LTI Systems (1A)

•

Young Won Lim
3/21/15

Please send corrections (or suggestions) to youngwlim@hotmail.com.

This document was produced by using OpenOffice and Octave.

An Improper Integration

$$F(s) = \int_0^\infty f(t) e^{-st} \, dt$$

Complex Number

Real Number

$$s = \sigma + i\omega$$

$\Re\{s\}$ real part

$\Im\{s\}$ imag part

Real Number

t Integration Variable

The improper integral **converges** if the limit defining it exists.

An Integration Function

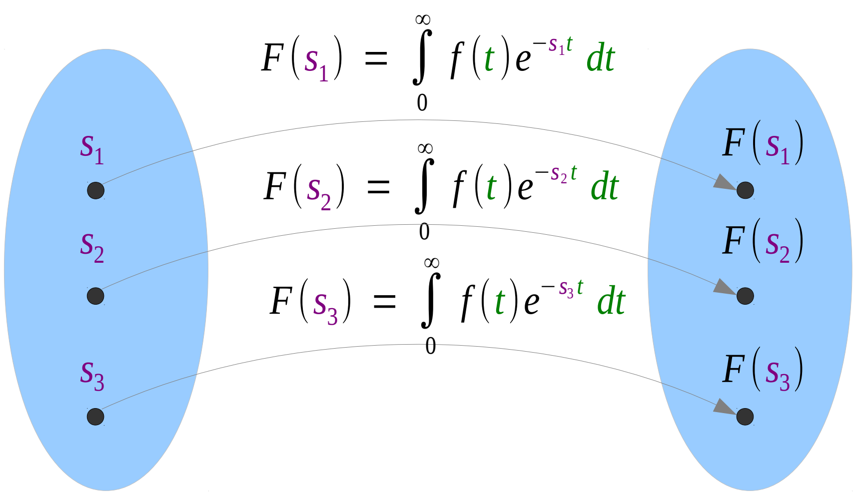
For a given function f(t)

$$F(s) = \int_0^\infty f(t) e^{-st}\, dt$$

Complex Number — Real Number

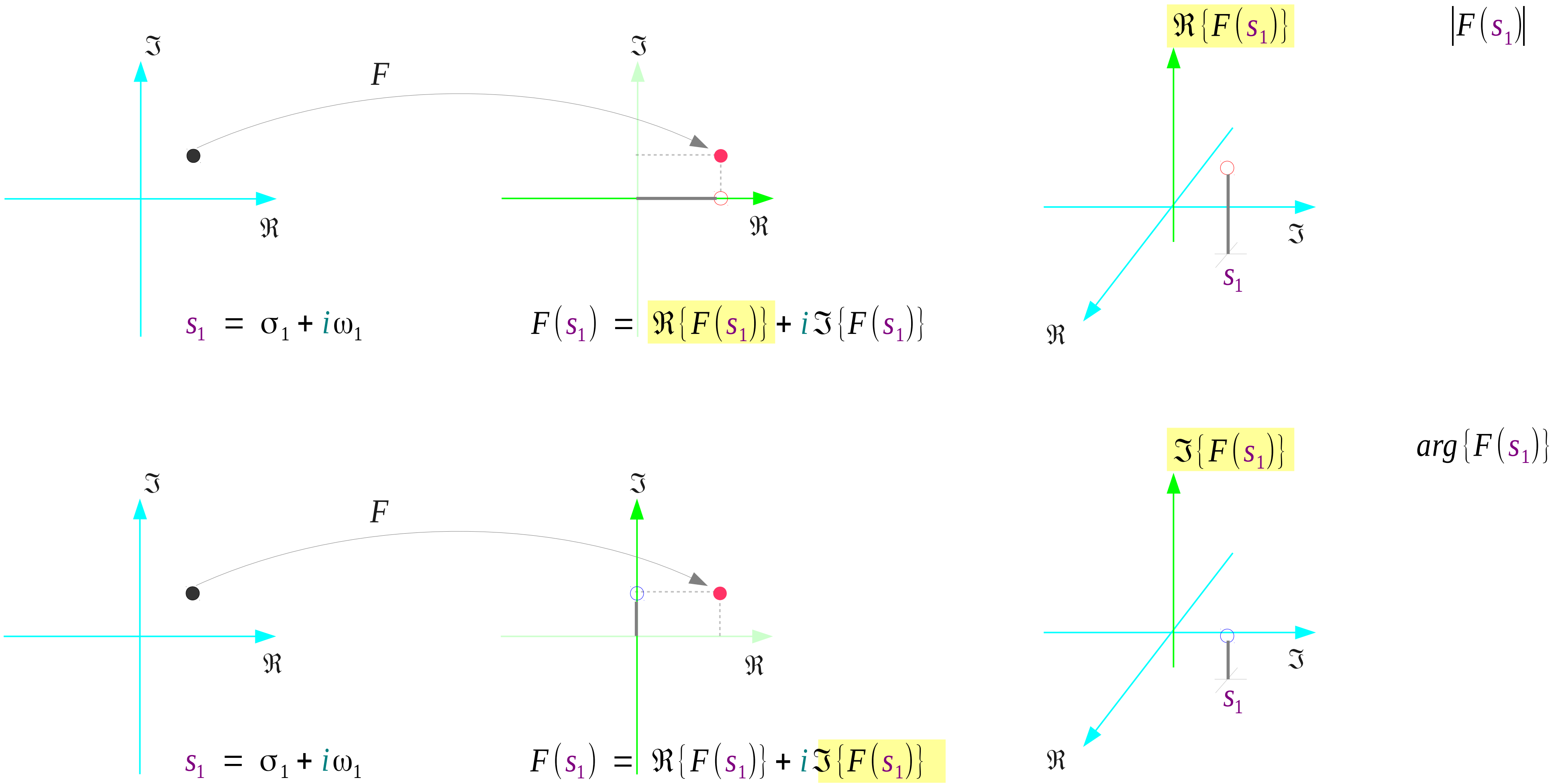
$$s = \sigma + i\omega$$

t — Real Number, Integration Variable



Complex Numbers — Complex Numbers

Complex Function Plot



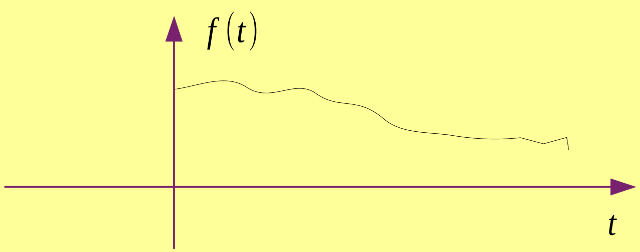
Two Functions: f(t) & F(s)

For a given function f(t)

there exits a unique F(s)

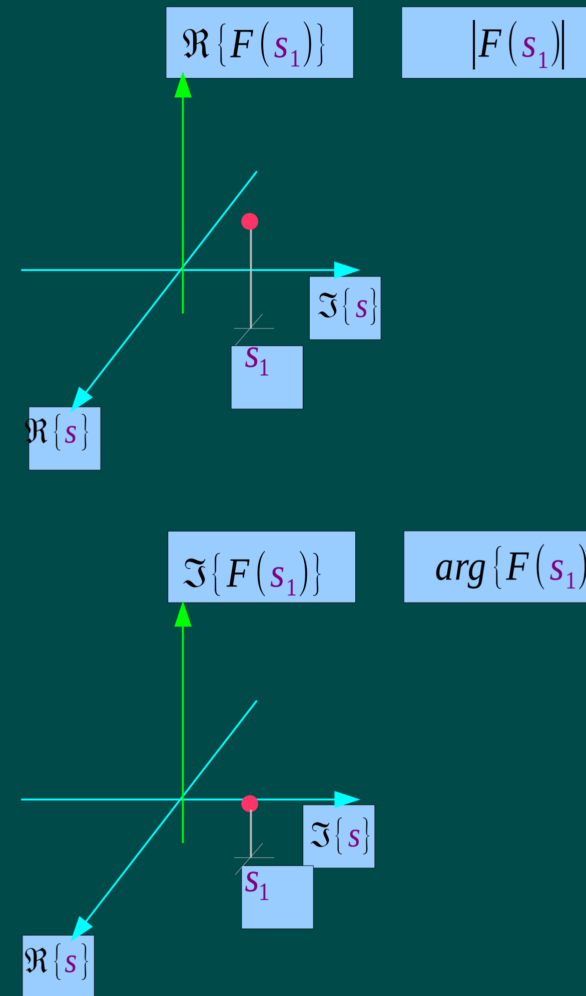
$$f(t) \Longleftrightarrow F(s)$$

t-domain function **f(t)**



Real number domain function f(t)

s-domain function **F(s)**



Complex number domain function F(s)

Laplace Transform

$f_a(x) \rightarrow$ **L** $\rightarrow F_A(s)$

$f_b(t) \rightarrow$ **L** $\rightarrow F_B(s)$

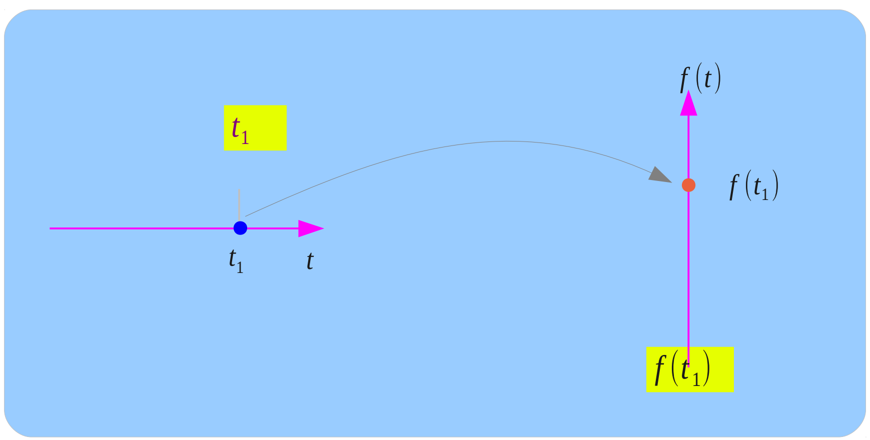
$f_c(t) \rightarrow$ **L** $\rightarrow F_C(s)$

