Laurent Series with Residue Theorem

20170713

Based on

T.J. Cavicchi, Digital Signal Processing

Complex Analysis for Mathematics and Engineering
J. Mathews

Residue Theorem

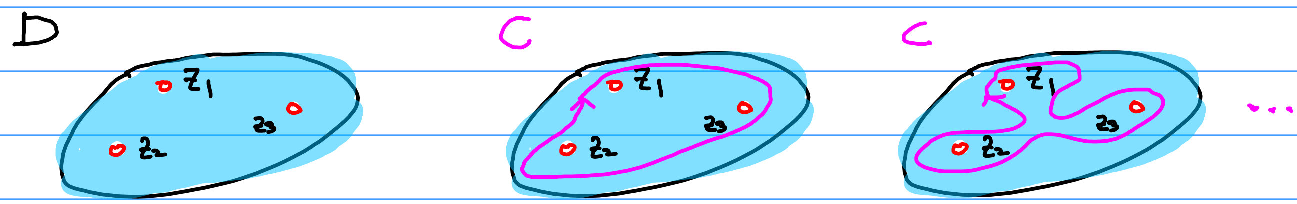
D: simply connected domain
C: simple closed contour (CCW) in D

if $f(z)$ is analytic inside C and on C
except at the points $\boxed{z_1, z_2, \cdots, z_k}$ in C

then

$$\boxed{\frac{1}{2\pi i}\int_C f(z)\,dz = \sum_{j=1}^{k} \mathrm{Res}\left(f(z), z_j\right)}$$

singular points of $f(z)$: $z_1, z_2, \cdots, z_k$ inside C



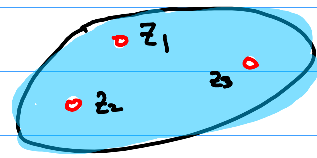
Integration of a function of a complex var.

$$\boxed{\oint_C f(z)\,dz} = 2\pi i \sum_{k=1}^{n} \mathbf{Res}(f(z), z_k)$$

finite number n of

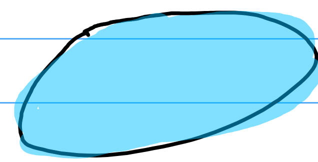
singular points z_k

residue theorem



$$\boxed{\oint_C f(z)\,dz} = 0$$ if $f(z)$ is analytic within and on C

no singularity



$$\boxed{\oint_C f(z)\,dz} = 0$$ if $\boxed{f(z) = F'(z)}$ on C

: $F(z)$ is an antiderivative of $f(z)$

fundamental theorem of calculus

Thomas J. Cavicchi
Digital Signal Processing, Wiley, 2000

$\boxed{\oint_c f(z)\,dz} = 0$ if $f(z)$ is continuous in D and

$\boxed{f(z) = F'(z)}$: $F(z)$ is an antiderivative of $f(z)$

fundamental theorem of calculus

Series Expansion

can expand $f(z)$ about **any point z_m**
over powers of $(z - z_m)$

whether or not $f(z)$ is singular at z_m
or at other point between z and z_m

$$\boxed{f(z) = \sum_{n=n_1}^{\infty} a_n^{\{m\}} (z - z_m)^n}$$

① **Laurent Series** Expansion of $f(z)$ at z_m
general n_1 — depend on $f(z)$ and z_m

② **z-transform** of $a_n^{\{m\}}$
general n_1 — depend on $f(z)$
$\boxed{z_m = 0}$

③ **Taylor Series** Expansion of $f(z)$ at z_m
positive n_1 — depend on $f(z)$ and z_m $(n_1 > 0)$

④ **MacLaurin Series** Expansion of $f(z)$ at z_m
positive n_1 — depend on $f(z)$ $(n_1 > 0)$
$\boxed{z_m = 0}$

Expansion Center, Signs of Powers

$$\boxed{f(z) = \sum_{n=n_1}^{\infty} a_n^{\{m\}} (z - z_m)^n}$$

$n_1 \geq 0$ non-negative power only

	① Laurent Series	③ Taylor Series
$\boxed{z_m = 0}$	② z-transform	④ MacLaurin Series

$a_n^{\{m\}} \longleftarrow z_m$

$a_n^{\{1\}} \longleftarrow z_1$

$a_n^{\{2\}} \longleftarrow z_2$

$\vdots \qquad \vdots$

* Expansion of $f(z)$ about any point z_m over powers of $(z - z_m)$

$$\boxed{f(z) = \sum_{n=n_1}^{\infty} a_n^{\{m\}} (z - z_m)^n}$$

$$a_n^{\{m\}} = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_m)^{n+1}}\, dz$$ for general $f(z)$

$$a_n^{\{m\}} = \sum_k \text{Res}\left[\frac{f(z)}{(z-z_m)^{n+1}},\ z_k\right]$$ for general $f(z)$

$$\boxed{a_n^{\{m\}} = \frac{1}{n!} f^{(n)}(z_m) \qquad n_1 \geq 0}$$ for analytic $f(z)$ within C

analytic $f(z)$ $\longrightarrow$ $\dfrac{f(z)}{(z-z_m)^{n+1}}$ has a **pole** at z_m

order of $n+1$ $(n+1 > 0)$

$(n \geq 0)$

$n_1 \geq 0$

non-negative n_1

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$$\boxed{f(z) = \sum_{n=n_1}^{\infty} a_n^{\{m\}} (z - z_m)^n}$$

$\boxed{z_m}$: possible poles of $f(z)$

(can be non-singular)

$$a_n^{\{m\}} = \frac{1}{2\pi i} \oint_C \frac{f(z')}{(z'-z_m)^{n+1}} \, dz'$$

$$= \sum_k \operatorname{Res}\left(\frac{f(z)}{(z-z_m)^{n+1}}, z_k \right)$$

$\boxed{z_k}$: poles of $\dfrac{f(z)}{(z-z_m)^{n+1}}$

enclosed by C

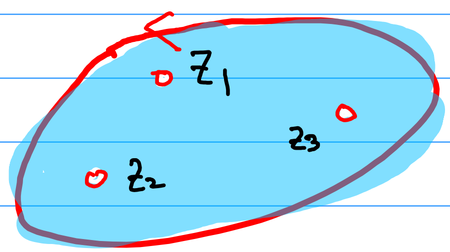
$$= \frac{1}{n!} f^{(n)}(z_m) \qquad n_1 \geq 0$$

