

# CMOS Delay-7 (H.7) Elmore Delay

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# References

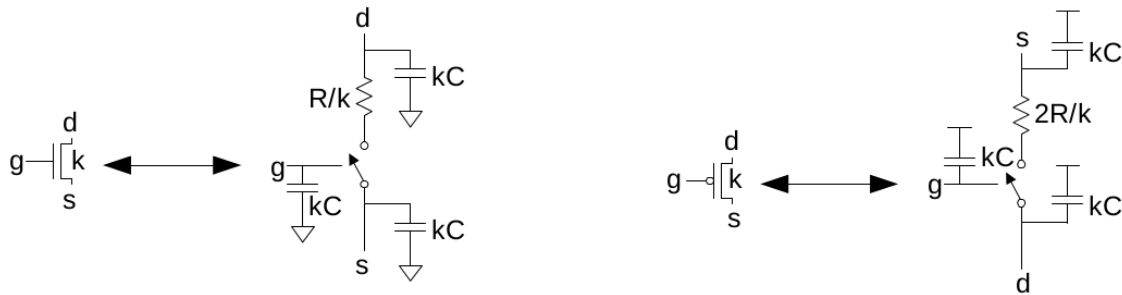
Some Figures from the following sites

[1] <http://pages.hmc.edu/harris/cmosvlsi/4e/index.html>  
Weste & Harris Book Site

[2] [en.wikipedia.org](http://en.wikipedia.org)

# RC Delay Model

- ❑ Use equivalent circuits for MOS transistors
  - Ideal switch + capacitance and ON resistance
  - Unit nMOS has resistance  $R$ , capacitance  $C$
  - Unit pMOS has resistance  $2R$ , capacitance  $C$
- ❑ Capacitance proportional to width
- ❑ Resistance inversely proportional to width