$1 \rightarrow \int_0^\infty f_1(t) e^{-st}\, dt = F_1(s) \rightarrow \frac{1}{s}$

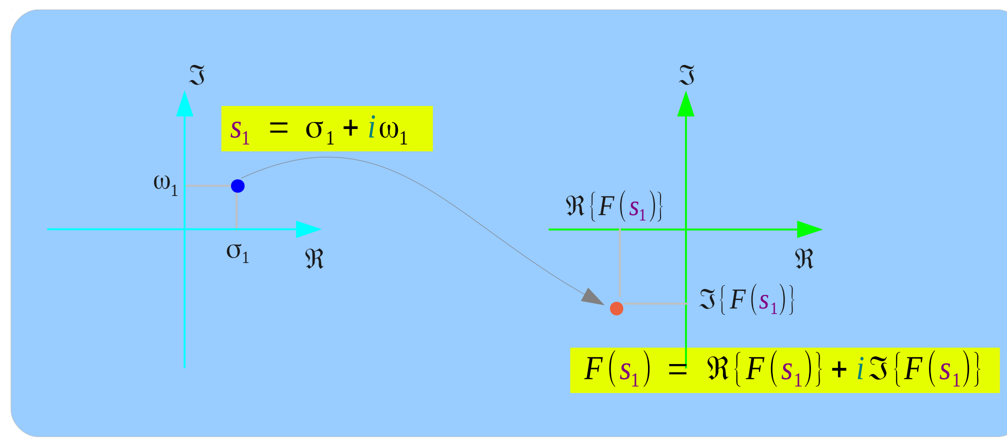
$e^{-at} \rightarrow \int_0^\infty f_2(t) e^{-st}\, dt = F_2(s) \rightarrow \frac{1}{s+a}$

$\cos(kt) \rightarrow \int_0^\infty f_3(t) e^{-st}\, dt = F_3(s) \rightarrow \frac{s}{s^2+k^2}$

$f_a(x)$ Real-valued Function



$F_A(s)$ Complex Function



Some Laplace Transform Pairs

1	$\Longleftrightarrow$	$\dfrac{1}{s}$	e^{-at}	$\Longleftrightarrow$	$\dfrac{1}{s+a}$
t	$\Longleftrightarrow$	$\dfrac{1}{s^2}$	$\sin(\omega t)$	$\Longleftrightarrow$	$\dfrac{\omega}{s^2+\omega^2}$
t^2	$\Longleftrightarrow$	$\dfrac{2}{s^3}$	$\cos(\omega t)$	$\Longleftrightarrow$	$\dfrac{s}{s^2+\omega^2}$
t^3	$\Longleftrightarrow$	$\dfrac{6}{s^4}$	$\sinh(\omega t)$	$\Longleftrightarrow$	$\dfrac{\omega}{s^2-\omega^2}$
t^n	$\Longleftrightarrow$	$\dfrac{n!}{s^{n+1}}$	$\cosh(\omega t)$	$\Longleftrightarrow$	$\dfrac{s}{s^2-\omega^2}$

Partial Fraction Methods

$$\frac{1}{\cdots(ax+b)\cdots} \Rightarrow \cdots + \frac{A}{(ax+b)} + \cdots$$

$$\frac{1}{\cdots(ax+b)^k\cdots} \Rightarrow \cdots + \frac{A_1}{(ax+b)} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_k}{(ax+b)^k} + \cdots$$

$$\frac{1}{\cdots(ax^2+bx+c)\cdots} \Rightarrow \cdots + \frac{Ax+b}{(ax^2+bx+c)} + \cdots$$

$$\frac{1}{\cdots(ax^2+bx+c)^k\cdots} \Rightarrow \cdots + \frac{A_1x+B_1}{(ax^2+bx+c)} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_kx+B_k}{(ax^2+bx+c)^k} + \cdots$$

Cover-up Method (1)

$$\frac{2s-1}{(s+3)(s-2)} = \frac{A}{(s+3)} + \frac{B}{(s-2)}$$

$$\frac{2s-1}{(s+3)(s-2)}(s+3) \quad \Rightarrow \quad A = \frac{2s-1}{(s-2)}\Big|_{s=-3} = \frac{-6-1}{-3-2} = +\frac{7}{5}$$

$$\frac{2s-1}{(s+3)(s-2)}(s-2) \quad \Rightarrow \quad B = \frac{2s-1}{(s+3)}\Big|_{s=2} = \frac{4-1}{2+3} = +\frac{3}{5}$$

Examples

$$\frac{2s+5}{(s-3)^2} = \frac{A_1}{(s-3)} + \frac{A_2}{(s-3)^2}$$

$$\frac{2s+5}{(s-3)^2}(s-3)^2 = \frac{A_1}{(s-3)}(s-3)^2 + \frac{A_2}{(s-3)^2}(s-3)^2 = A_1(s-3) + A_2$$

$$\left.\frac{2s+5}{(s-3)^2}(s-3)^2\right|_{s=3} = (2s+5)\big|_{s=3} = 11 = A_2$$

$$\frac{d}{ds}\left(\frac{2s+5}{(s-3)^2}(s-3)^2\right) = \frac{d}{ds}\left(A_1(s-3) + A_2\right) = A_1$$

$$\left.\frac{d}{ds}\left(\frac{2s+5}{(s-3)^2}(s-3)^2\right)\right|_{s=3} = 2 = A_1$$

Examples

$$X(s) = \frac{P(s)}{(s+p)(s+r)^k} = \frac{K}{(s+p)} + \frac{A_0}{(s+r)^k} + \frac{A_1}{(s+r)^{k-1}} + \cdots + \frac{A_{k-1}}{(s+r)^1}$$

$$A_0 = X(s)(s+r)^k \big|_{s=-r}$$

$$A_1 = \frac{d}{ds}[X(s)(s+r)^k]\big|_{s=-r}$$

$$A_2 = \frac{1}{2!}\frac{d^2}{ds^2}[X(s)(s+r)^k]\big|_{s=-r}$$

$$A_m = \frac{1}{m!}\frac{d^m}{ds^m}[X(s)(s+r)^k]\big|_{s=-r}$$

Differentiation in the s-domain

$$f(t) \Longleftrightarrow F(s) \qquad F(s) = \int_0^\infty f(t)\cdot e^{-st}\, dt$$

$$-t f(t) \Longleftrightarrow F'(s) \qquad \frac{d}{ds}F(s) = \int_0^\infty \frac{\partial}{\partial s}\left[f(t)\cdot e^{-st}\right] dt = \int_0^\infty (-t) f(t)\cdot e^{-st}\, dt$$

$$+t^2 f(t) \Longleftrightarrow F''(s) \qquad \frac{d^2}{ds^2}F(s) = \int_0^\infty \frac{\partial^2}{\partial s^2}\left[f(t)\cdot e^{-st}\right] dt = \int_0^\infty (-t)^2 f(t)\cdot e^{-st}\, dt$$

$$-t^3 f(t) \Longleftrightarrow F^{(3)}(s) \qquad \frac{d^3}{ds^3}F(s) = \int_0^\infty \frac{\partial^3}{\partial s^3}\left[f(t)\cdot e^{-st}\right] dt = \int_0^\infty (-t)^3 f(t)\cdot e^{-st}\, dt$$

$$t^n f(t) \Longleftrightarrow (-1)^n \frac{d^n}{ds^n} F(s)$$

Differentiation in the t-domain (1)

$f(t) \longleftrightarrow F(s)$ $\qquad F(s) = \int_0^\infty f(t) \cdot e^{-st}\, dt$

$$\int_0^\infty f'(t) \cdot e^{-st}\, dt = \left[f(t) \cdot e^{-st}\right]_0^\infty - \int_0^\infty (-s) f(t) \cdot e^{-st}\, dt$$

$$= -f(0) + s\int_0^\infty f(t) \cdot e^{-st}\, dt = s F(s) - f(0)$$

$f'(t) \longleftrightarrow sF(s) - f(0)$

$f''(t) \longleftrightarrow s(sF(s) - f(0)) - f'(0)$

$f^{(3)}(t) \longleftrightarrow s(s(sF(s) - f(0)) - f'(0)) - f''(0)$

Differentiation in the t-domain (2)

$$f(t) \Longleftrightarrow F(s)$$

$$f'(t) \Longleftrightarrow sF(s) - f(0)$$

$$f''(t) \Longleftrightarrow s^2F(s) - sf(0) - f'(0)$$

$$f^{(3)}(t) \Longleftrightarrow s^3F(s) - s^2f(0) - sf'(0) - f''(0)$$

$$f^{(n)}(t) \Longleftrightarrow s^nF(s) - s^{(n-1)}f(0) - s^{(n-2)}f'(0) - \cdots - f^{(n-1)}(0)$$

Integration in the t-domain

$$f(t) \longleftrightarrow F(s)$$

$$\int_0^t f(\tau)\,d\tau \longleftrightarrow \frac{F(s)}{s}$$

$$f(t) = \frac{d}{dt}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{d}{dt} g(t)$$

$$g(t) = \int_0^t f(\tau)\,d\tau$$

$$g(0) = \int_0^0 f(\tau)\,d\tau = 0$$

$$g(t) \longleftrightarrow G(s)$$

$$g(t) = \int_0^t f(\tau)\,d\tau \qquad G(s) = ?$$

$$f(t) \longleftrightarrow F(s)$$

$$f(t) = g'(t) \longleftrightarrow F(s) = sG(s) - g(0)$$

$$F(s) = \frac{F(s)}{s}$$

Translation in the s-domain

$f(t) \longleftrightarrow F(s)$ $\qquad F(s) = \int_0^{\infty} f(t) \cdot e^{-st}\, dt$

