_0 a^n + \frac{b a^n}{a-1}$$

$$y[n] = \frac{b}{1-a} + y_0 a^n + \frac{b a^n}{a-1} = y_0 a^n + b\frac{a^n - 1}{a-1}$$

Repeated Dosage Drug Level

IV z-Transform Method - Convolution Method

$$y[n+1] = a\,y[n] + b$$

$$\boxed{y[n+1] - a\,y[n] = \boxed{x[n+1]}}$$

$$z(Y(z) - \cancel{y_0}) = aY(z) + z(X(z) - \cancel{x_0})$$

$$\boxed{H(z) = \frac{Y(z)}{X(z)} = \frac{z}{z-a}}$$

$$\boxed{h[n] = a^n} \qquad H(z) = \frac{z}{z-a}$$

$$\boxed{x[n] = b} \qquad X(z) = \frac{bz}{z-1}$$

ZSR $\quad \boxed{y_{zs}[n] = \mathcal{Z}^{-1}[H(z)X(z)]} = \mathcal{Z}^{-1}\left[\frac{z}{z-a}\,\frac{bz}{z-1}\right]$

$$\frac{z^2}{(z-a)(z-1)} = 1 + \frac{(a+1)z - a}{(z-a)(z-1)}$$

$$\frac{(a+1)z - a}{(z-a)(z-1)} = \frac{A}{z-a} + \frac{B}{z-1}$$

$$A = (z-a)\frac{(a+1)z - a}{(z-a)(z-1)}\bigg|_{z=a} = \frac{a^2}{a-1}$$

$$B = (z-1)\frac{(a+1)z - a}{(z-a)(z-1)}\bigg|_{z=1} = \frac{1}{1-a} = \frac{-1}{a-1}$$

$$\frac{z^2}{(z-a)(z-1)} = 1 + \left(\frac{a^2}{a-1}\right)\frac{1}{z-a} - \left(\frac{1}{a-1}\right)\frac{1}{z-1}$$

$$y_{zs}[n] = \mathcal{Z}^{-1}\left[b + \left(\frac{a^2 b}{a-1}\right)\frac{1}{z-a} - \left(\frac{b}{a-1}\right)\frac{1}{z-1}\right]$$

ZSR $y_{zs}[n] = \mathcal{Z}^{-1}\left[b + \left(\frac{a^2 b}{a-1}\right)\frac{1}{z-a} - \left(\frac{b}{a-1}\right)\frac{1}{z-1} \right]$

$$= b\,\delta[n] + \left(\frac{a^2 b}{a-1}\right) a^{n-1} u[n-1] - \left(\frac{b}{a-1}\right) u[n-1]$$

$$= b\,\delta[n] + \left(\frac{b a^{n+1}}{a-1}\right) u[n-1] - \left(\frac{b}{a-1}\right) u[n-1]$$

*

$n=0$	$b + 0 + 0 = b$	$= b\left(\frac{a-1}{a-1}\right)$
$n=1$	$0 + \frac{ba^2}{a-1} - \frac{b}{a-1}$	$= b\left(\frac{a^2-1}{a-1}\right)$
$n=2$	$0 + \frac{ba^3}{a-1} - \frac{b}{a-1}$	$= b\left(\frac{a^3-1}{a-1}\right)$

ZSR $y_{zs}[n] = \left(\frac{a^{n+1}-1}{a-1}\right) b = \left(\frac{a^{n+1}-1}{a-1}\right) b\, u[n]$

ZSR $y_{zs}[n] = x[n] * h[n] = \sum_{i=0}^{n} x[n-i]\,h[i]$

$$= b * a^n$$

ZSR $y_{zs}[n] = \sum_{i=0}^{n} b a^i = \left(\frac{a^{n+1}-1}{a-1}\right) b$

ZSR $y_{zs}[0] = \sum_{i=0}^{0} b a^i = b$

ZIR has the same form as y_h

$y_{zi}[n] = C_1 a^n$

$$y[n] = y_{ZI}[n] + y_{ZS}[n] = c_1 a^n + \sum_{i=0}^{n} b a^i$$

$$y[n] = c_1 a^n + \left(\frac{a^{n+1}-1}{a-1}\right) b$$

I.C.