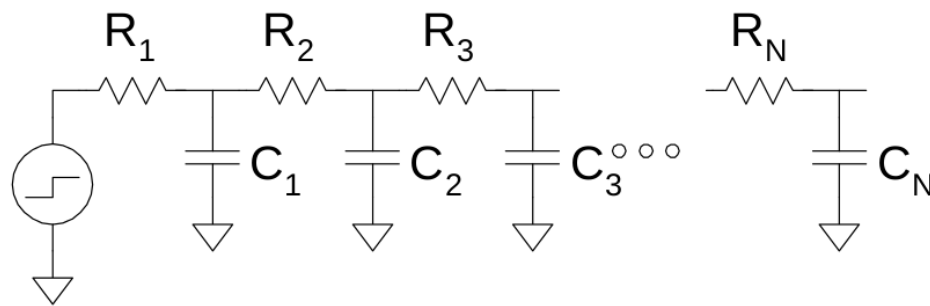


# Elmore Delay

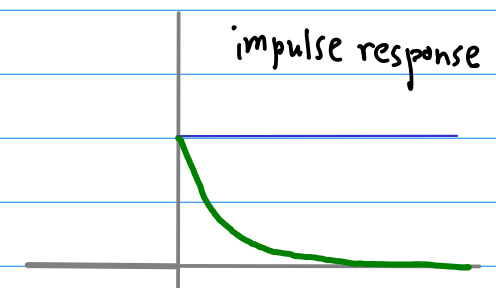
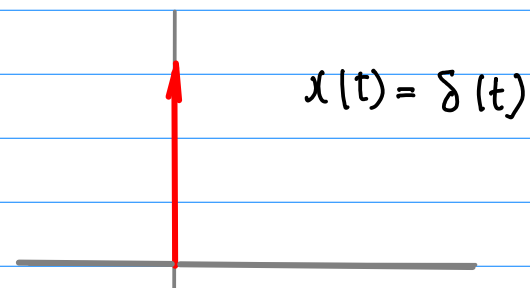
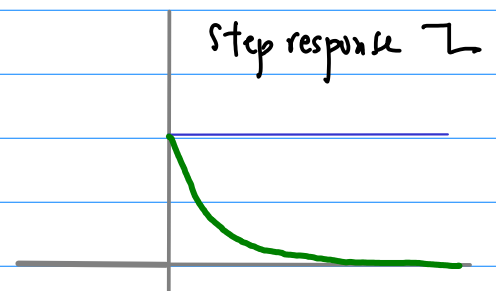
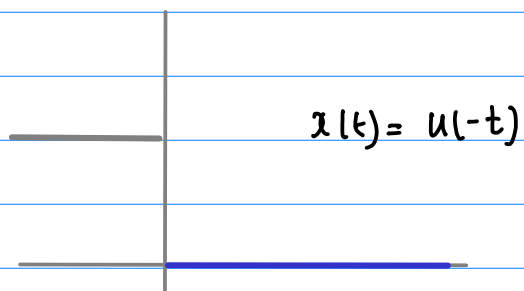
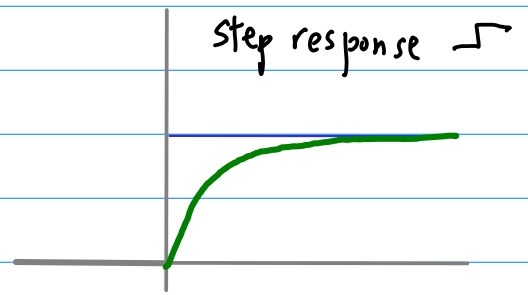
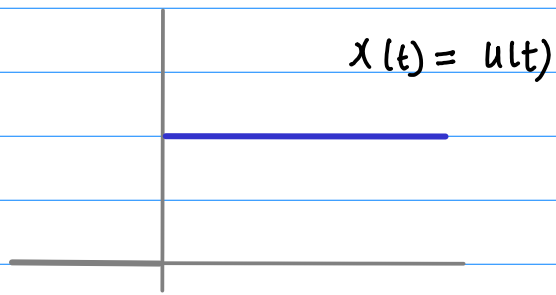
- ❑ ON transistors look like resistors
- ❑ Pullup or pulldown network modeled as *RC ladder*
- ❑ Elmore delay of RC ladder

$$t_{pd} \approx \sum_{\text{nodes } i} R_{i\text{-to-source}} C_i$$

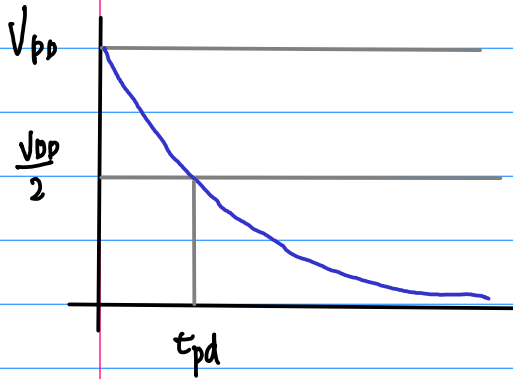
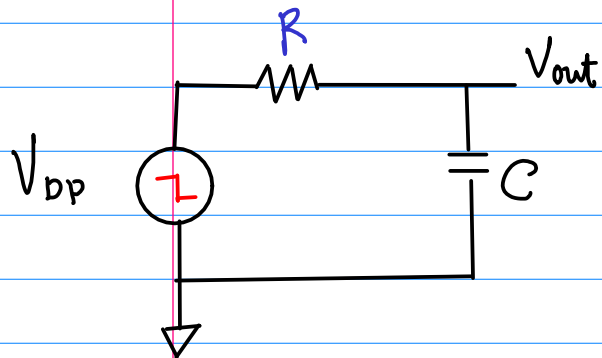
$$= R_1 C_1 + (R_1 + R_2) C_2 + \dots + (R_1 + R_2 + \dots + R_N) C_N$$