$e^{+at} f(t) \longleftrightarrow F(s-a)$ $\qquad F(s-a) = \int_0^{\infty} f(t) \cdot e^{-(s-a)t}\, dt = \int_0^{\infty} \left[e^{+at} f(t)\right] e^{-st}\, dt$

$e^{\pm at} f(t) \longleftrightarrow F(s \mp a)$

Translation in the t-domain

$f(t) \quad \Longleftrightarrow \quad F(s)$

$$F(s) = \int_0^\infty f(t) \cdot e^{-st}\, dt$$

$f(t-a)u(t-a) \quad \Longleftrightarrow \quad e^{-as}F(s)$

$f(t \mp a)u(t \mp a) \quad \Longleftrightarrow \quad e^{\mp as}F(s)$

$$\int_0^\infty f(t-a)u(t-a) \cdot e^{-st}\, dt$$

$$= \int_0^a f(t-a)u(t-a) \cdot e^{-st}\, dt + \int_a^\infty f(t-a)u(t-a) \cdot e^{-st}\, dt$$

$$= \int_a^\infty f(t-a) \cdot e^{-st}\, dt$$

$\nu = t - a \qquad d\nu = dt$

$0 = a - a$

$$= \int_0^\infty f(\nu) \cdot e^{-s(\nu+a)}\, d\nu$$

$$= e^{-as} \cdot \int_0^\infty f(\nu) \cdot e^{-s\nu}\, d\nu$$

$$= e^{-as} \cdot F(s)$$

shift right : always o.k.
shift left: only when no information is lost during improper integration by the left shift

Initial Value Theorem

$$\lim_{t \to 0^+} f(t) = \lim_{s \to \infty} sF(s)$$

$$F(s) = \int_{0^-}^{\infty} f(t) \cdot e^{-st}\, dt$$

$$sF(s) - f(0^-) = \int_{0^-}^{\infty} f'(t) \cdot e^{-st}\, dt$$

$$\lim_{s \to \infty} sF(s)$$

$$= f(0^-) + \lim_{s \to \infty} \int_{0^-}^{\infty} f'(t) \cdot e^{-st}\, dt$$

$$= f(0^-) + f(0^+) - f(0^-)$$

$$= f(0^+)$$

Final Value Theorem

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$$

$$F(s) = \int_{0^-}^{\infty} f(t) \cdot e^{-st}\, dt$$

$$sF(s) - f(0^-) = \int_{0^-}^{\infty} f'(t) \cdot e^{-st}\, dt$$

$$\lim_{s \to 0} \left[sF(s) - f(0^-) \right] = \lim_{s \to 0} \int_{0^-}^{\infty} f'(t) \cdot e^{-st}\, dt$$

$$\lim_{s \to \infty} sF(s) = f(0^-) + \lim_{s \to 0} \int_{0^-}^{\infty} f'(t) \cdot e^{-st}\, dt$$

$$= f(0^-) + f(\infty) - f(0^-)$$

$$= f(\infty)$$

Laplace Transform and ODE's

$$y'' + 3y' + 2y = e^{-3x} \qquad y(0) = k_1, \quad y'(0) = k_2$$

$$y(x) \longleftrightarrow Y(s)$$

$$y'(x) \longleftrightarrow sY(s) - y(0)$$

$$y''(x) \longleftrightarrow s^2 Y(s) - s\,y(0) - y'(0)$$

$$e^{-3x} \longleftrightarrow \frac{1}{(s+3)}$$

$$\left[s^2 Y(s) - s\,y(0) - y'(0)\right] + 3\left[sY(s) - y(0)\right] + 2\left[Y(s)\right] = \frac{1}{s+3}$$

$$(s^2 + 3s + 2)Y(s) = k_1 s + k_2 + 3k_1 + \frac{1}{s+3}$$

Partitioning

$$y'' + 3y' + 2y = e^{-3x} \qquad y(0) = k_1, \quad y'(0) = k_2$$

$$\left[s^2 Y(s) - s\,y(0) - y'(0)\right] + 3\left[s\,Y(s) - y(0)\right] + 2\left[Y(s)\right] = \frac{1}{s+3}$$

$$(s^2 + 3s + 2)\,Y(s) = (k_1 s + k_2 + 3k_1) + \frac{1}{s+3}$$

depends only on initial conditions k_1, k_2

depends only on input e^{-3x}

Decomposed Y(s)

output ↑ input ↓

$$\left(s^2 + 3s + 2\right) Y_{zi}(s) = \left(k_1 s + k_2 + 3k_1\right)$$

depends only on initial conditions k_1, k_2

No Input

output ↑ input ↓

$$\left(s^2 + 3s + 2\right) Y_{zs}(s) = \frac{1}{s+3}$$

depends only on input e^{-3x}

No State

output ↑ input ↓

$$\left(s^2 + 3s + 2\right) Y(s) = \left(k_1 s + k_2 + 3k_1\right) + \frac{1}{s+3}$$

depends only on initial conditions k_1, k_2

depends only on input e^{-3x}

ZIR & ZSR

$y_{zi}(x) \Longleftrightarrow Y_{zi}(s) = \dfrac{k_1 s + k_2 + 3k_1}{(s+1)(s+2)}$

Zero Input Response

$y_{zs}(x) \Longleftrightarrow Y_{zs}(s) = \dfrac{1}{(s+1)(s+2)(s+3)}$

Zero State Response

$y(x) \Longleftrightarrow Y(s) = \dfrac{k_1 s + k_2 + 3k_1}{(s+1)(s+2)} + \dfrac{1}{(s+1)(s+2)(s+3)}$

$y(x) \Longleftrightarrow Y(s) = Y_{zi}(s) + Y_{zs}(s)$

Laplace Transform and IVP's

$$y'' + 3y' + 2y = 0 \qquad y(0) = k_1, \quad y'(0) = k_2$$

ZIR IVP

$$y_{zi}(x) \quad \Longleftrightarrow \quad Y_{zi}(s) = \frac{k_1 s + k_2 + 3k_1}{(s+1)(s+2)}$$

$$y'' + 3y' + 2y = e^{-3x} \qquad y(0) = 0, \quad y'(0) = 0$$

ZSR IVP

$$y_{zs}(x) \quad \Longleftrightarrow \quad Y_{zs}(s) = \frac{1}{(s+1)(s+2)(s+3)}$$

$$y'' + 3y' + 2y = e^{-3x} \qquad y(0) = k_1, \quad y'(0) = k_2$$

To include impulse inputs

$$y'' + 3y' + 2y = e^{-3x} \qquad y(0) = k_1, \quad y'(0) = k_2$$

$$y(x) \longleftrightarrow Y(s)$$

$$y'(x) \longleftrightarrow sY(s) - y(0^-)$$

$$y''(x) \longleftrightarrow s^2 Y(s) - s\,y(0^-) - y'(0^-)$$

$$e^{-3x} \longleftrightarrow \frac{1}{(s+3)}$$

$$\left[s^2 Y(s) - s\,y(0^-) - y'(0^-)\right] + 3\left[sY(s) - y(0^-)\right] + 2\left[Y(s)\right] = \frac{1}{s+3}$$

ODEs with an input g(x)

$$y'' + 3y' + 2y = e^{-3x} \qquad y(0) = k_1, \;\; y'(0) = k_2$$

usually known i.c. $\qquad y(0^-) = k_1, \;\; y'(0^-) = k_2$

i.c. to be calculated $\qquad y(0^+) = m_1, \;\; y'(0^+) = m_2$

solution to be found $\qquad y(t) \quad (t > 0)$

ZIR & ZSR

$$y'' + 3y' + 2y = x(t) \quad y(0) = k_1, \quad y'(0) = k_2$$

$$s^2 Y(s) - s\,y(0) - y'(0) + 3(s\,Y(s) - y(0)) + 2\,Y(s) = X(s)$$

$$(s^2 + 3s + 2)\,Y(s) = k_1 s + k_2 + 3k_1 + X(s)$$

$$Y(s) = \frac{k_1 s + k_2 + 3k_1}{(s+1)(s+2)} + \frac{X(s)}{(s+1)(s+2)}$$

Zero Input Response

$$x(t) = 0$$

$$Y(s) = \frac{s+5}{(s+1)(s+2)}$$

$$= +4\frac{1}{(s+1)} - 3\frac{1}{(s+2)}$$

$$\longleftrightarrow \quad y = 4e^{-t} - 3e^{-2t}$$

Zero State Response

$$y(0) = 0, \quad y'(0) = 0$$

$$Y(s) = \frac{X(s)}{(s+1)(s+2)} \qquad x(t) = e^{+t}$$

$$= +\frac{1}{3}\frac{1}{(s+2)} - \frac{1}{2}\frac{1}{(s+1)} + \frac{1}{6}\frac{1}{(s-1)}$$

$$\longleftrightarrow \quad y = -\frac{1}{2}e^{-t} + \frac{1}{3}e^{-2t} + \frac{1}{6}e^{+t}$$