$$y[0] = c_1 a^0 + \left(\frac{a^1-1}{a-1}\right) b = c_1 + b = y_0 \Rightarrow \quad c_1 = y_0 - b$$

$$y[n] = \underbrace{\boxed{(y_0 - b)a^n}}_{\text{ZIR}} + \underbrace{\boxed{\left(\frac{a^{n+1}-1}{a-1}\right) b}}_{\text{ZSR}}$$

$$y[n] = (y_0 - b)\, a^n + \left(\frac{a^{n+1}-1}{a-1}\right) b$$

$$= y_0 a^n + \left(\frac{a^{n+1}-1}{a-1} - a^n\right) b$$

$$\frac{a^{n+1}-1-a^{n+1}+a^n}{a-1}$$

$$y[n] = y_0 a^n + \left(\frac{a^n-1}{a-1}\right) b$$

$$y[n] = \boxed{\left(y_0 + \frac{b}{a-1}\right) a^n} + \boxed{\left(\frac{-1}{a-1}\right) b}$$

Natural Response

Forced Response

(lumped all char. modes)

P582

Ambardar

$$y[n] - \frac{1}{2}y[n-1] = 2\left(\frac{1}{4}\right)^n x[n] \qquad y[-1] = -2$$

$$y_{zs}[n] = -2\left(\frac{1}{4}\right)^n u[n] + 4\left(\frac{1}{2}\right)^n u[n]$$

$$y_{zi}[n] = -\left(\frac{1}{2}\right)^n u[n]$$

$$y_h[n] = 3\left(\frac{1}{2}\right)^n u[n]$$

$$y_p[n] = -2\left(\frac{1}{4}\right)^n u[n]$$

$(n, n-1)$

$$\boxed{y[n] - \frac{1}{2} y[n-1] = 2\left(\frac{1}{4}\right)^n u[n]} \qquad \boxed{y[-1] = -2}$$

$$Y(z) - \frac{1}{2}\left(z^{-1} Y(z) + y[-1]\right) = 2 \frac{z}{z - \frac{1}{4}}$$

$$\left(1 - \frac{1}{2} z^{-1}\right) Y(z) = \frac{2z}{z - \frac{1}{4}} + \frac{1}{2} y[-1] = \frac{2z}{z - \frac{1}{4}} - 1 = \frac{z + \frac{1}{4}}{z - \frac{1}{4}}$$

$$Y(z) = \frac{z + \frac{1}{4}}{z - \frac{1}{4}} \cdot \frac{1}{1 - \frac{1}{2} z^{-1}} = \frac{z\left(z + \frac{1}{4}\right)}{\left(z - \frac{1}{4}\right)\left(z - \frac{1}{2}\right)}$$

$$z^{-1} Y(z) = \frac{\left(z + \frac{1}{4}\right)}{\left(z - \frac{1}{4}\right)\left(z - \frac{1}{2}\right)} = \frac{A}{z - \frac{1}{4}} + \frac{B}{z - \frac{1}{2}}$$

$$A = \left(z - \frac{1}{4}\right) \frac{\left(z + \frac{1}{4}\right)}{\left(z - \frac{1}{4}\right)\left(z - \frac{1}{2}\right)} \Bigg|_{z = \frac{1}{4}} = \frac{\frac{1}{2}}{-\frac{1}{4}} = -2$$

$$B = \left(z - \frac{1}{2}\right) \frac{\left(z + \frac{1}{4}\right)}{\left(z - \frac{1}{4}\right)\left(z - \frac{1}{2}\right)} \Bigg|_{z = \frac{1}{2}} = \frac{\frac{3}{4}}{\frac{1}{4}} = 3$$

$$z^{-1} Y(z) = \frac{-2}{z - \frac{1}{4}} + \frac{3}{z - \frac{1}{2}}$$

$$Y(z) = \frac{-2z}{z - \frac{1}{4}} + \frac{3z}{z - \frac{1}{2}}$$

$$y[n] = -2\left(\frac{1}{4}\right)^n + 3\left(\frac{1}{2}\right)^n \qquad n \geq 0$$

$$y[-1] = -2\left(\frac{1}{4}\right)^{-1} + 3\left(\frac{1}{2}\right)^{-1}$$
$$= -8 + 6 = -2 \quad \text{o.K.}$$

$$x[n-1] \longleftrightarrow z^{-1} X(z) + x[-1]$$

$$y[n] - \frac{1}{2}y[n-1] = 2\left(\frac{1}{4}\right)^n u[n] \qquad y[-1] = -2$$