# Step Response & Impulse Response



# Transient Response : 1<sup>st</sup> Order RC Systems



Step response to  $\downarrow$

$$\frac{\frac{1}{sC}}{R + \frac{1}{sC}} = \frac{1}{1 + sRC}$$

$$H(s) = \frac{1}{1 + sRC}$$

$$V_{out}(t) = V_{DP} e^{-t/\tau}$$

$$\tau = RC$$

$$\frac{V_{DP}}{2} = V_{DP} e^{-t/\tau}$$

$$\frac{1}{2} = e^{-t/\tau}$$

$$-t/\tau = \ln 2^{-1}$$

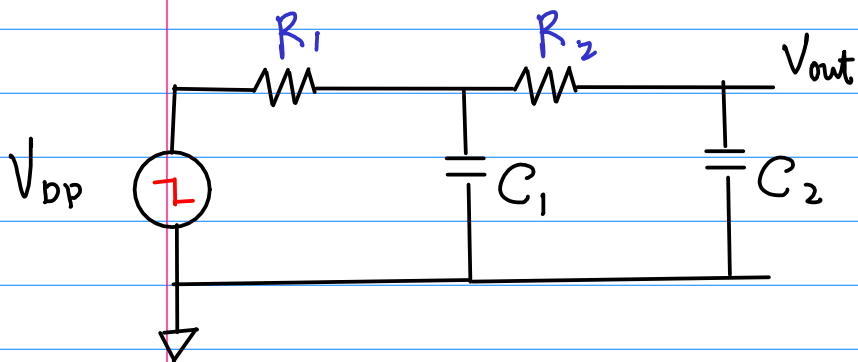
$$t = \ln 2 \tau$$

$$t_{pd} = RC \ln 2$$

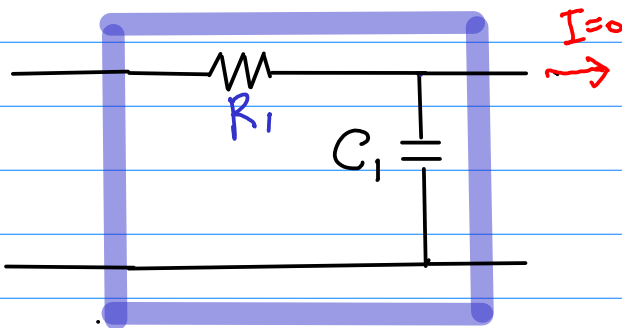