Transfer Function

$$y'' + 3y' + 2y = x(t) \quad y(0) = k_1, \quad y'(0) = k_2$$

$$s^2 Y(s) - s\,y(0) - y'(0) + 3(s\,Y(s) - y(0)) + 2\,Y(s) = X(s)$$

$$(s^2 + 3s + 2)\,Y(s) = k_1 s + k_2 + 3k_1 + X(s)$$

$$Y(s) = \frac{k_1 s + k_2 + 3k_1}{(s+1)(s+2)} + \frac{X(s)}{(s+1)(s+2)}$$

Zero State Response

$$y(0) = 0, \quad y'(0) = 0$$

$$Y(s) = \frac{X(s)}{(s+1)(s+2)}$$

$$\frac{Y(s)}{X(s)} = \frac{1}{(s+1)(s+2)}$$

Transfer Function

Transfer Function & Impulse Response

$$y'' + 3y' + 2y = x(t) \quad y(0) = k_1, \quad y'(0) = k_2$$

$$s^2 Y(s) - s\,y(0) - y'(0) + 3(s\,Y(s) - y(0)) + 2\,Y(s) = X(s)$$

$$(s^2 + 3s + 2)Y(s) = k_1 s + k_2 + 3k_1 + X(s)$$

$$Y(s) = \frac{k_1 s + k_2 + 3k_1}{(s+1)(s+2)} + \frac{X(s)}{(s+1)(s+2)}$$

Transfer Function

$$\frac{Y(s)}{X(s)} = H(s)$$

$$Y(s) = H(s)X(s) \quad \longleftrightarrow \quad y(t) = h(t) * x(t)$$

$$H(s) \quad \longleftrightarrow \quad h(t)$$

Transfer Function — **Impulse Response**

Zero State Response

$$y(0) = 0, \quad y'(0) = 0$$

$$Y(s) = \frac{X(s)}{(s+1)(s+2)}$$

$$\frac{Y(s)}{X(s)} = \frac{1}{(s+1)(s+2)}$$

Transfer Function

Everlasting & Causal Exponential Function

exponential function

$e^{st} = e^{\sigma t + i\omega t}$

$s = \sigma + i\omega$

sinusoid function

$e^{st} = e^{i\omega t}$

$s = i\omega$

- **everlasting exponential** function

  applied at $t = -\infty$

  e^{st}

- **causal exponential** function

  applied at $t = 0$

  $e^{st}\, u(t)$

- **everlasting sinusoid** function

  applied at $t = -\infty$

  $e^{i\omega t}$

- **causal sinusoid** function

  applied at $t = 0$

  $e^{i\omega t}\, u(t)$

Sinusoidal Functions and Initial Conditions

- **everlasting sinusoid function**

  applied at $t = -\infty$

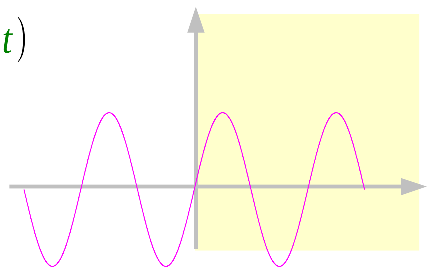
  zero state (no initial conditions)

- **causal sinusoid function**

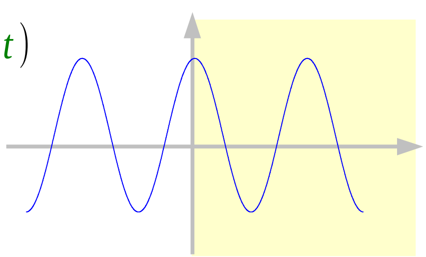
  applied at $t = 0$ $e^{i\omega t}\, u(t)$

  non-zero state at t = 0+

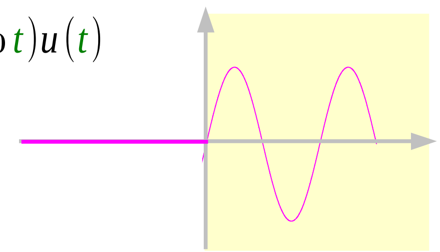
$\sin(\omega t)$



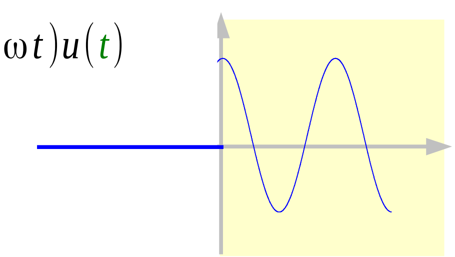
$\cos(\omega t)$

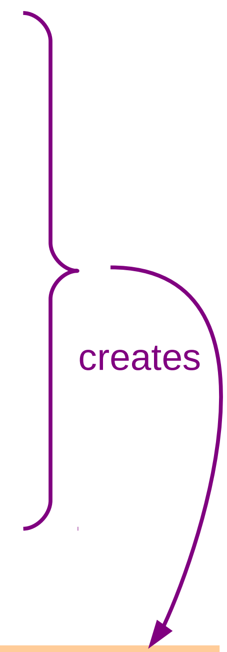


$\sin(\omega t) u(t)$



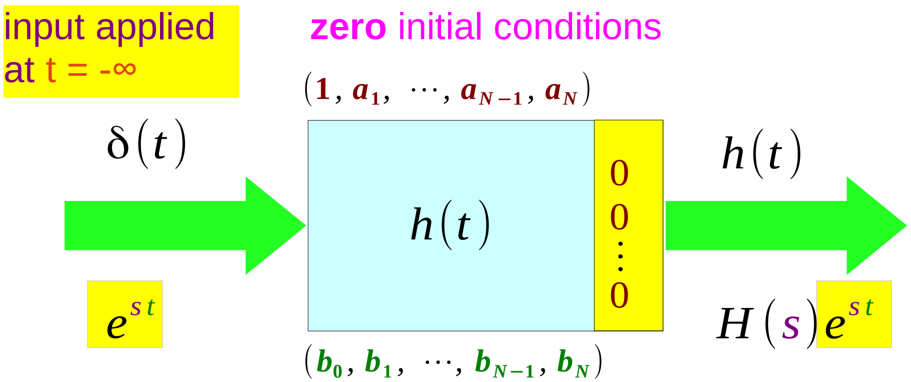
$\cos(\omega t) u(t)$





zero conditions at time $t = 0^-$ → non-zero conditions at time $t = 0^+$

ZSR to an everlasting exponential input



$$y(t) = h(t) * e^{st}$$

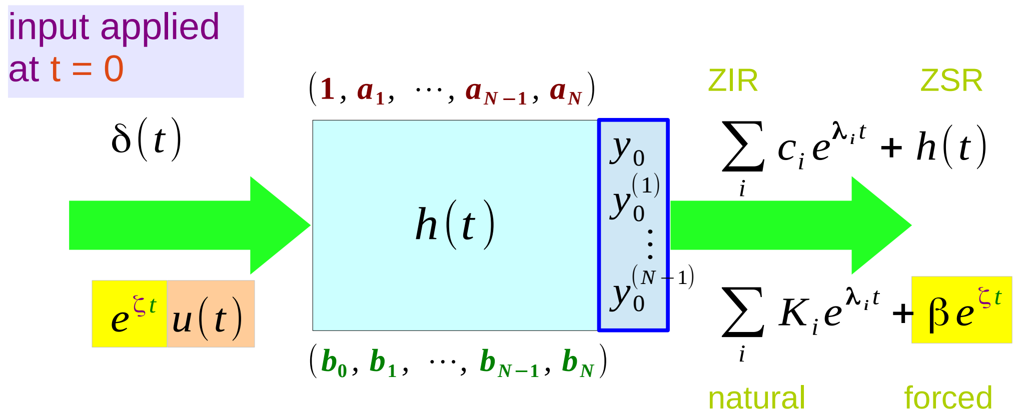
$$= \int_{-\infty}^{+\infty} h(\tau) e^{s(t-\tau)} d\tau$$

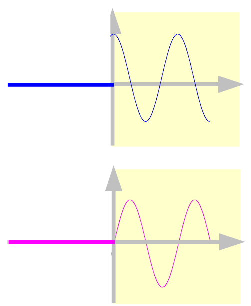
$$= e^{st} \cdot \int_{-\infty}^{+\infty} h(\tau) e^{-s\tau} d\tau$$

$h(t) = 0$
$(t < 0)$

$$= e^{st} \cdot H(s)$$

$$H(s) = \int_{-\infty}^{+\infty} h(\tau) e^{-s\tau} d\tau$$





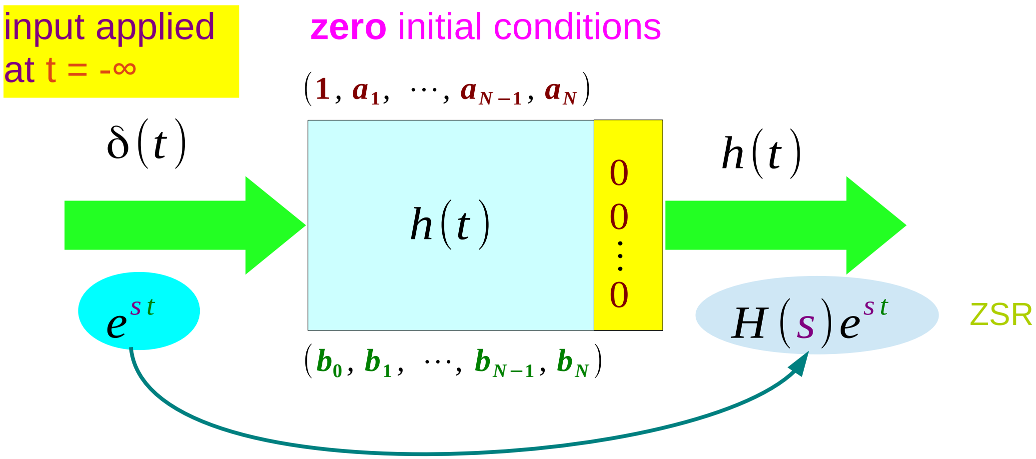
initial conditions at time $t = 0^-$

initial conditions at time $t = 0^+$

$$y_n(t) = K_1 e^{\lambda_1 t} + K_2 e^{\lambda_2 t} + \cdots + K_N e^{\lambda_N t} = \sum_i K_i e^{\lambda_i t}$$

$$y_n(t) + y_p(t) \quad \{y^{(N-1)}(0^+), \cdots, y^{(1)}(0^+), y(0^+)\} \Rightarrow K_i$$