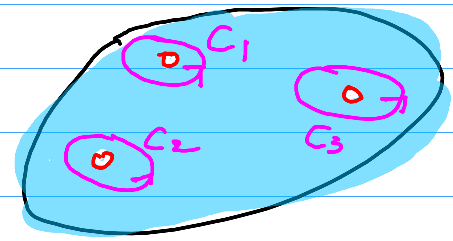
Punctured Open Disks and Residues

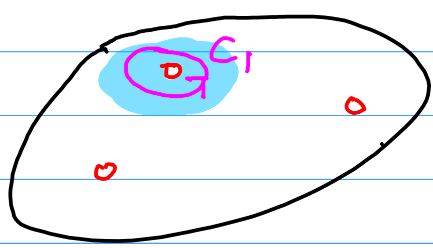
assumed there are K singularities (poles) of $f(z)$ in a region

let $\boxed{C_k}$ is taken to enclose only one pole z_k

C $\qquad\qquad$ C_1, C_2, C_3



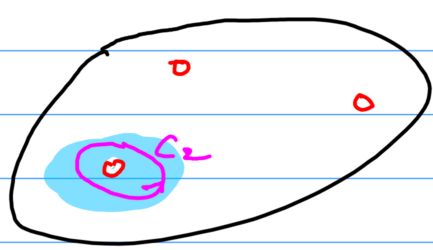




$\tilde{a}_n^{\{1\}}$: expanded at z_1

C_1 encloses z_1 only

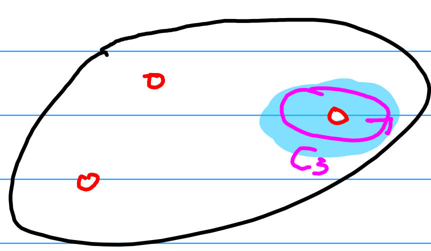
$\tilde{a}_{-1}^{\{1\}} = \mathbf{Res}(f(z), z_1)$



$\tilde{a}_n^{\{2\}}$: expanded at z_2

C_2 encloses z_2 only

$\tilde{a}_{-1}^{\{2\}} = \mathbf{Res}(f(z), z_2)$



$\tilde{a}_n^{\{3\}}$: expanded at z_3

C_3 encloses z_3 only

$\tilde{a}_{-1}^{\{3\}} = \mathbf{Res}(f(z), z_3)$

Cauchy's Residue Theorem

$f(z)$: analytic on and within C

except a finite number of singular points

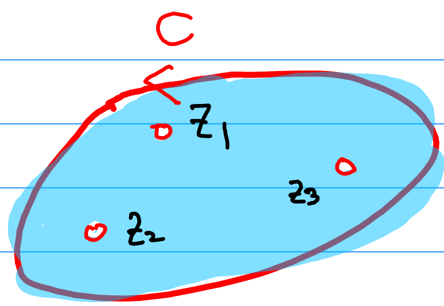
$z_1, z_2, \cdots, z_k$ within C

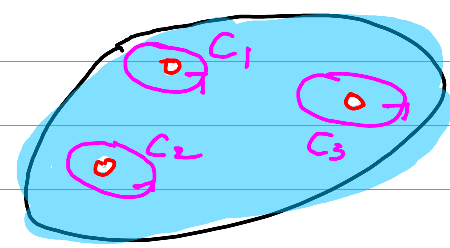
then

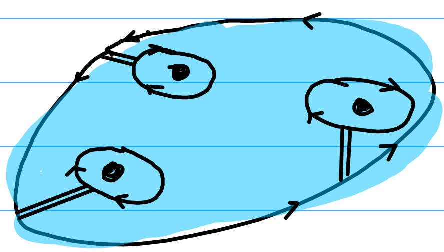
$$\boxed{\int_C f(z)\,dz} = 2\pi i \boxed{\sum_{k=1}^{K} \mathrm{Res}(f(z), z_k)}$$

D : a simply connected domain

C : a simple closed contour in D



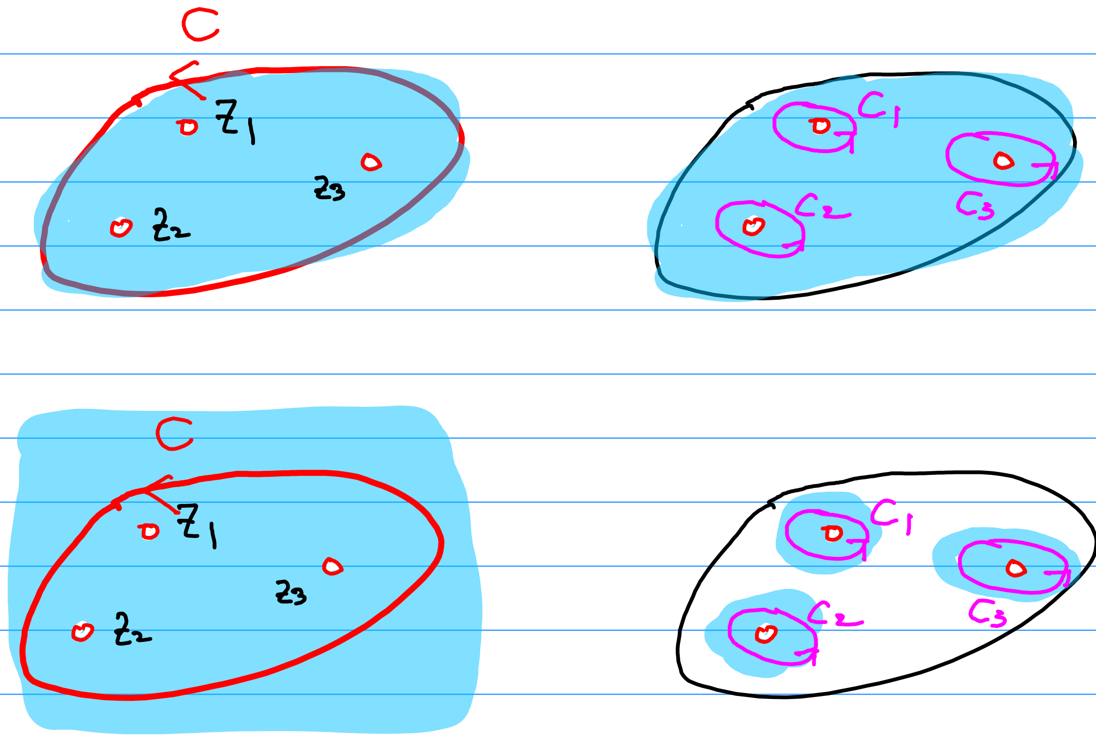


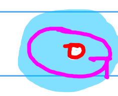


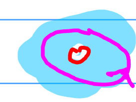
$z_1, z_2, \cdots, z_k$:

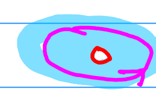
singular points

enclosed by C



the punctured open disk 1 at z_1  C_1

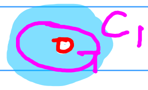
the punctured open disk 2 at z_2  C_2

the punctured open disk 3 at z_3  C_3

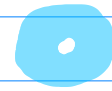
$\rightarrow$ different ROC $\rightarrow$ different Laurent Series

Different Residue, Different Laurent Series

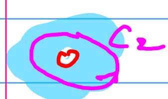
Different Poles, Different ROC's

z_1 

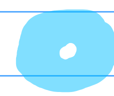
$$f(z) = \sum_{k=-\infty}^{+\infty} a_k^{\{1\}} (z - z_1)^k \quad \text{expanded around } z_1$$

$\boxed{\text{valid}}$ in the punctured open disk at z_1 

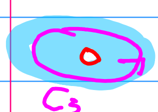
$$a_{-1}^{\{1\}} = \frac{1}{2\pi i} \oint_{C_1} f(s)\, ds = \mathbf{Res}(f(z), z_1)$$

z_2 

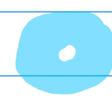
$$f(z) = \sum_{k=-\infty}^{+\infty} a_k^{\{2\}} (z - z_2)^k \quad \text{expanded around } z_2$$