$$= R \ln 2 \cdot C$$

$$= R' C$$

# Transient Response : 2<sup>nd</sup> Order RC Systems

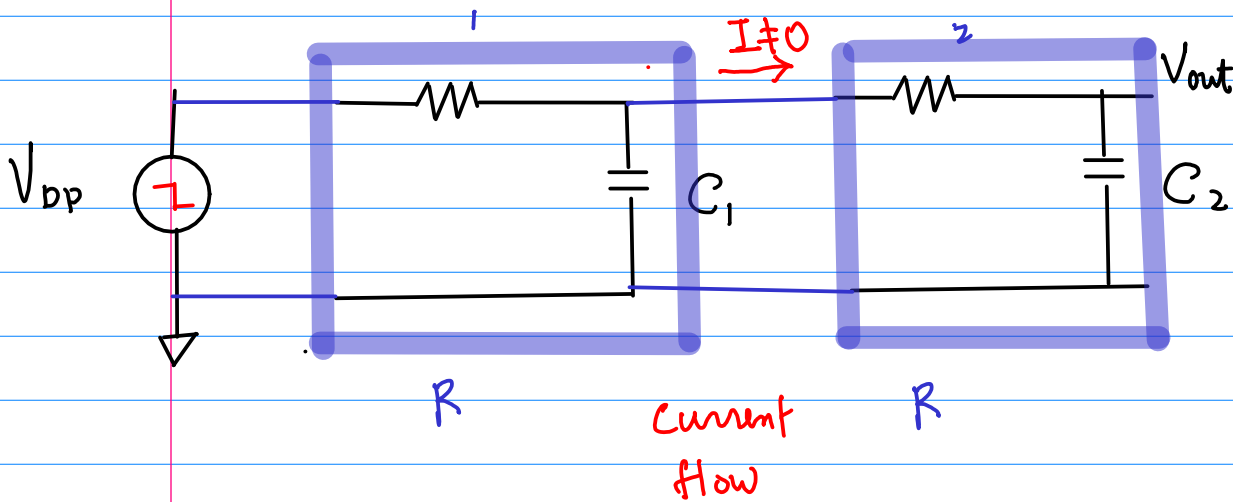


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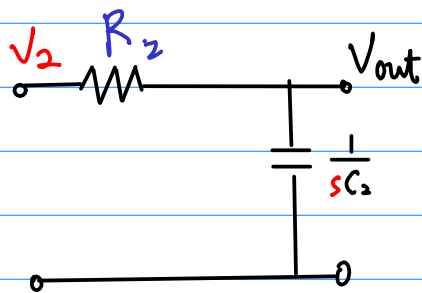
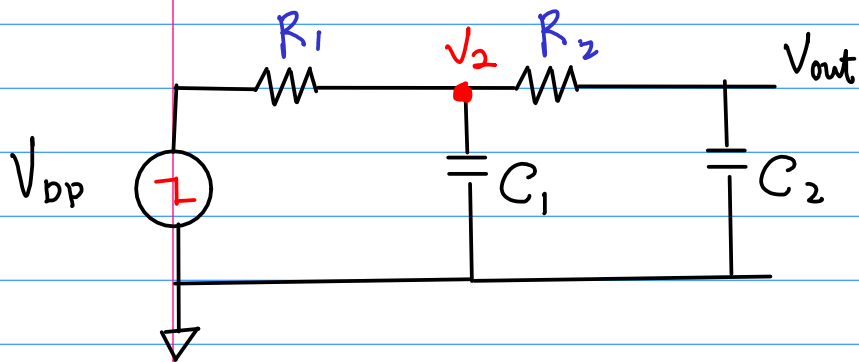
$$H(s) = \frac{1}{1 + sRC}$$

no current flow

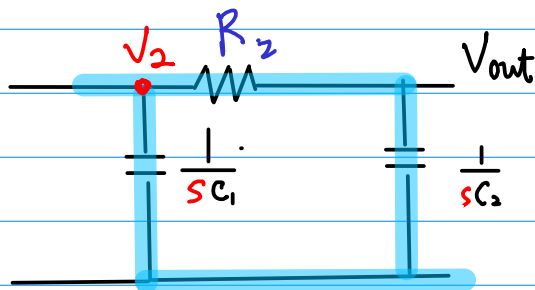


$$H(s) = \cancel{\frac{1}{1 + sR_1C_1}} \cdot \frac{1}{1 + sR_2C_2} = \frac{1}{1 + s(R_1C_1 + R_2C_2) + s^2R_1R_2C_1C_2}$$