Everlasting Exponential Total Response



$$\text{for a given } s = \zeta$$

ZSR

$$y(t) = H(\zeta)e^{\zeta t} \qquad -\infty < t < +\infty$$

$$y(t) = H(\zeta)x(t) \qquad x(t) = e^{\zeta t}$$

$$Y(s) = H(s)X(s)$$

Laplace Transform of h(t)

$$H(s) = \int_0^{+\infty} h(\tau)e^{-s\tau}d\tau$$

Polynomials of Differential Equation

$$H(s) = \frac{P(s)}{Q(s)}$$

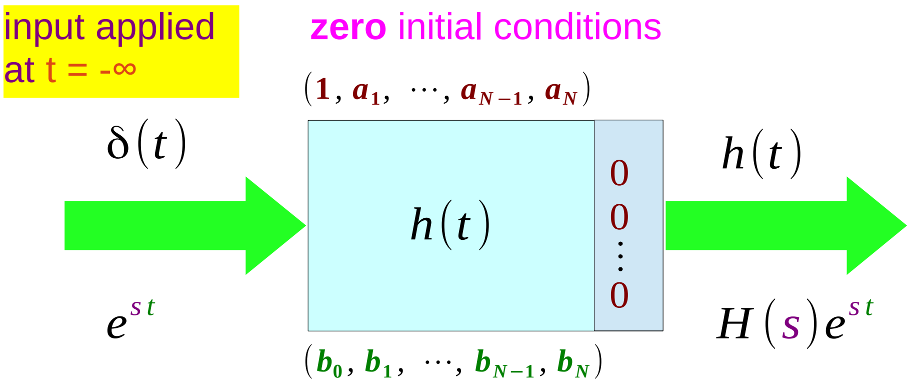
Transfer function (t-domain)

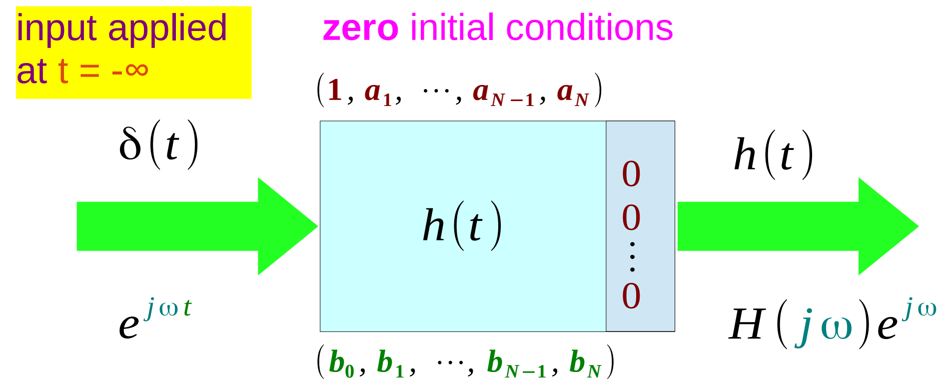
$$H(s) = \left[\frac{y(t)}{x(t)}\right]_{x(t)=e^{st}}$$

Transfer function (s-domain)

$$H(s) = \frac{Y(s)}{X(s)}$$

Transfer Function & Frequency Response





$s = \sigma + j\omega$

$$
\begin{aligned}
y(t) &= h(t) * e^{st} \\
&= \int_{-\infty}^{+\infty} h(\tau) e^{s(t-\tau)} d\tau \\
&= e^{st} \cdot \int_{-\infty}^{+\infty} h(\tau) e^{-s\tau} d\tau \\
&= e^{st} \cdot H(s)
\end{aligned}
$$

$h(t) = 0$
$(t < 0)$

$$H(s) = \int_{-\infty}^{+\infty} h(\tau) e^{-s\tau} d\tau$$

Transfer function

$s = j\omega$

$$
\begin{aligned}
y(t) &= h(t) * e^{j\omega t} \\
&= \int_{-\infty}^{+\infty} h(\tau) e^{j\omega(t-\tau)} d\tau \\
&= e^{j\omega t} \cdot \int_{-\infty}^{+\infty} h(\tau) e^{-j\omega\tau} d\tau \\
&= e^{j\omega t} \cdot H(j\omega)
\end{aligned}
$$

$h(t) = 0$
$(t < 0)$

$$H(j\omega) = \int_{-\infty}^{+\infty} h(\tau) e^{-j\omega\tau} d\tau$$

Frequency response

Frequency Response

Laplace Transform of h(t)

$$H(s) = \int_{-\infty}^{+\infty} h(\tau) e^{-s\tau} d\tau \qquad s = \sigma + j\omega$$

Fourier Transform of h(t)

$$H(j\omega) = \int_{-\infty}^{+\infty} h(\tau) e^{-j\omega\tau} d\tau \qquad s = j\omega$$

Polynomials of Differential Equation

$$H(s) = \frac{P(s)}{Q(s)} \qquad s = \sigma + j\omega$$

Polynomials of Differential Equation

$$H(j\omega) = \frac{P(j\omega)}{Q(j\omega)} \qquad s = j\omega$$

Transfer function (t-domain)

$$H(s) = \left[\frac{y(t)}{x(t)}\right]_{x(t)=e^{st}} \qquad s = \sigma + j\omega$$

Frequency response (t-domain)

$$H(j\omega) = \left[\frac{y(t)}{x(t)}\right]_{x(t)=e^{j\omega t}} \qquad s = j\omega$$

Transfer function (s-domain)

$$H(s) = \frac{Y(s)}{X(s)} \qquad s = \sigma + j\omega$$

Frequency response (ω-domain)

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} \qquad s = j\omega$$

ZSR to everlasting sinusoidal inputs

$e^{+j\omega t}$ → $h(t)$ → $H(+j\omega)\,e^{+j\omega t} = |H(+j\omega)|\,e^{+j\,arg\{H(+j\omega)\}}\cdot e^{+j\omega t}$

$e^{-j\omega t}$ → $h(t)$ → $H(-j\omega)\,e^{-j\omega t} = |H(-j\omega)|\,e^{+j\,arg\{H(-j\omega)\}}\cdot e^{-j\omega t}$

$e^{+j\omega t} + e^{-j\omega t}$ → $h(t)$ → $|H(+j\omega)|\,e^{[+j\omega t + arg\{H(+j\omega)\}]} + |H(-j\omega)|\,e^{[-j\omega t + arg\{H(-j\omega)\}]}$

$$= |H(+j\omega)|\left\{e^{[+j\omega t + arg\{H(+j\omega)\}]} + e^{[-j\omega t - arg\{H(+j\omega)\}]}\right\}$$

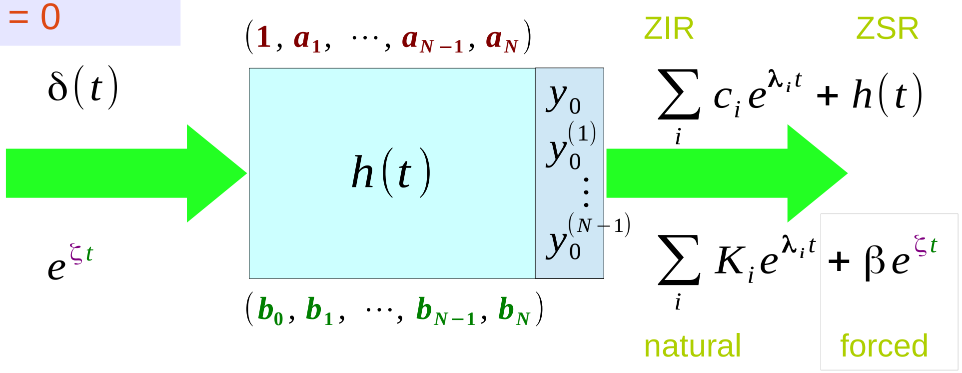
$2\cos(\omega t)$ → $h(t)$ → $|H(j\omega)|\,2\cos(\omega t + arg\{H(j\omega)\})$

$A\cos(\omega t + \alpha)$ → $h(t)$ → $A\,|H(j\omega)|\cos(\omega t + \alpha + arg\{H(j\omega)\})$

$A\sin(\omega t + \alpha)$ → $h(t)$ → $A\,|H(j\omega)|\sin(\omega t + \alpha + arg\{H(j\omega)\})$

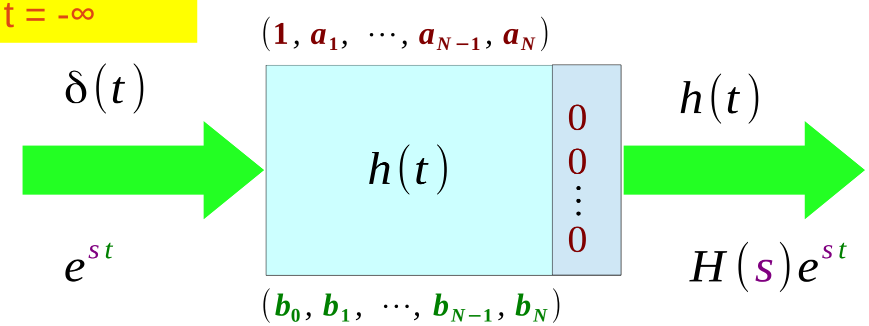
Sinusoidal Steady State Response (1)

input applied at t = 0



input applied at t = -∞

zero initial conditions



total response

natural forced

$$y(t) = \sum_i K_i e^{\lambda_i t} + H(\zeta) e^{\zeta t} \qquad t \geq 0$$

$$y(t) = \sum_i K_i e^{\lambda_i t} + H(\zeta) x(t) \qquad x(t) = e^{\zeta t}$$

$$Y(s) = \left[\sum_i \frac{K_i}{(s - \lambda_i)} + H(s) \right] X(s)$$

steady state response

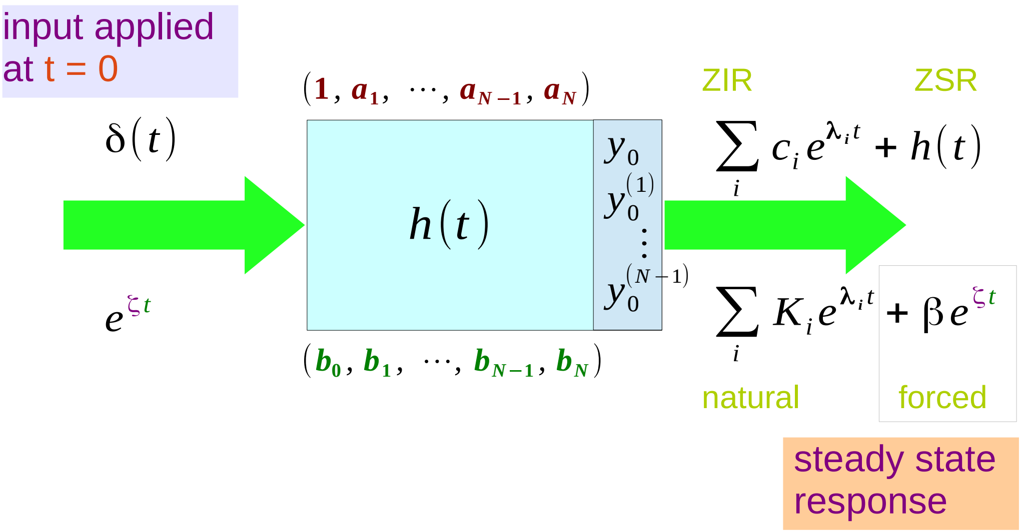
forced

$$y_{ss}(t) = H(\zeta) e^{\zeta t} \qquad t \to \infty$$

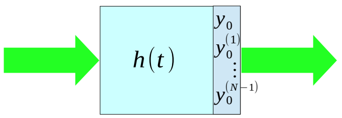
$$y_{ss}(t) = H(\zeta) x(t) \qquad x(t) = e^{\zeta t}$$

$$Y_{ss}(s) = H(s) X(s)$$

Sinusoidal Steady State Response (2)