$\boxed{\text{valid}}$ in the punctured open disk at z_2 

$$a_{-1}^{\{2\}} = \frac{1}{2\pi i} \oint_{C_2} f(s)\, ds = \mathbf{Res}(f(z), z_2)$$

z_3 

$$f(z) = \sum_{k=-\infty}^{+\infty} a_k^{\{3\}} (z - z_3)^k \quad \text{expanded around } z_3$$

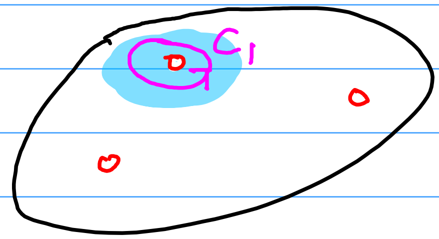
$\boxed{\text{valid}}$ in the punctured open disk at z_3 

$$a_{-1}^{\{3\}} = \frac{1}{2\pi i} \oint_{C_3} f(s)\, ds = \mathbf{Res}(f(z), z_3)$$

Residue at a pole → Laurent series expanded at that pole

$\boxed{z_1}$ Laurent series expansion at z_1

$$\boxed{f(z)=\sum_{n=-\infty}^{+\infty}\tilde{a}_n^{\{1\}}(z-z_1)^n}$$

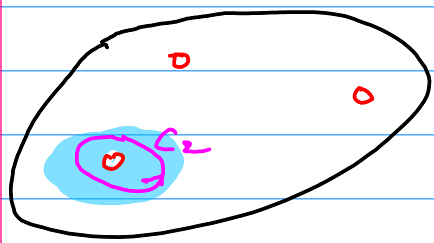


$$\tilde{a}_{-1}^{\{1\}} = \mathbf{Res}(f(z), z_1)$$

$$= \frac{1}{2\pi i}\oint_{C_1} f(z)\,dz$$

$\boxed{z_2}$ Laurent series expansion at z_2

$$\boxed{f(z)=\sum_{n=-\infty}^{+\infty}\tilde{a}_n^{\{2\}}(z-z_2)^n}$$

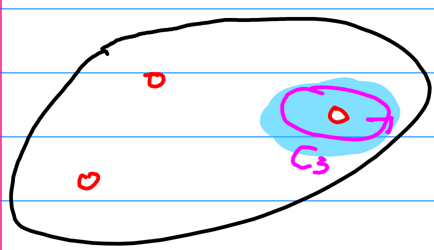


$$\tilde{a}_{-1}^{\{2\}} = \mathbf{Res}(f(z), z_2)$$

$$= \frac{1}{2\pi i}\oint_{C_2} f(z)\,dz$$

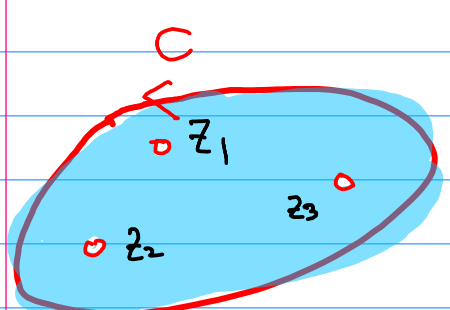
$\boxed{z_3}$ Laurent series expansion at z_3

$$\boxed{f(z)=\sum_{n=-\infty}^{+\infty}\tilde{a}_n^{\{3\}}(z-z_3)^n}$$



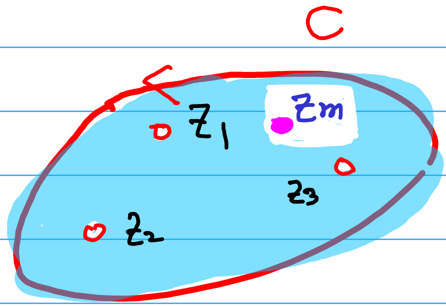
$$\tilde{a}_{-1}^{\{3\}} = \mathbf{Res}(f(z), z_3)$$

$$= \frac{1}{2\pi i}\oint_{C_3} f(z)\,dz$$



$$\int_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \mathbf{Res}(f(z), z_k)$$

Residue Theorem + Laurent Series



* whether (z_m) is singular or not

$$f(z) = \sum_{n=n_1}^{\infty} \boxed{a_n^{\{m\}}} (z - z_m)^n$$

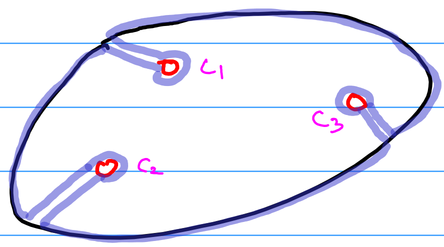
$$\boxed{a_n^{\{m\}}} = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_m)^{n+1}} dz$$

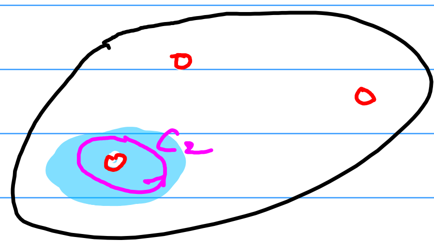
$$= \sum_k \mathbf{Res}\left(\frac{f(z)}{(z-z_m)^{n+1}}, z_k\right)$$

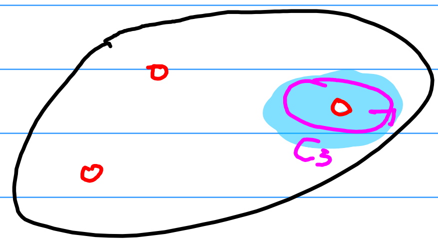
$n=-1$

$$a_{-1}^{\{m\}} = \frac{1}{2\pi i} \oint_C f(z)\, dz$$

$$= \sum_k \mathbf{Res}\left(f(z), z_k\right)$$







$\tilde{a}_{-1}^{\{1\}} = \mathbf{Res}(f(z), z_1)$ $\quad$ $\tilde{a}_{-1}^{\{2\}} = \mathbf{Res}(f(z), z_2)$ $\quad$ $\tilde{a}_{-1}^{\{3\}} = \mathbf{Res}(f(z), z_3)$

$$\boxed{a_{-1}^{\{m\}} = \tilde{a}_{-1}^{\{1\}} + \tilde{a}_{-1}^{\{2\}} + \tilde{a}_{-1}^{\{3\}}}$$

$$a_{-1}^{\{m\}} = \mathbf{Res}(f(z), z_1) + \mathbf{Res}(f(z), z_2) + \mathbf{Res}(f(z), z_3)$$