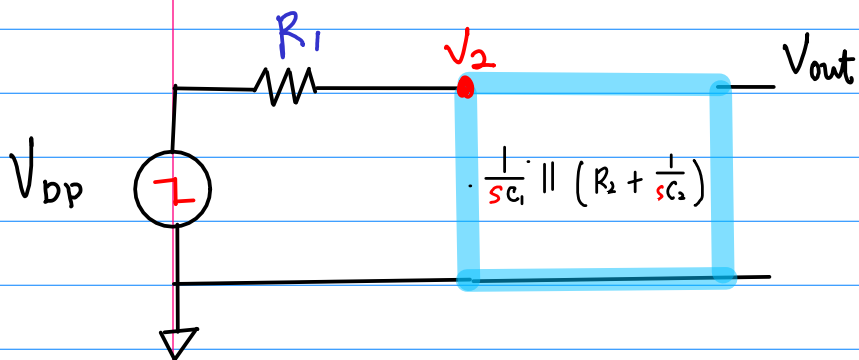
# Voltage Divider



$$V_{out} = \frac{\frac{1}{sC_2}}{R_2 + \frac{1}{sC_2}} V_2$$



$$\frac{1}{sC_1} \parallel \left( R_2 + \frac{1}{sC_2} \right)$$

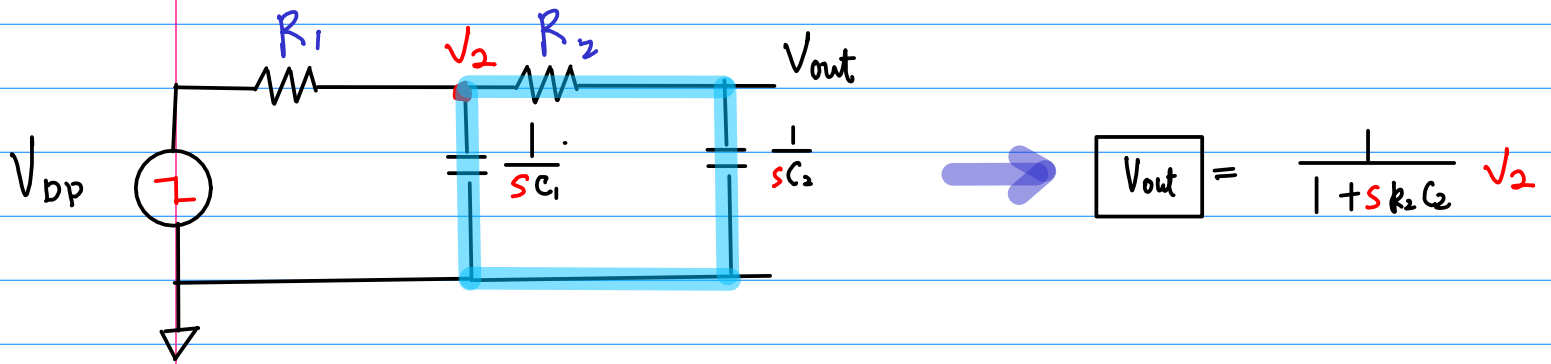


$$V_2 = \frac{\frac{1}{sC_1} \parallel \left( R_2 + \frac{1}{sC_2} \right)}{R_1 + \frac{1}{sC_1} \parallel \left( R_2 + \frac{1}{sC_2} \right)} V_{dp}$$

→ 
$$V_{out} = \frac{1}{1 + sR_2C_2} V_2$$



# Transfer Function $H(s)$



$$V_{out} = \frac{1}{1 + sR_2C_2} V_2$$

$$\frac{1}{sC_1} \parallel \left( R_2 + \frac{1}{sC_2} \right) = \frac{1}{sC_1 + \frac{1}{R_2 + \frac{1}{sC_2}}} = \frac{1}{sC_1 + \frac{sC_2}{sR_2C_2 + 1}}$$

$$= \frac{sR_2C_2 + 1}{s^2R_2C_1C_2 + sC_1 + sC_2} = \frac{sR_2C_2 + 1}{s^2R_2C_1C_2 + s(C_1 + C_2)}$$

$$V_2 = \frac{\frac{sR_2C_2 + 1}{s^2R_2C_1C_2 + s(C_1 + C_2)} V_{dp}}{R_1 + \frac{sR_2C_2 + 1}{s^2R_2C_1C_2 + s(C_1 + C_2)}} \quad V_{dp} = \frac{sR_2C_2 + 1}{s^2R_1R_2C_1C_2 + s(R_1C_1 + R_1C_2 + R_2C_2) + 1} V_{dp}$$

$$V_{out} = \frac{1}{1 + sR_2C_2} V_2 = \frac{1}{\cancel{(1 + sR_2C_2)} s^2R_1R_2C_1C_2 + s(R_1C_1 + R_1C_2 + R_2C_2) + 1} V_{dp}$$

$$H(s) = \frac{1}{1 + s[R_1C_1 + (R_1 + R_2)C_2] + s^2R_1C_1R_2C_2}$$

# Quadratic Equations

the reciprocals of the roots

$$a s^2 + b s + c = 0$$

$$s = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\frac{1}{s} = \frac{2a}{-b \pm \sqrt{b^2 - 4ac}} \rightarrow \frac{1}{2c} \left[ -b \mp \sqrt{b^2 - 4ac} \right]$$

