

# Power Density Spectrum - Discrete Time

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Based on  
Probability, Random Variables and Random Signal Principles,  
P.Z. Peebles,Jr. and B. Shi



# Bilateral z-Transform of $R_{XX}[n]$

$N$  Gaussian random variables

## Definition

$$S_{XX}(z) = \sum_{n=-\infty}^{\infty} R_{XX}[n]z^{-n}$$

# Discrete Time Fourier Transform of $R_{XX}[n]$

$N$  Gaussian random variables

## Definition

$$S_{XX}(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} R_{XX}[n] e^{-jn\Omega}$$

$$R_{XX}[n] = \frac{1}{2\pi} \int_{-\pi}^{+\pi} S_{XX}(e^{j\Omega}) e^{jn\Omega} d\Omega$$

# Properties of Power Density Spectrum - DT

$N$  Gaussian random variables

- 1  $S_{XX}(e^{j\Omega}) \geq 0$
- 2  $S_{XX}(e^{-j\Omega}) = S_{XX}(e^{+j\Omega})$  for real  $X[n]$
- 3  $S_{XX}(e^{+j\Omega})$  is real
- 4  $\frac{1}{2\pi} \int_{-\pi}^{+\pi} S_{XX}(e^{j\Omega}) d\Omega = E[X^2[n]]$

# Estimating the Power Density Spectrum

$N$  Gaussian random variables

## Definition

$$\hat{R}_N[k] = \frac{1}{N} \sum_{n=0}^{N-1-|k|} X[n]X[n+|k|] \quad |k| < N$$

## Definition

$$X_N(\Omega_k) = \sum_{n=0}^{N-1} X[n] e^{-j\Omega_k n} \quad k = 0, 1, \dots, N-1$$

$$\Omega_k = \frac{2\pi k}{N} \quad k = 0, 1, \dots, N-1$$



# Periodogram

$N$  Gaussian random variables

## Definition

Periodogram : the estimate of the power density spectrum

$$\hat{S}_N(\Omega_k) = \frac{1}{N} |X_N(\Omega_k)|^2 \quad k = 0, 1, \dots, N-1$$

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$$\lim_{N \rightarrow \infty} E \left[ \hat{S}_N(\Omega_k) \right] = S_{XX}(\Omega_k) \quad k = 0, 1, \dots, N-1$$

# Periodogram Proof (1)

$N$  Gaussian random variables

## Definition

$$\begin{aligned} E \left[ \widehat{S}_N(\Omega_k) \right] &= \frac{1}{N} E \left[ \left\{ \sum_{n=0}^{N-1} X[n] e^{-j\Omega_k n} \right\} \left\{ \sum_{m=0}^{N-1} X[m] e^{-j\Omega_k m} \right\} \right] \\ &= \frac{1}{N} \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} E[X[n]X[m]] e^{-j\Omega_k(n-m)} \\ &= \frac{1}{N} \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} R_{XX}[n-m] e^{-j\Omega_k(n-m)} \end{aligned}$$

# Periodogram Proof (2)

$N$  Gaussian random variables

## Definition

$$\begin{aligned} E \left[ \widehat{S}_N(\Omega_k) \right] &= \frac{1}{N} \sum_{n=-(N-1)}^{+(N-1)} (N - |k|) R_{XX}[k] e^{-j\Omega_k k} \\ &= \frac{N}{N} \sum_{n=-(N-1)}^{+(N-1)} R_{XX}[k] e^{-j\Omega_k k} - \frac{1}{N} \sum_{n=-(N-1)}^{+(N-1)} |k| R_{XX}[k] e^{-j\Omega_k k} \end{aligned}$$

# The Variance of the Periodogram

$N$  Gaussian random variables

## Definition

$$\sigma_S^2 = E \left[ \widehat{S}_N(\Omega_k)^2 \right] - E \left[ \widehat{S}_N(\Omega_k) \right]^2$$

$$E \left[ \widehat{S}_N(\Omega_k)^2 \right] = E \left[ \left( \frac{1}{N} \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} E[X[n]X[m]] e^{-j\Omega_k(n-m)} \right)^2 \right]$$

$$= \frac{1}{N^2} \sum_{n,m,p,q} E[X[n]X[m]X[p]X[q]] e^{-j\Omega_k(n-m)} e^{-j\Omega_k(q-p)}$$