$(n+1,\ n)$

$$\boxed{y[n+1] - \frac{1}{2}y[n] = 2\left(\frac{1}{4}\right)^{n+1} u[n+1]} \qquad \boxed{y[0] = 1}$$

$$y[-1+1] - \frac{1}{2}y[-1] = 2\left(\frac{1}{4}\right)^{n+1} u[-1+1]$$

$$y[0] - \frac{1}{2}\cdot(-2) = 2 \qquad y[0] = 1$$

$$\mathcal{Z}\left[\left(\frac{1}{4}\right)^{n+1} u[n+1]\right] = z\left(\mathcal{Z}\left[\left(\frac{1}{4}\right)^n u[n]\right] - \left[\left(\frac{1}{4}\right)^0 u[0]\right]\right)$$

$$= z\left(\frac{z}{z-\frac{1}{4}} - 1\right) = z\left(\frac{\frac{1}{4}}{z-\frac{1}{4}}\right) = \frac{\frac{1}{4}z}{z-\frac{1}{4}}$$

$$y[n+1] - \frac{1}{2}y[n] = 2\left(\frac{1}{4}\right)^{n+1} u[n+1] \qquad y[0] = 1$$

$$z\left(Y(z) - Y[0]\right) - \frac{1}{2}Y(z) = 2\,\frac{\frac{1}{4}z}{z-\frac{1}{4}}$$

$$\left(z - \frac{1}{2}\right)Y(z) = z + \frac{\frac{1}{2}z}{z-\frac{1}{4}} = z\left(1 + \frac{\frac{1}{2}}{z-\frac{1}{4}}\right) = \frac{z\left(z+\frac{1}{4}\right)}{\left(z-\frac{1}{4}\right)}$$

$$z^{-1}Y(z) = \frac{\left(z+\frac{1}{4}\right)}{\left(z-\frac{1}{4}\right)\left(z-\frac{1}{2}\right)} = \frac{A}{z-\frac{1}{4}} + \frac{B}{z-\frac{1}{2}}$$

$$A = \left(z-\frac{1}{4}\right)\frac{\left(z+\frac{1}{4}\right)}{\left(z-\frac{1}{4}\right)\left(z-\frac{1}{2}\right)}\Bigg|_{z=\frac{1}{4}} = \frac{\frac{1}{2}}{-\frac{1}{4}} = -2$$

$$B = \left(z-\frac{1}{2}\right)\frac{\left(z+\frac{1}{4}\right)}{\left(z-\frac{1}{4}\right)\left(z-\frac{1}{2}\right)}\Bigg|_{z=\frac{1}{2}} = \frac{\frac{3}{4}}{\frac{1}{4}} = 3$$

$$z^{-1}Y(z) = \frac{-2}{z-\frac{1}{4}} + \frac{3}{z-\frac{1}{2}}$$

$$Y(z) = \frac{-2z}{z-\frac{1}{4}} + \frac{3z}{z-\frac{1}{2}}$$

$$y[n] = -2\left(\frac{1}{4}\right)^n + 3\left(\frac{1}{2}\right)^n \qquad n \geq 0$$

$$x[n+1] \Longleftrightarrow z^{+1}X(z) - x[0]z$$

ZSR & ZIR

$$y[n] - \frac{1}{2} y[n-1] = 2\left(\frac{1}{4}\right)^n u[n] \qquad y[-1] = -2$$

$$Y(z) - \frac{1}{2}\left(z^{-1} Y(z) + y[-1] \right) = 2 \frac{z}{z-\frac{1}{4}}$$

$$\left(1 - \frac{1}{2} z^{-1}\right) Y(z) = \frac{2z}{z-\frac{1}{4}} + \frac{1}{2} y[-1] = \frac{2z}{z-\frac{1}{4}} - 1$$

$$Y(z) = \underbrace{\boxed{\frac{z}{z-\frac{1}{2}} \frac{2z}{z-\frac{1}{4}}}}_{ZSR} \underbrace{\boxed{- \frac{z}{z-\frac{1}{2}}}}_{ZIR}$$

$$z^{-1} Y_{zs}(z) = \frac{2z}{(z-\frac{1}{2})(z-\frac{1}{4})} = \frac{A}{(z-\frac{1}{2})} + \frac{B}{(z-\frac{1}{4})}$$