Laurent Series coefficient $\tilde{a}_{-1}^{\{i\}}$

- singular center z_i
- punctured open disk

We do not say this $a_{-1}^{\{m\}}$ is a residue

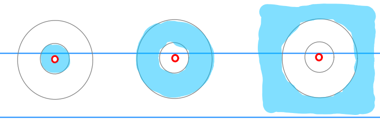
because it is not

isolalated singular center nor punctured open disk ROC

Laurent Series

Annular Region of Convergence

no singularity in this region

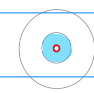


can be expanded at a singular/non-singular center

this point need not be in the convergence region

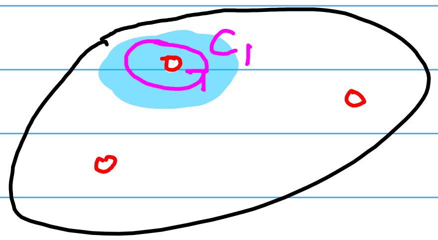
Residue

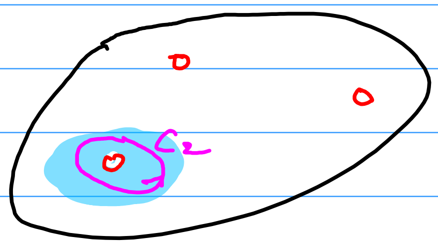
over a punctured open disk

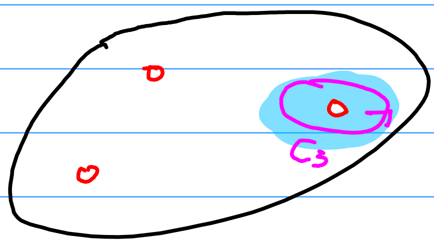


and thus annular region

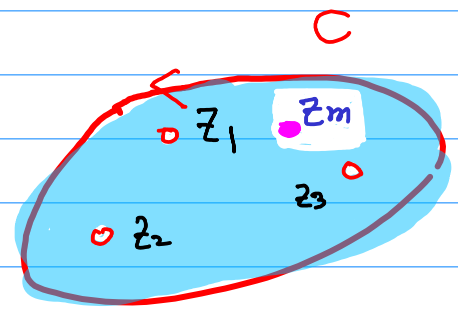
must expanded at a pole (a singular point)







$\tilde{a}^{\{1\}}_{-1} = \mathbf{Res}(f(z), z_1)$ $\tilde{a}^{\{2\}}_{-1} = \mathbf{Res}(f(z), z_2)$ $\tilde{a}^{\{3\}}_{-1} = \mathbf{Res}(f(z), z_3)$



$\tilde{a}^{\{m\}}_{-1} = \mathbf{Res}(f(z), z_m)$

when z_m is a pole

$\cancel{\tilde{a}^{\{m\}}_{-1} = \mathrm{Res}(f(z), z_m)}$

when z_m is not a pole

$$\dots,\ a_{-2}^{\{m\}},\ a_{-1}^{\{m\}},\ a_{0}^{\{m\}},\ a_{+1}^{\{m\}},\ a_{+2}^{\{m\}},\ \dots$$

$$\boxed{f(z) = \sum_{n=n_1}^{\infty} a_n^{\{m\}} (z - z_m)^n}$$

$$\boxed{\begin{aligned} a_n^{\{m\}} &= \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_m)^{n+1}}\, dz \\ &= \sum_k \mathrm{Res}\left(\frac{f(z)}{(z-z_m)^{n+1}},\ z_k\right) \end{aligned}}$$

(general formula)

no specific ROC's

$$\boxed{\begin{aligned} a_{-1}^{\{m\}} &= \frac{1}{2\pi i} \oint_C f(z)\, dz \\ &= \sum_k \mathrm{Res}\left(f(z),\ z_k\right) \end{aligned}}$$

$\vdots$

z_k : poles of $\dfrac{f(z)}{(z-z_m)^{n+1}}$

$$a_{-3}^{\{m\}} = \sum_k \mathrm{Res}\left(f(z)(z-z_m)^2,\ z_k\right)$$

$$a_{-2}^{\{m\}} = \sum_k \mathrm{Res}\left(f(z)(z-z_m)^1,\ z_k\right)$$

$$a_{-1}^{\{m\}} = \sum_k \mathrm{Res}\left(f(z),\ z_k\right)$$

$$a_{0}^{\{m\}} = \sum_k \mathrm{Res}\left(\frac{f(z)}{(z-z_m)^1},\ z_k\right)$$

$$a_{1}^{\{m\}} = \sum_k \mathrm{Res}\left(\frac{f(z)}{(z-z_m)^2},\ z_k\right)$$

$$a_{2}^{\{m\}} = \sum_k \mathrm{Res}\left(\frac{f(z)}{(z-z_m)^3},\ z_k\right)$$

$\vdots$

Involved Laurent Series

$\vdots$

$$a_{-3}^{\{m\}} = \sum_k \mathrm{Res}\left(f(z)(z-z_m)^2 ,\ z_k \right)$$

Laurent Series of $\boxed{f(z)(z-z_m)^2}$

z_k : poles of $\boxed{f(z)(z-z_m)^2}$

$$a_{-2}^{\{m\}} = \sum_k \mathrm{Res}\left(f(z)(z-z_m)^1 ,\ z_k \right)$$

Laurent Series of $\boxed{f(z)(z-z_m)^1}$

z_k : poles of $\boxed{f(z)(z-z_m)^1}$

$$a_{-1}^{\{m\}} = \sum_k \mathrm{Res}\left(f(z) ,\ z_k \right)$$

Laurent Series of $\boxed{f(z)}$

at z_k : poles of $\boxed{f(z)}$

$$a_{0}^{\{m\}} = \sum_k \mathrm{Res}\left(\frac{f(z)}{(z-z_m)^1} ,\ z_k \right)$$

Laurent Series of $\boxed{\frac{f(z)}{(z-z_m)^1}}$

at z_k : poles of $\boxed{\frac{f(z)}{(z-z_m)^1}}$

$$a_{1}^{\{m\}} = \sum_k \mathrm{Res}\left(\frac{f(z)}{(z-z_m)^2} ,\ z_k \right)$$

Laurent Series of $\boxed{\frac{f(z)}{(z-z_m)^2}}$

at z_k : poles of $\boxed{\frac{f(z)}{(z-z_m)^2}}$

$$a_{2}^{\{m\}} = \sum_k \mathrm{Res}\left(\frac{f(z)}{(z-z_m)^3} ,\ z_k \right)$$

Laurent Series of $\boxed{\frac{f(z)}{(z-z_m)^3}}$

at z_k : poles of $\boxed{\frac{f(z)}{(z-z_m)^3}}$

$\vdots$

Computing $a_n^{\{m\}}$

$$\boxed{\oint_C \frac{1}{(z-z_0)^n}\,dz = \begin{cases} 2\pi i & n=1 \\ 0 & n\neq 1 \end{cases}}$$

— simple pole z_0
— deformation of contour

$$\oint_C \left[\boxed{\cdots (z-z_m)^{-3} + (z-z_m)^{-2}}^{\to 0} + \boxed{\frac{1}{(z-z_m)}} + \boxed{1 + (z-z_m) + (z-z_m)^2 + \cdots}^{\to 0}\right] dz$$

$$= \oint_C \boxed{\frac{1}{(z-z_m)}}\,dz = 2\pi i$$

for a given n

$$\boxed{\oint_C \frac{f(z)}{(z-z_m)^{n+1}}\,dz} = \oint_C \sum_{k=n_1}^{\infty} a_k^{\{m\}} (z-z_m)^{k-n-1}\,dz$$

$$= \boxed{\sum_{k=n_1}^{\infty} \oint_C a_k^{\{m\}} (z-z_m)^{k-n-1}\,dz}$$

Only one term left $\boxed{k=n}$ $\leftarrow n$

$$\boxed{\oint_C \frac{f(z)}{(z-z_m)^{n+1}}\,dz} = \boxed{\oint_C a_n^{\{m\}} \frac{1}{(z-z_m)}\,dz} = \boxed{2\pi i \cdot a_n^{\{m\}}}$$