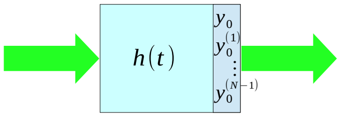


$$x(t) = A$$
$$\xi = 0$$



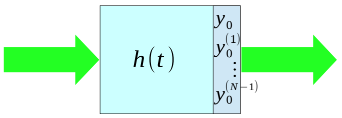
$$y_{ss}(t) = H(0) \cdot A e^{0t} = A \cdot H(0)$$

$$x(t) = A e^{j\omega t}$$
$$\xi = j\omega$$



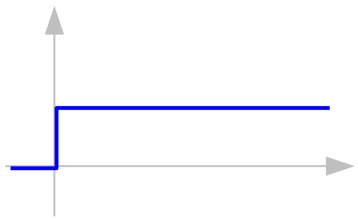
$$y_{ss}(t) = H(j\omega) \cdot A e^{j\omega t} = A \cdot H(j\omega) e^{j\omega}$$

$$x(t) = A\cos(\omega t)$$
$$\xi = j\omega$$



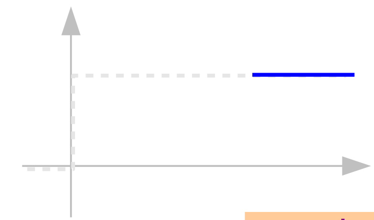
$$y_{ss}(t) = H(j\omega) \cdot A\cos(\omega t) = A \cdot H(j\omega)\cos(\omega t)$$

Sinusoidal Steady State Response (3)



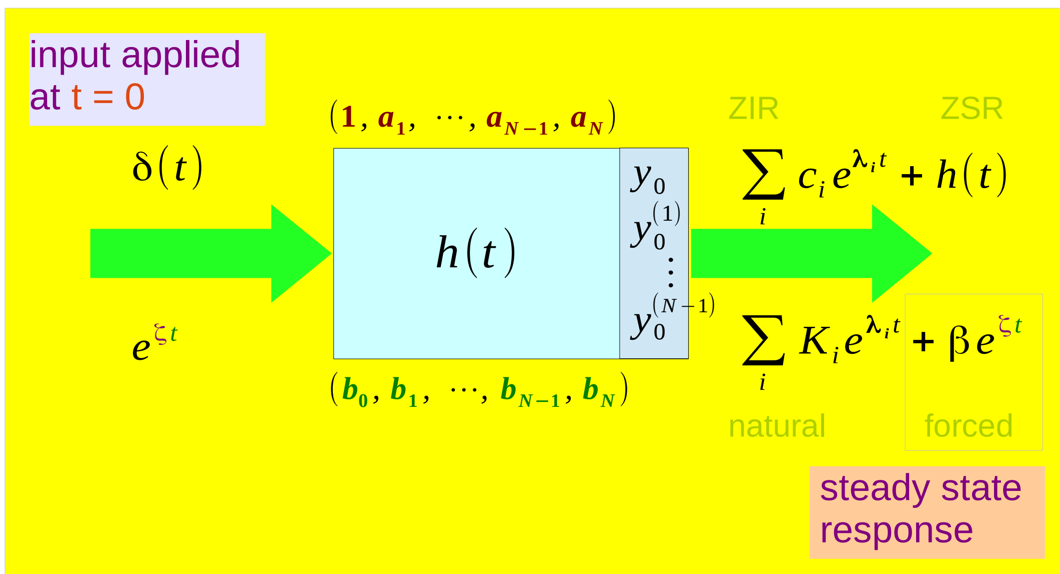
$$y_{ss}(t) = H(0)\cdot A e^{0t} = A\cdot H(0)$$

$$x(t) = A \qquad \xi = 0$$



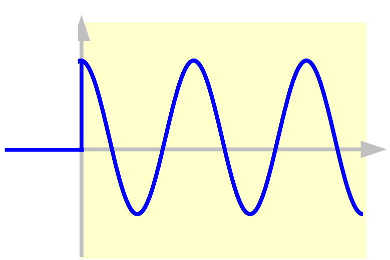
steady state response

Forced response



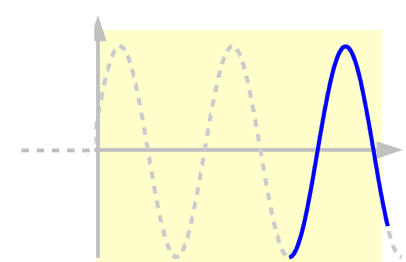
Forced response

steady state response



$$y_{ss}(t) = H(j\omega)\cdot A\cos(\omega t) = A\cdot H(j\omega)\cos(\omega t)$$

$$x(t) = A\cos(\omega t) \qquad \xi = j\omega$$



Frequency Response

$$y_{ss}(t) = H(j\omega)\cdot A\cos(\omega t) = A\cdot H(j\omega)\cos(\omega t)$$

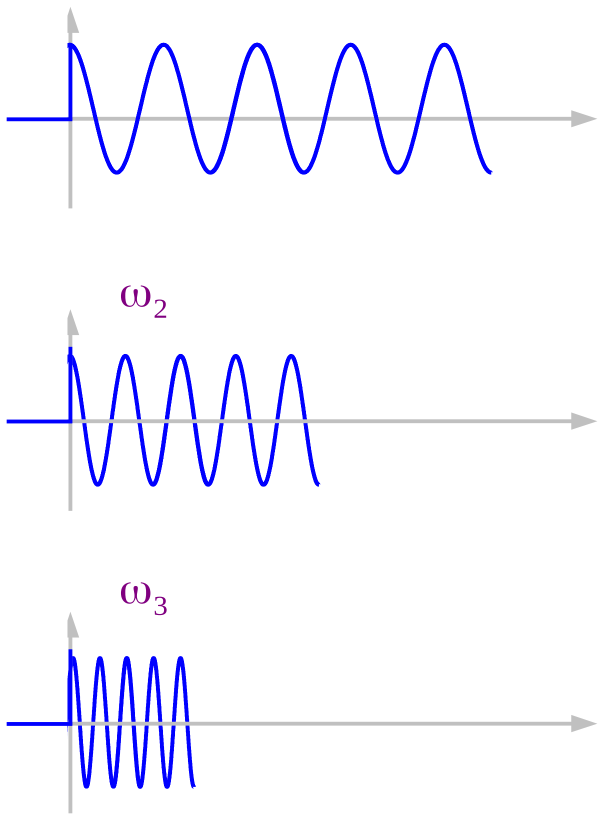
$$x(t) = A\cos(\omega t)$$
$$\xi = j\omega$$