$$A = (z-\frac{1}{2}) \frac{2z}{(z-\frac{1}{2})(z-\frac{1}{4})} \Bigg|_{z=\frac{1}{2}} = \frac{1}{\frac{1}{4}} = 4$$

$$B = (z-\frac{1}{4}) \frac{2z}{(z-\frac{1}{2})(z-\frac{1}{4})} \Bigg|_{z=\frac{1}{4}} = \frac{\frac{1}{2}}{-\frac{1}{4}} = -2$$

$$Y_{zs}(z) = 4 \frac{z}{(z-\frac{1}{2})} - 2 \frac{z}{(z-\frac{1}{4})}$$

$$y_{zs}[n] = 4\left(\frac{1}{2}\right)^n u[n] - 2\left(\frac{1}{4}\right)^n u[n]$$

$$Y_{zI}(z) = - \frac{z}{z-\frac{1}{2}}$$

$$y_{zI}[n] = -\left(\frac{1}{2}\right)^n u[n]$$

$$y[n] = y_{zs}[n] + y_{zI}[n] = 4\left(\frac{1}{2}\right)^n u[n] - 2\left(\frac{1}{4}\right)^n u[n] - \left(\frac{1}{2}\right)^n u[n]$$

$$= 3\left(\frac{1}{2}\right)^n u[n] - 2\left(\frac{1}{4}\right)^n u[n]$$

Natural and Forced Response

$y[n] - \frac{1}{2}y[n-1] = 2\left(\frac{1}{4}\right)^n u[n]$ $\qquad$ $y[-1] = -2$

force

$y[n] = y_{zs}[n] + y_{zi}[n] = 4$

$= 3\left(\frac{1}{2}\right)^n u[n] - 2\left(\frac{1}{4}\right)^n u[n]$

forced response

natural response

Transfer Function

Steady State Input and Output Relations

$\lim_{n\to\infty} y[n]$, $\lim_{n\to\infty} x[n]$

No transient response by initial conditions

$H(z) = \dfrac{Y(z)}{X(z)}$ with zero state conditions

$$y[n] - \tfrac{1}{2} y[n-1] = 2\left(\tfrac{1}{4}\right)^n u[n] \qquad y[-1] = -2$$

$$Y(z) - \tfrac{1}{2}\left(z^{-1} Y(z) + \cancel{y[-1]}\right) = 2\,\frac{z}{z-\frac{1}{4}}$$

$$\left(1 - \tfrac{1}{2} z^{-1}\right) Y(z) = X(z) = 2\,\frac{z}{z-\frac{1}{4}}$$

$$\frac{Y(z)}{X(z)} = \frac{2z^2}{(z-\frac{1}{2})(z-\frac{1}{4})} = H(z)$$

$Y_{zs}(z)$ $$z^{-1} H(z) = \frac{2z}{(z-\frac{1}{2})(z-\frac{1}{4})} = \frac{A}{(z-\frac{1}{2})} + \frac{B}{(z-\frac{1}{4})}$$

$$A = \left(z-\tfrac{1}{2}\right)\frac{2z}{(z-\frac{1}{2})(z-\frac{1}{4})}\Bigg|_{z=\frac{1}{2}} = \frac{1}{\frac{1}{4}} = 4$$

$$B = \left(z-\tfrac{1}{4}\right)\frac{2z}{(z-\frac{1}{2})(z-\frac{1}{4})}\Bigg|_{z=\frac{1}{4}} = \frac{\frac{1}{2}}{-\frac{1}{4}} = -2$$

$Y_{zs}(z)$ $$H(z) = 4\,\frac{z}{(z-\frac{1}{2})} - 2\,\frac{z}{(z-\frac{1}{4})}$$

$y_{zs}[n]$ $$h[n] = 4\left(\tfrac{1}{2}\right)^n u[n] - 2\left(\tfrac{1}{4}\right)^n u[n]$$

583p
Ambardar

$y[n] - \alpha y[n-1] = x[n]$

$Y(z) - \alpha z^{-1} Y(z) = X(z)$

$H(z) = \dfrac{Y(z)}{X(z)} = \dfrac{1}{1-\alpha z^{-1}} = \dfrac{z}{z-\alpha}$

$x[n] = \alpha^n u[n]$

$X(z) = \dfrac{z}{z-\alpha}$

$Y(z) = H(z)X(z) = \dfrac{z^2}{(z-\alpha)^2}$

$y[n] = (n+1)\alpha^n u[n]$

Step Response