$$f(z) = \sum_{n=n_1}^{\infty} a_n^{\{m\}} (z - z_m)^n$$

$$\boxed{n \leftarrow k}$$

$$f(z) = \sum_{k=n_1}^{\infty} a_k^{\{m\}} (z - z_m)^k$$

for a given n

$$\frac{f(z)}{(z-z_m)^{n+1}} = \sum_{k=n_1}^{\infty} a_k^{\{m\}} (z - z_m)^{k-n-1}$$

k : index variable
n : fixed value

$$\boxed{\oint_C \frac{f(z)}{(z-z_m)^{n+1}} dz} = \oint_C \sum_{k=n_1}^{\infty} a_k^{\{m\}} (z - z_m)^{k-n-1} dz$$

$$= \boxed{\sum_{k=n_1}^{\infty} \oint_C a_k^{\{m\}} (z - z_m)^{k-n-1} dz}$$

$$\boxed{k = n}$$

$$\boxed{\oint_C \frac{f(z)}{(z-z_m)^{n+1}} dz} = \boxed{\oint_C a_n^{\{m\}} \frac{1}{(z - z_m)} dz} = \boxed{2\pi i \cdot a_n^{\{m\}}}$$

$$\boxed{a_n^{\{m\}} = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_m)^{n+1}} dz}$$

cf) Cauchy's Integral Formula

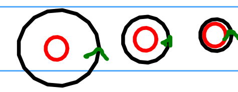
$f(z)$: analytic in a simply connected domain D,

C : a simple closed contour that is entirely within D

z_0 : any point within C

deformation

$z \to z_0$



$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0} dz$$

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz \qquad \frac{d}{dz_0} f(z_0)$$

$$f(z) = \sum_{n=n_1}^{\infty} a_n (z - z_m)^n \qquad a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_m)^{n+1}} dz$$

	$n=-1$	$n=0$	$n=1$
$f(z)$	$\boxed{f(z)/(z-z_m)^0}$	$\boxed{f(z)/(z-z_m)^1}$	$\boxed{f(z)/(z-z_m)^2}$
$= \cdots$	$= \cdots$	$= \cdots$	$= \cdots$
$+ a_{-2} (z-z_m)^{-2}$	$+ a_{-2} (z-z_m)^{-2}$	$+ a_{-2} (z-z_m)^{-3}$	$+ a_{-2} (z-z_m)^{-4}$
$+ a_{-1} (z-z_m)^{-1}$	$+ a_{-1} (z-z_m)^{-1}$	$+ a_{-1} (z-z_m)^{-2}$	$+ a_{-1} (z-z_m)^{-3}$
$+ a_0 (z-z_m)^0$	$+ a_0 (z-z_m)^0$	$+ a_0 (z-z_m)^{-1}$	$+ a_0 (z-z_m)^{-2}$
$+ a_1 (z-z_m)^1$	$+ a_1 (z-z_m)^1$	$+ a_1 (z-z_m)^0$	$+ a_1 (z-z_m)^{-1}$
$+ a_2 (z-z_m)^2$	$+ a_2 (z-z_m)^2$	$+ a_2 (z-z_m)^1$	$+ a_2 (z-z_m)^0$
$\cdots$	$\cdots$	$\cdots$	$\cdots$
	$\downarrow$ $\boxed{a_{-1}}$	$\downarrow$ $\boxed{a_0}$	$\downarrow$ $\boxed{a_1}$

$$f(z) = \sum_{n=n_1}^{\infty} a_n (z - z_m)^n$$

$$= \cdots + a_{-1}(z-z_m)^{-1} + a_0 (z-z_m)^0 + a_1 (z-z_m)^1 + \cdots$$

* if $f(z)$ analytic within C, then no poles → no negative powers $\cancel{(n<0)}$

$$f(z) = a_0 (z-z_m)^0 + a_1 (z-z_m)^1 + \cdots$$

$$\boxed{f(z_m) = a_0} \Rightarrow \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_m)^{0+1}}\, dz$$

$$f(z_m) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_m)^1}\, dz$$

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0}\, dz$$

* if $f(z)$ analytic within C, then no poles

$\downarrow$ no negative powers $\cancel{(n<0)}$

$$f(z) = a_0 (z - z_m)^0 + a_1 (z - z_m)^1 + \cdots$$

$n < 0 \qquad n = -1, -2, -3 \cdots$

$-n - 1 \geq 0 \qquad -n - 1 = 0, 1, 2, 3$

$(z - z_m)^{n+1}$ positive power $\rightarrow$ no pole

$f(z)$ analytic within C (assumed)

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_m)^{n+1}} \, dz$$

$$= \frac{1}{2\pi i} \oint_C f(z) \, (z - z_m)^{n+1} \, dz$$

no poles   not poles

$a_n = 0 \qquad n < 0 \qquad n = -1, -2, -3 \cdots$

$a_n = 0 \Rightarrow$ no negative powers

Computing $a_n^{\{m\}}$ using Residues

expansion at z_m

$$\boxed{\begin{aligned} a_n^{\{m\}} &= \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_m)^{n+1}}\,dz \\ &= \sum_k \mathrm{Res}\left(\frac{f(z)}{(z-z_m)^{n+1}}, z_k\right) \end{aligned}}$$

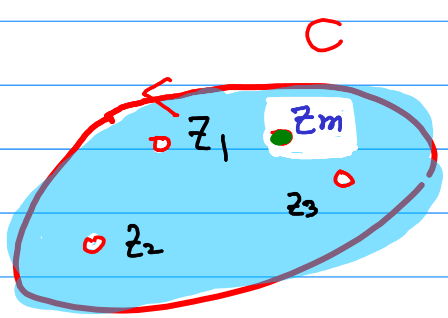
$\boxed{n=-1}$ $n+1=0$ $(z-z_m)^{n+1}=1$

$$\boxed{\begin{aligned} a_{-1}^{\{m\}} &= \frac{1}{2\pi i}\oint_C f(z)\,dz \\ &= \sum_k \mathrm{Res}\left(f(z), z_k\right) \end{aligned}}$$

$$a_{-1}^{\{m\}} = \frac{1}{2\pi i}\oint_C f(z)\,dz = \sum_k \mathrm{Res}(f(z), z_k)$$

$$a_{-1}^{\{m\}} = \cancel{\mathrm{Res}(f(z), z_m)}$$

We do not say this is a residue

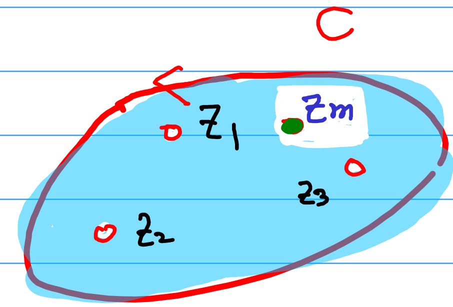


$$\left\{\begin{array}{l} \text{singular } z_m \\ \text{non-singular } z_m \end{array}\right.$$

$\boxed{\begin{array}{l} z_1, z_2, \cdots, z_k: \\ \text{singular points} \\ \text{enclosed by } C \end{array}}$