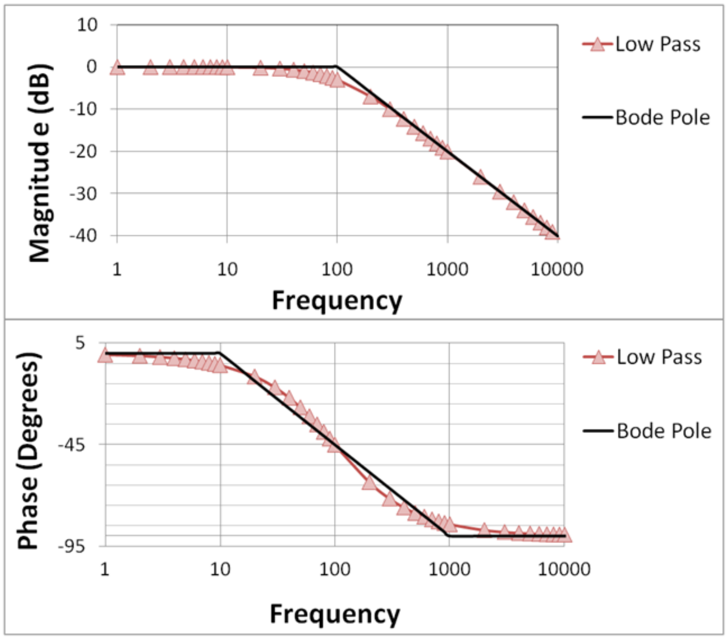
ω_1 ω_2 ω_3

ω_1

ω_2

ω_3



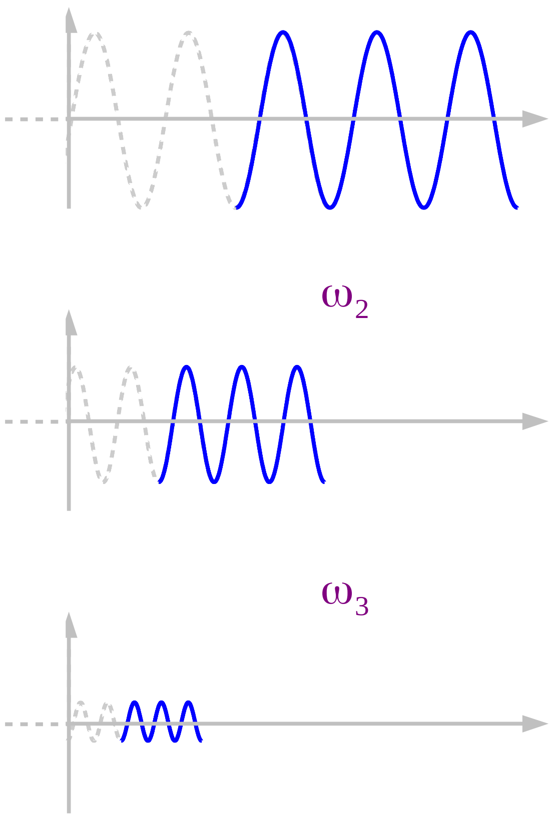


http://en.wikipedia.org/wiki/File:Bode_Low-Pass.PN(

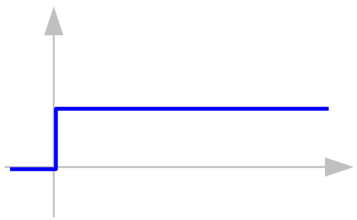
ω_1

ω_2

ω_3



Transient Response

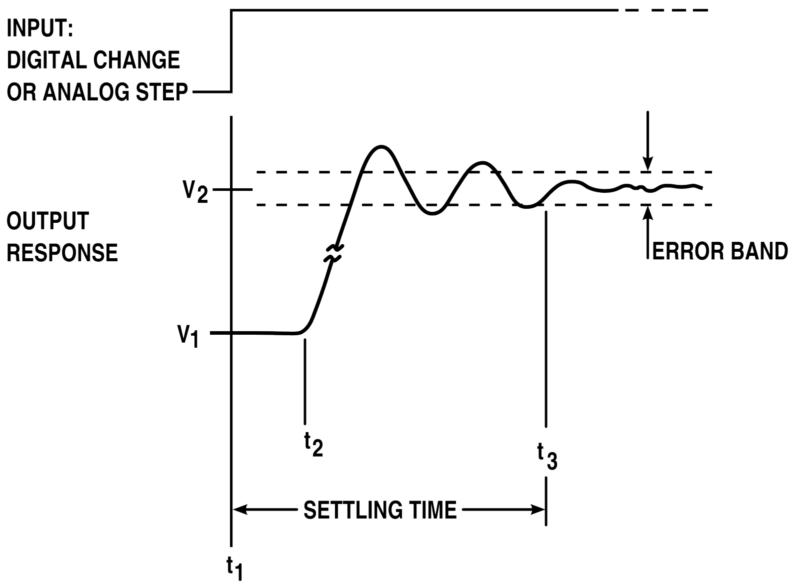


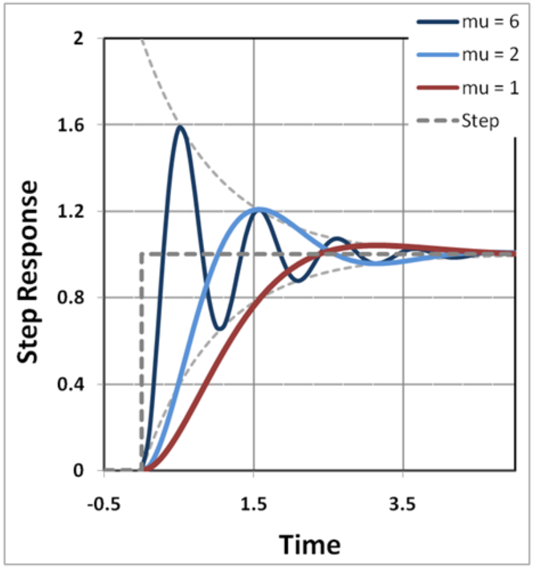
$$y_{ss}(t) = H(0) \cdot A e^{0t} = A \cdot H(0)$$

$$x(t) = A$$
$$\xi = 0$$

Natural + Forced response

transient response





http://en.wikipedia.org/wiki/File:High_accuracy_settling_time_measurements_figure_1.png
http://en.wikipedia.org/wiki/File:Step_response_for_two-pole_feedback_amplifier.PNG

Frequency Response in Control Theory (1)

$$\sqrt{A^2+B^2}\cos(\omega t-\theta) \;\rightarrow\; G(s) \;\rightarrow\; |G(j\omega)|\,\sqrt{A^2+B^2}\,\cos\big(\omega t-\theta+arg\{G(j\omega)\}\big)$$

$$A\cos(\omega t)+B\sin(\omega t)$$

$$=\sqrt{A^2+B^2}\left[\frac{A}{\sqrt{A^2+B^2}}\cos(\omega t)+\frac{B}{\sqrt{A^2+B^2}}\sin(\omega t)\right]$$

$$=\sqrt{A^2+B^2}\left[\cos(\theta)\cos(\omega t)+\sin(\theta)\sin(\omega t)\right]$$

$$=\sqrt{A^2+B^2}\cos(\theta-\omega t)$$

$$=\sqrt{A^2+B^2}\cos(\omega t-\theta)$$

$$A\cos(\omega t)+B\sin(\omega t)$$

$$=\sqrt{A^2+B^2}\cos(\omega t-\theta)$$

$$\cos(\theta)=\frac{A}{\sqrt{A^2+B^2}}$$

$$\sin(\theta)=\frac{B}{\sqrt{A^2+B^2}}$$

Frequency Response in Control Theory (2)

$$\sqrt{A^2+B^2}\cos(\omega t-\theta) \rightarrow G(s) \rightarrow |G(j\omega)|\,\sqrt{A^2+B^2}\,\cos\big(\omega t-\theta+arg\{G(j\omega)\}\big)$$

$$A\cos(\omega t)+B\sin(\omega t) = \sqrt{A^2+B^2}\cos(\omega t-\theta)$$

$$\longleftrightarrow$$

$$\frac{A\,s}{s^2+\omega^2}+\frac{B\,\omega}{s^2+\omega^2} = \frac{A\,s+B\,\omega}{s^2+\omega^2}$$

$$Y(s) = \frac{A\,s+B\,\omega}{s^2+\omega^2}G(s) = \frac{A\,s+B\,\omega}{(s+j\omega)(s-j\omega)}G(s) = \frac{K_1}{s+j\omega}+\frac{K_2}{s-j\omega}+F(s)$$

Partial Fraction

$$K_1 = \left[\frac{A\,s+B\,\omega}{s^2+\omega^2}(s+j\omega)G(s)\right]_{s=-j\omega} = \left[\frac{A\,s+B\,\omega}{(s-j\omega)}G(s)\right]_{s=-j\omega} = \frac{-A\,j\omega+B\,\omega}{-2\,j\omega}G(-j\omega) = \frac{1}{2}(A+jB)G(-j\omega)$$

$$K_2 = \left[\frac{A\,s+B\,\omega}{s^2+\omega^2}(s-j\omega)G(s)\right]_{s=+j\omega} = \left[\frac{A\,s+B\,\omega}{(s+j\omega)}G(s)\right]_{s=+j\omega} = \frac{A\,j\omega+B\,\omega}{+2\,j\omega}G(+j\omega) = \frac{1}{2}(A-jB)G(+j\omega)$$

Frequency Response in Control Theory (3)

$\sqrt{A^2+B^2}\cos(\omega t-\theta)$ → $G(s)$ → $|G(j\omega)|\,\sqrt{A^2+B^2}\,\cos(\omega t-\theta+\arg\{G(j\omega)\})$

$$Y(s) = \frac{K_1}{s+j\omega} + \frac{K_2}{s-j\omega} + F(s)$$

$$K_1 = \frac{1}{2}(A+jB)G(-j\omega) \qquad K_2 = \frac{1}{2}(A-jB)G(+j\omega)$$

$$A \pm jB = \sqrt{A^2+B^2}\left[\frac{A}{\sqrt{A^2+B^2}} \pm j\frac{B}{\sqrt{A^2+B^2}}\right]$$

$$= \sqrt{A^2+B^2}\,[\cos\theta \pm j\sin\theta]$$

$$= \sqrt{A^2+B^2}\,e^{\pm j\theta}$$

Stable System

$e^{p_i} \to 0 \qquad e^{\sigma_i} \to 0$

system modes

$p_i<0,\ \sigma_i<0$

$$\lim_{t\to\infty} f(t) = 0 \qquad \lim_{s\to\infty} sF(s) = 0$$

Ignore $F(s)$

$$Y_{ss}(s) = \frac{K_1}{s+j\omega} + \frac{K_2}{s-j\omega}$$

$$K_1 = \frac{1}{2}\sqrt{A^2+B^2}\,e^{+j\theta}G(-j\omega) \qquad K_2 = \frac{1}{2}\sqrt{A^2+B^2}\,e^{-j\theta}G(-j\omega)$$

Frequency Response in Control Theory (4)

$$\sqrt{A^2+B^2}\cos(\omega t-\theta) \rightarrow G(s) \rightarrow |G(j\omega)|\sqrt{A^2+B^2}\cos(\omega t-\theta+arg\{G(j\omega)\})$$

$$Y_{ss}(s) = \frac{K_1}{s+j\omega} + \frac{K_2}{s-j\omega} \quad = \frac{1}{2}(A+jB)G(-j\omega)\frac{1}{s+j\omega} + \frac{1}{2}(A-jB)G(+j\omega)\frac{1}{s-j\omega}$$

$$= \frac{1}{2}\sqrt{A^2+B^2}e^{+j\theta}G(-j\omega)\frac{1}{s+j\omega} + \frac{1}{2}\sqrt{A^2+B^2}e^{-j\theta}G(+j\omega)\frac{1}{s-j\omega}$$

$$y_{ss}(t) = \frac{\sqrt{A^2+B^2}}{2}\left[G(-j\omega)e^{-j\omega t}e^{+j\theta} + G(+j\omega)e^{+j\omega t}e^{-j\theta}\right]$$

$$= \frac{\sqrt{A^2+B^2}}{2}\left[G(+j\omega)e^{-j(\omega t-\theta)} + G(+j\omega)e^{+j(\omega t-\theta)}\right]$$

$$= \sqrt{A^2+B^2}\ \Re\{G(+j\omega)e^{+j(\omega t-\theta)}\}$$

$$= \sqrt{A^2+B^2}\ |G(+j\omega)|\cos(\omega t-\theta+arg\{G(+j\omega)\})$$

Using the natural frequency of the simple harmonic oscillator $\omega_n = \sqrt{k/m}$ and the definition of the damping ratio above, we can rewrite this as:

$$\frac{d^2x}{dt^2} + 2\zeta\omega_n \frac{dx}{dt} + \omega_n^2 x = 0.$$

This equation can be solved with the approach.

