Power Density Spectrum - Discrete Time

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Based on
Probability, Random Variables and Random Signal Principles,
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Outline

Definition

$$S_{XX}(z) = \sum_{n=-\infty}^{\infty} R_{XX}[n] z^{-n}$$

Definition

$$S_{XX}(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} R_{XX}[n] e^{-jn\Omega}$$

$$R_{XX}[n] = \frac{1}{2\pi} \int_{-\pi}^{+\pi} S_{XX}(e^{j\Omega}) e^{jn\Omega} d\Omega$$

1. $S_{XX}(e^{j\Omega}) \geq 0$
2. $S_{XX}(e^{-j\Omega}) = S_{XX}(e^{+j\Omega})$ for real $X[n]$
3. $S_{XX}(e^{+j\Omega})$ is real
4. $\frac{1}{2\pi}\int_{-\pi}^{+\pi} S_{XX}(e^{j\Omega})d\Omega = E[X^2[n]]$

Definition

$$\hat{R}_N[k] = \frac{1}{N} \sum_{n=0}^{N-1-|k|} X[n]X[n+|k|] \qquad |k| < N$$

Definition

$$X_N(\Omega_k) = \sum_{n=0}^{N-1} X[n] e^{-j\Omega_k n} \qquad k = 0, 1, ..., N-1$$

$$\Omega_k = \frac{2\pi k}{N} \qquad k = 0, 1, ..., N-1$$

Definition

Periodogram : the estimate of the power density spectrum

$$\widehat{S}_N(\Omega_k) = \frac{1}{N}|X_N(\Omega_k)|^2 \qquad k = 0, 1, ..., N-1$$

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$$\lim_{N\to\infty} E\left[\widehat{S}_N(\Omega_k)\right] = S_{XX}(\Omega_k) \qquad k = 0, 1, ..., N-1$$

Periodogram Proof (1)

N Gaussian random variables

Definition

$$E\left[\widehat{S}_N(\Omega_k)\right] = \frac{1}{N} E\left[\left\{\sum_{n=0}^{N-1} X[n] e^{-j\Omega_k n}\right\}\left\{\sum_{m=0}^{N-1} X[m] e^{-j\Omega_k m}\right\}\right]$$

$$= \frac{1}{N} \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} E\left[X[n] X[m]\right] e^{-j\Omega_k (n-m)}$$

$$= \frac{1}{N} \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} R_{XX}\left[n-m\right] e^{-j\Omega_k (n-m)}$$

Definition

$$E\left[\widehat{S}_N(\Omega_k)\right] = \frac{1}{N} \sum_{n=-(N-1)}^{+(N-1)} (N - |k|) R_{XX}[k] e^{-j\Omega_k k}$$

$$= \frac{N}{N} \sum_{n=-(N-1)}^{+(N-1)} R_{XX}[k] e^{-j\Omega_k k} - \frac{1}{N} \sum_{n=-(N-1)}^{+(N-1)} |k| R_{XX}[k] e^{-j\Omega_k k}$$

Definition

$$\sigma_S^2 = E\left[\widehat{S}_N(\Omega_k)^2\right] - E\left[\widehat{S}_N(\Omega_k)\right]^2$$

$$E\left[\widehat{S}_N(\Omega_k)^2\right] = E\left[\left(\frac{1}{N}\sum_{n=0}^{N-1}\sum_{m=0}^{N-1} E\left[X[n]X[m]\right] e^{-j\Omega_k(n-m)}\right)^2\right]$$

$$= \frac{1}{N^2}\sum_{n,m,p,q} E\left[X[n]X[m]X[p]X[q]\right] e^{-j\Omega_k(n-m)} e^{-j\Omega_k(q-p)}$$