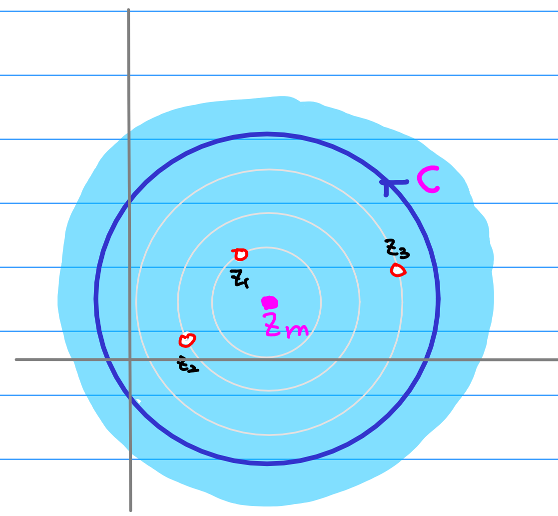
Residue → Laurent series → annular region → expanded at a pole ☆) a punctured open disk

Possible Region of Convergence and Contour C



$$f(z) = \sum_{n=n_1}^{\infty} a_n^{\{m\}} (z - z_m)^n$$

$$a_n^{\{m\}} = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_m)^{n+1}}\, dz$$
$$= \sum_k \mathrm{Res}\left(\frac{f(z)}{(z-z_m)^{n+1}},\, z_k\right)$$



$$a_{-1}^{\{m\}} = \frac{1}{2\pi i} \oint_C f(z)\, dz$$
$$= \sum_k \mathrm{Res}\left(f(z),\, z_k\right)$$