$$x(t) = Ce^{st},$$

where C and s are both complex constants. That approach assumes a solution that is oscillatory and/or decaying exponentially. Using it in the ODE gives a condition on the frequency of the damped oscillations,

$$s = -\omega_n(\zeta \pm \sqrt{\zeta^2 - 1}).$$

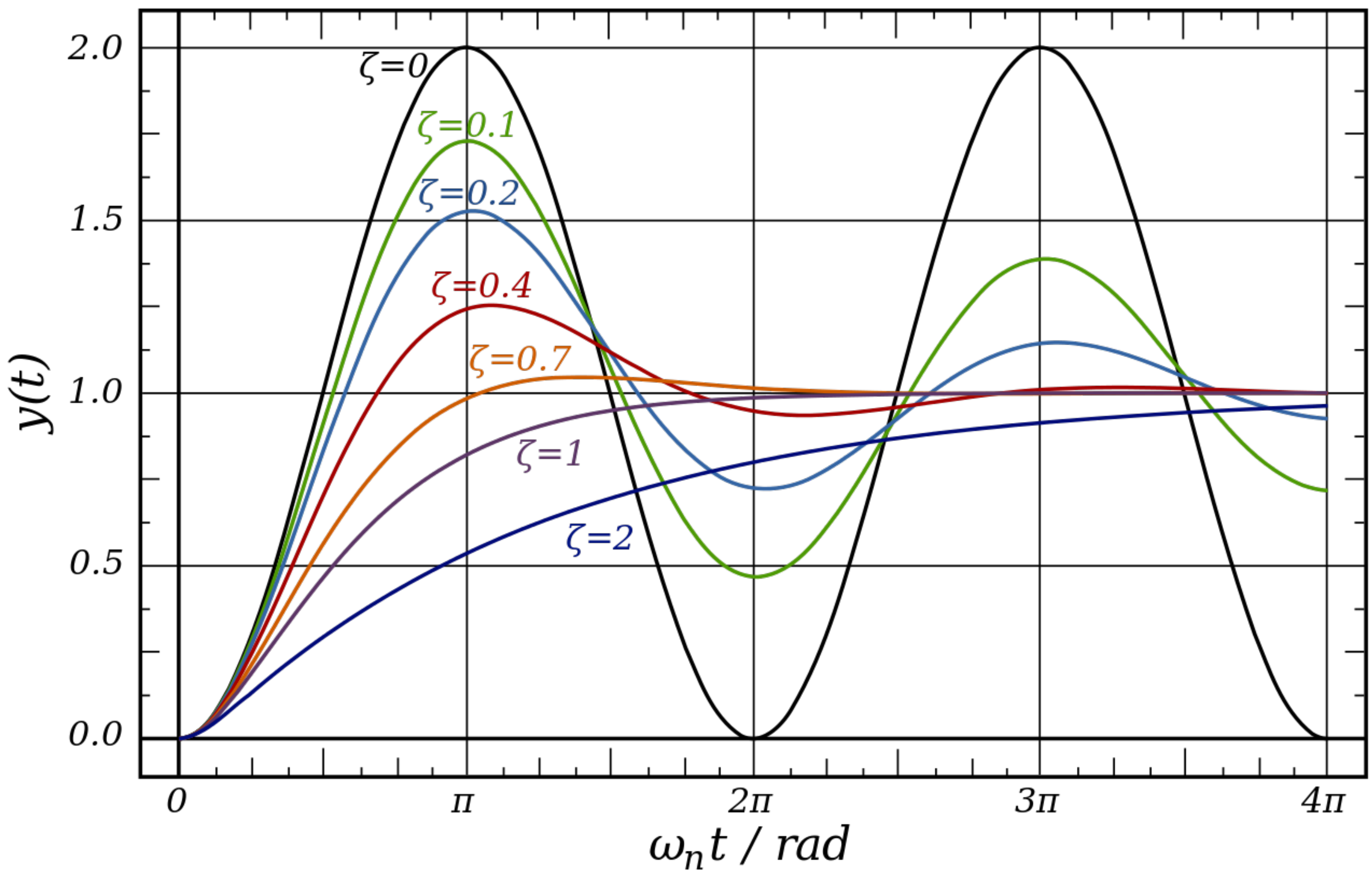
http://en.wikipedia.org/wiki/

Impulse Response h(t)

- **Undamped**: Is the case where $\zeta \to 0$ corresponds to the undamped simple harmonic oscillator, and in that case the solution looks like $\exp(i\omega_n t)$, as expected.
- **Underdamped**: If *s* is a complex number, then the solution is a decaying exponential combined with an oscillatory portion that looks like $\exp(i\omega_n \sqrt{1-\zeta^2 t})$. This case occurs for $\zeta < 1$, and is referred to as *underdamped*.
- **Overdamped**: If *s* is a real number, then the solution is simply a decaying exponential with no oscillation. This case occurs for $\zeta > 1$, and is referred to as *overdamped*.
- **Critically damped**:The case where $\zeta = 1$ is the border between the overdamped and underdamped cases, and is referred to as *critically damped*. This turns out to be a desirable outcome in many cases where engineering design of a damped oscillator is required (e.g., a door closing mechanism).

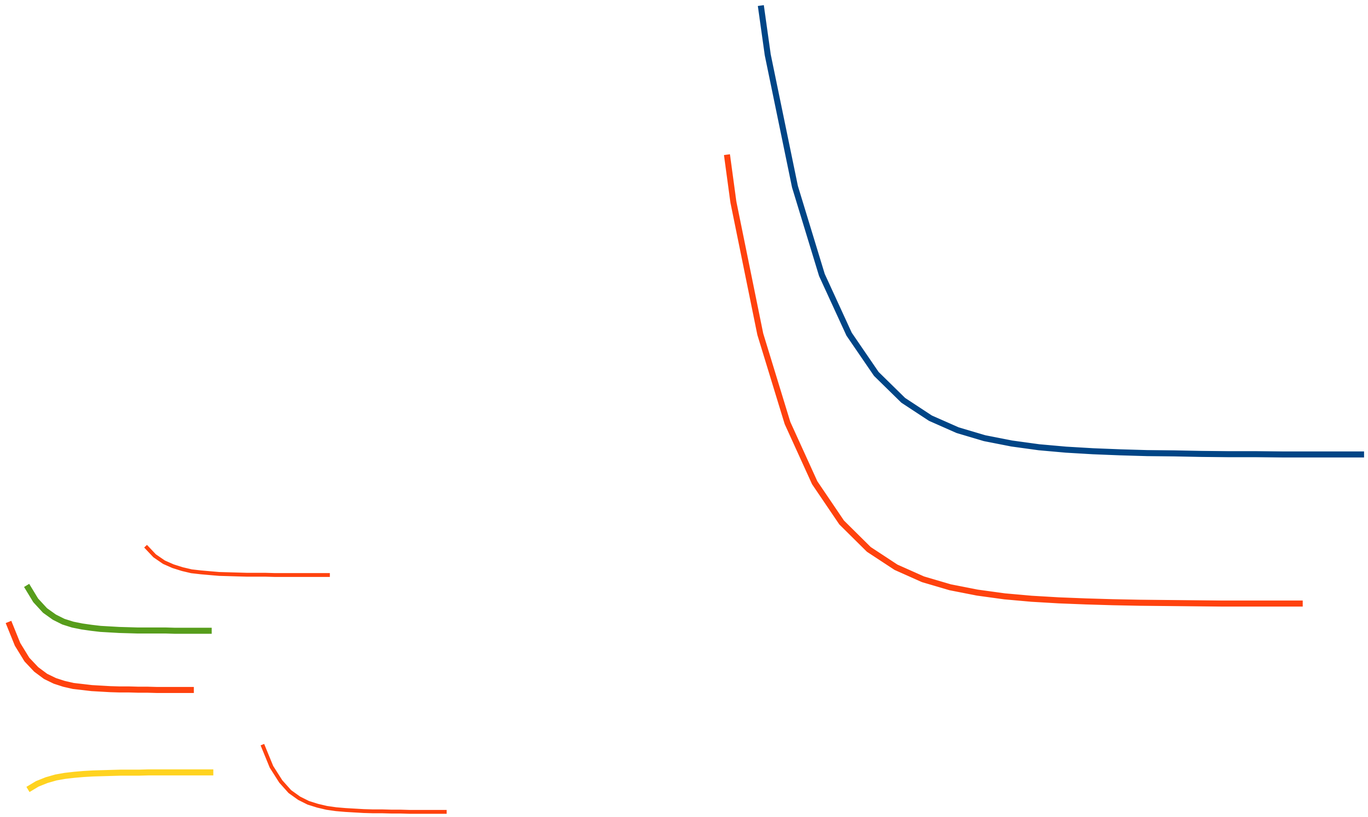
http://en.wikipedia.org/wiki/

Impulse Response h(t)



http://en.wikipedia.org/wiki/

Impulse Response h(t)



References

[1] http://en.wikipedia.org/
[2] J.H. McClellan, et al., Signal Processing First, Pearson Prentice Hall, 2003
[3] M. J. Roberts, Fundamentals of Signals and Systems
[4] S. J. Orfanidis, Introduction to Signal Processing
[5] B. P. Lathi, Signals and Systems