Cauchy Residue Theorem

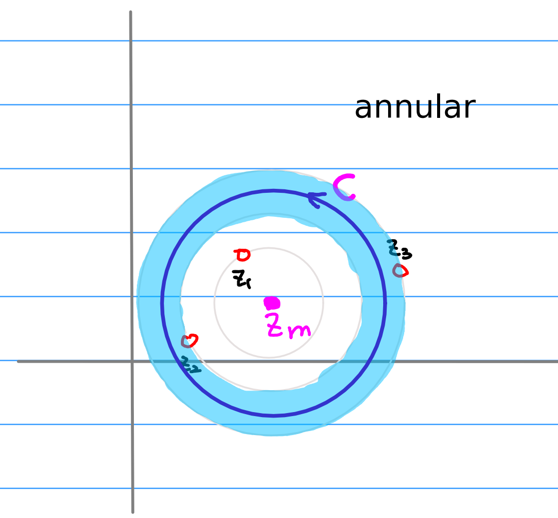
$f(z)$: analytic on and within C

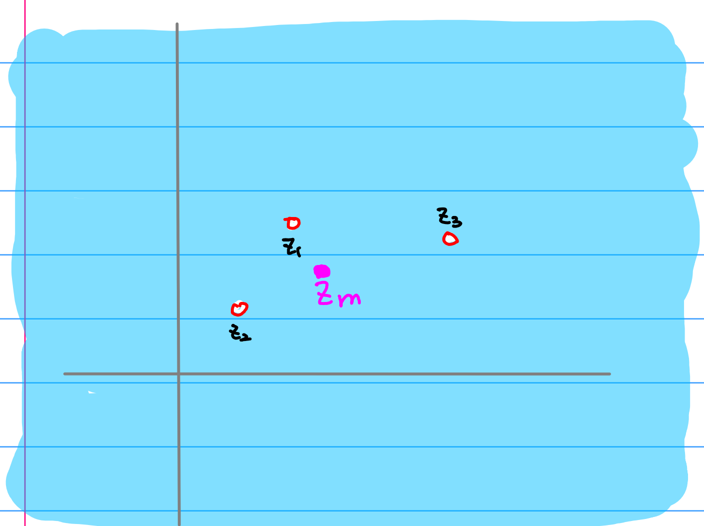
<u>except</u> a finite number of

singular points

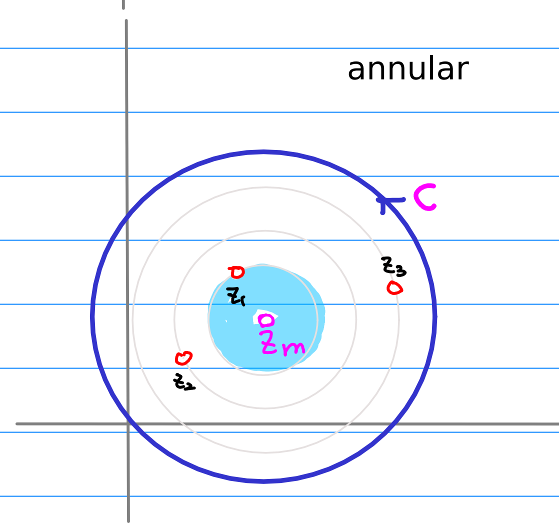
$z_1, z_2, \cdots, z_k$ within C

annular





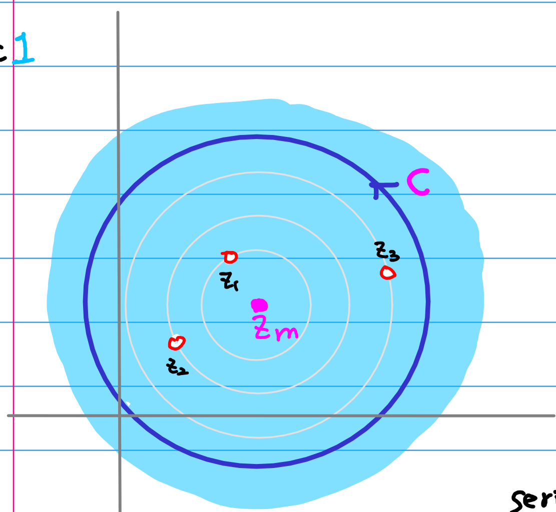
annular

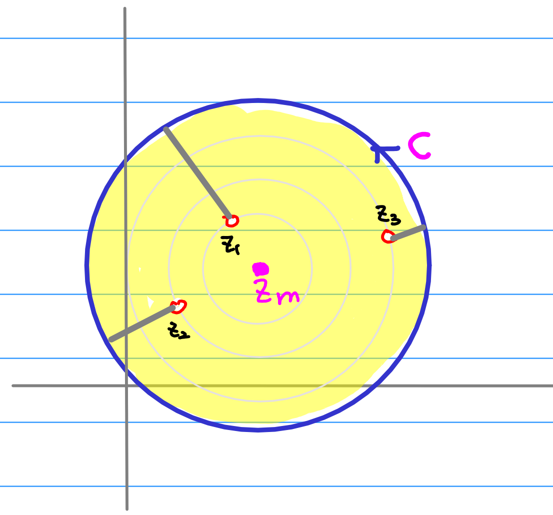


the same set of residues
different ROC's

Valid Region | **Cauchy Region**

Roc 1



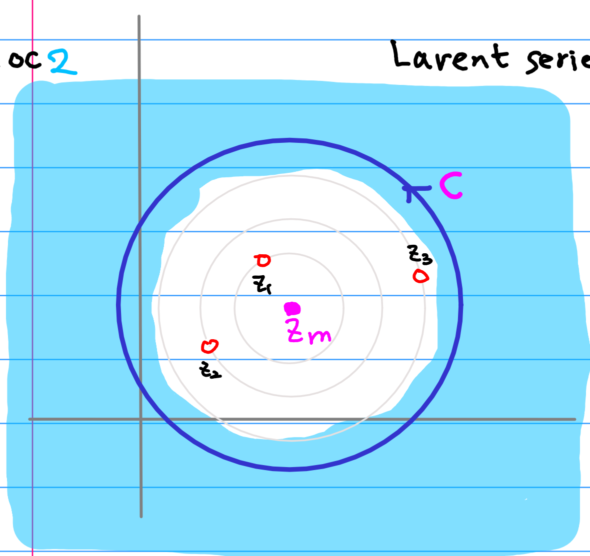


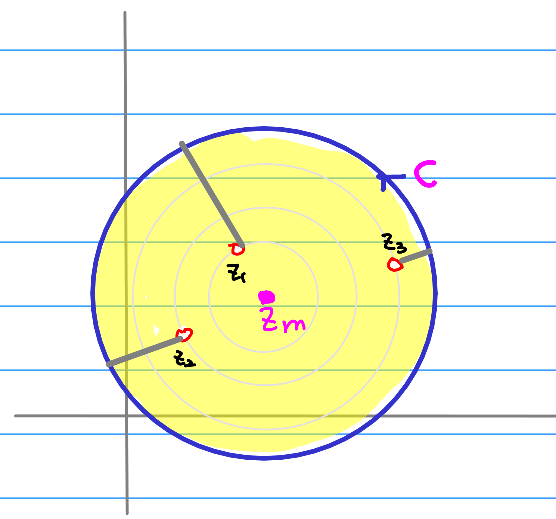
series w/ Roc 1

||| |||

Roc 2

Larent series w/ Roc 2





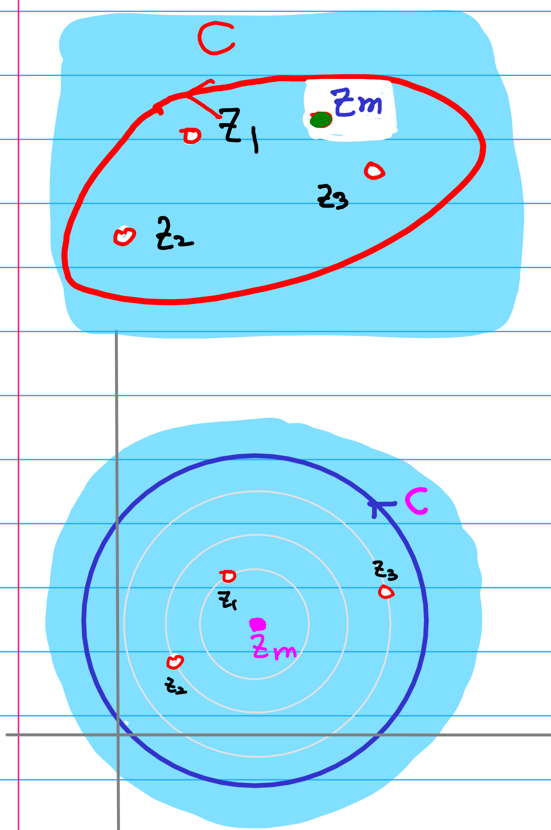
annular

$$\boxed{f(z) = \sum_{n=n_1}^{\infty} a_n^{\{m\}} (z - z_m)^n}$$

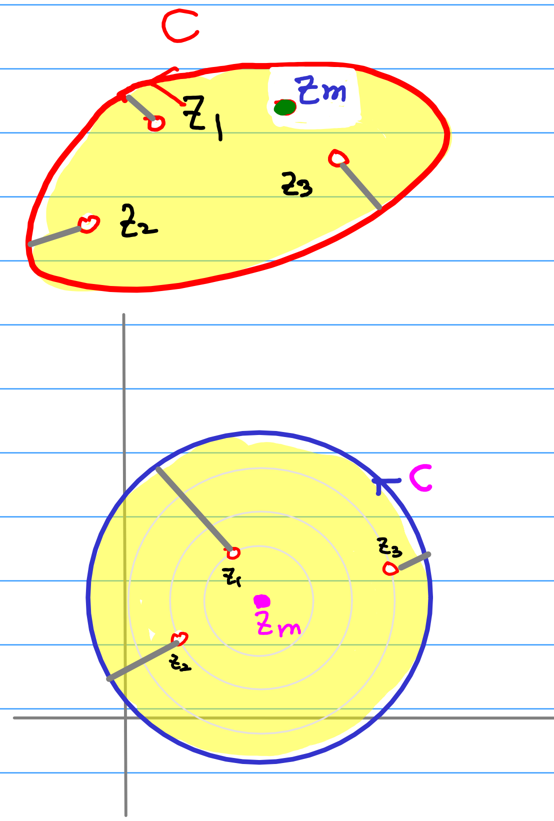
$$a_n^{\{m\}} = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_m)^{n+1}}\, dz$$

$$= \sum_k \mathrm{Res}\left(\frac{f(z)}{(z - z_m)^{n+1}},\ z_k \right)$$

ROC

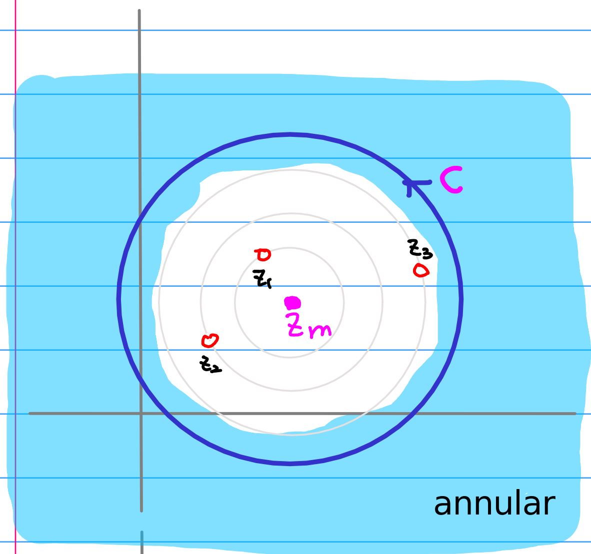


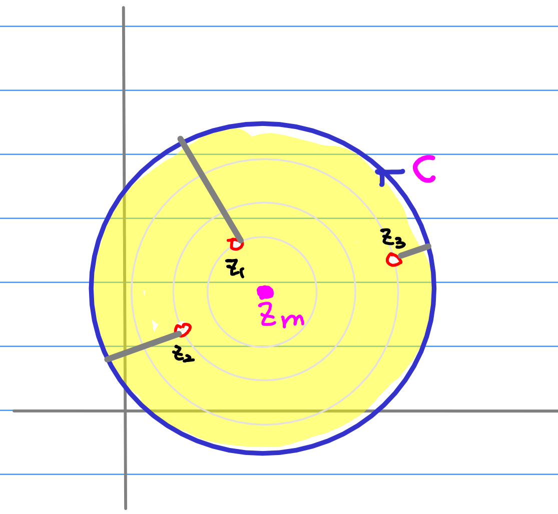
Poles to be counted

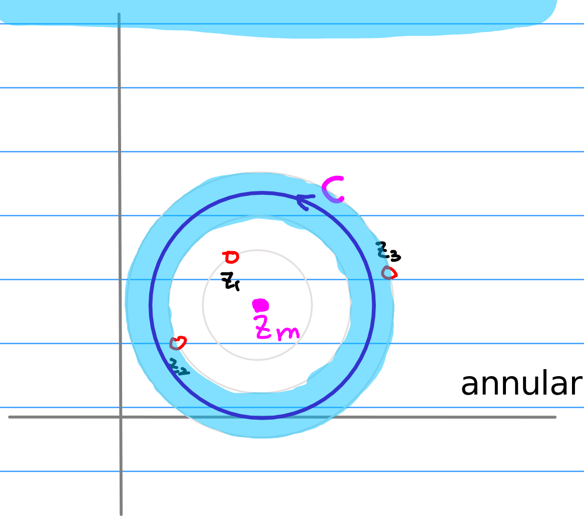


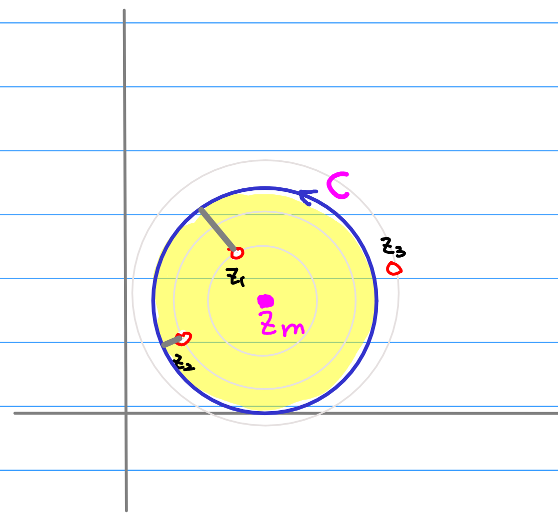
ROC

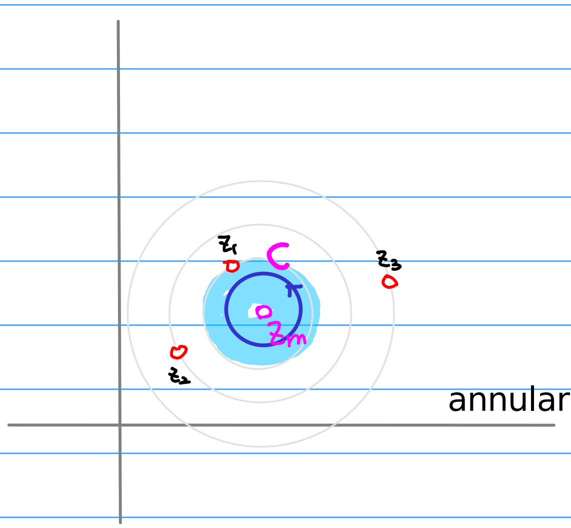
Poles to be counted

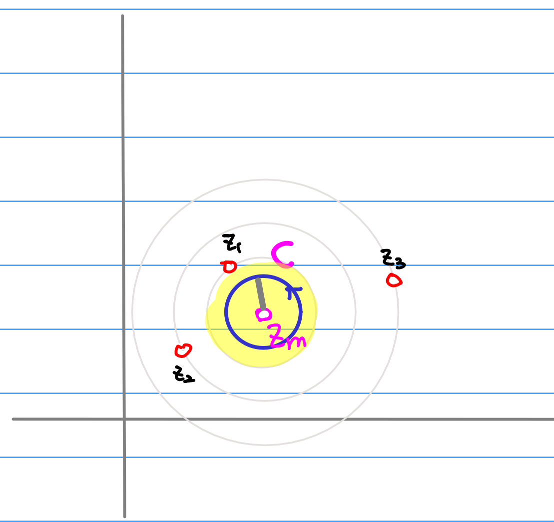












Poles used in Residue Computation

$$\boxed{f(z) = \sum_{n=n_1}^{\infty} a_n^{\{m\}} (z - z_m)^n}$$

$$\boxed{\begin{aligned} a_n^{\{m\}} &= \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_m)^{n+1}} \, dz \\ &= \sum_k \mathrm{Res}\left(\boxed{\frac{f(z)}{(z-z_m)^{n+1}}}, z_k \right) \end{aligned}}$$

z_k enclosed by C : singularities of $\boxed{\dfrac{f(z)}{(z-z_m)^{n+1}}}$

Ⓘ non-singular z_m

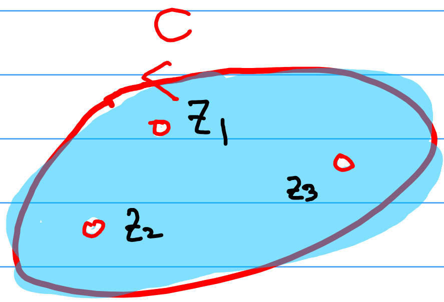
$$\begin{cases} n \geq 0 & \{\text{poles of } f(z)\} \cup \{ z_m \} \qquad n = 0, 1, 2 \cdots \\ n < 0 & \{\text{poles of } f(z)\} \qquad n = -1, -2, \cdots \end{cases}$$

Ⓘⓘ singular z_m

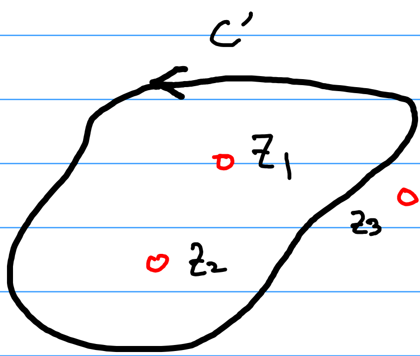
$\{\text{poles of } f(z)\}$

z_m included already

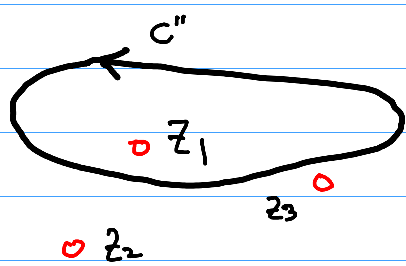
Various Contours (non-annular)



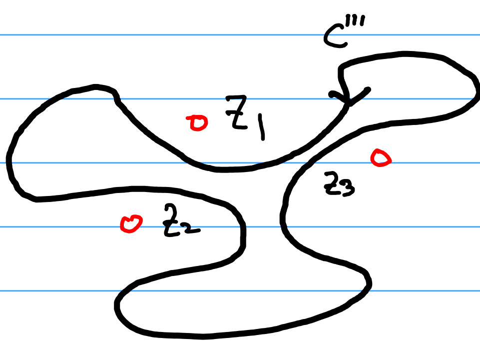
$$\int_C f(z)\,dz = 2\pi i \,\mathrm{Res}(f(z), z_1) + 2\pi i \,\mathrm{Res}(f(z), z_2) + 2\pi i \,\mathrm{Res}(f(z), z_3)$$



$$\int_{C'} f(z)\,dz = 2\pi i \,\mathrm{Res}(f(z), z_1) + 2\pi i \,\mathrm{Res}(f(z), z_2)$$



$$\int_{C''} f(z)\,dz = 2\pi i \,\mathrm{Res}(f(z), z_1)$$



$$\int_{C''} f(z)\,dz = 0$$