$$\frac{1}{-b \pm \sqrt{b^2 - 4ac}} = \frac{-b \mp \sqrt{b^2 - 4ac}}{(-b)^2 - (b^2 - 4ac)} = \frac{-b \mp \sqrt{b^2 - 4ac}}{4ac}$$

$$\frac{2a}{-b \pm \sqrt{b^2 - 4ac}} = 2a \cdot \frac{-b \mp \sqrt{b^2 - 4ac}}{4ac} = \frac{1}{2c} \left[ -b \mp \sqrt{b^2 - 4ac} \right]$$

the reciprocals of the poles of  $H(s)$

$$a s^2 + b s + c = 0$$

$$\frac{1}{s} = \frac{2a}{-b \pm \sqrt{b^2 - 4ac}} \rightarrow \frac{1}{2c} \left[ -b \mp \sqrt{b^2 - 4ac} \right]$$

$$1 + s \left[ R_1 C_1 + (R_1 + R_2) C_2 \right] + s^2 R_1 C_1 R_2 C_2 = (1 + s z_1) (1 + s z_2) = 0$$

$$s = -\frac{1}{z_1}, -\frac{1}{z_2} \quad \frac{1}{s} = z_1, z_2$$

$$a = R_1 C_1 R_2 C_2$$

$$b = \left[ R_1 C_1 + (R_1 + R_2) C_2 \right]$$

$$c = 1$$

$$\frac{1}{s} = \frac{1}{2c} \left[ -b \mp \sqrt{b^2 - 4ac} \right] \rightarrow$$

$$\frac{1}{s} = \frac{1}{2} \left[ -\left[ R_1 C_1 + (R_1 + R_2) C_2 \right] \mp \sqrt{\left[ R_1 C_1 + (R_1 + R_2) C_2 \right]^2 - 4 R_1 C_1 R_2 C_2} \right]$$

$$\frac{1}{s} = -\frac{1}{2} \left[ R_1 C_1 + (R_1 + R_2) C_2 \right] \left[ 1 \pm \sqrt{1 - \frac{4 R_1 C_1 R_2 C_2}{\left[ R_1 C_1 + (R_1 + R_2) C_2 \right]^2}} \right]$$

Time constants  $\tau_1$  &  $\tau_2$

$$1 + s[R_1 C_1 + (R_1 + R_2) C_2] + s^2 R_1 C_1 R_2 C_2 = (1 + s \tau_1)(1 + s \tau_2) = 0$$

$$s = -\frac{1}{\tau_1}, -\frac{1}{\tau_2}$$

$$\frac{1}{s} = -\frac{1}{2} [R_1 C_1 + (R_1 + R_2) C_2] \left[ 1 \pm \sqrt{1 - \frac{4 R_1 C_1 R_2 C_2}{[R_1 C_1 + (R_1 + R_2) C_2]^2}} \right]$$

$$\frac{R_2}{R_1} \equiv R'$$

$$\frac{C_2}{C_1} \equiv C'$$

$$\sqrt{1 - \frac{4 R_1 C_1 R_2 C_2}{[R_1 C_1 + (R_1 + R_2) C_2]^2}}$$

$$\equiv \sqrt{1 - \frac{4 \frac{R_2}{R_1} \frac{C_2}{C_1}}{\left[ 1 + \left( 1 + \frac{R_2}{R_1} \right) \frac{C_2}{C_1} \right]^2}}$$

$$\equiv \sqrt{1 - \frac{4 R' C'}{\left[ 1 + (1 + R') C' \right]^2}}$$

$$\tau_1, \tau_2 = \frac{1}{2} [R_1 C_1 + (R_1 + R_2) C_2] \left[ 1 \pm \sqrt{1 - \frac{4 R' C'}{\left[ 1 + (1 + R') C' \right]^2}} \right]$$

# Unit Step Response

$$H(s) = \frac{1}{1 + s[R_1 C_1 + (R_1 + R_2)C_2] + s^2 R_1 C_1 R_2 C_2}$$
$$= \frac{1}{(1 + s z_1)(1 + s z_2)} = \left[ \frac{A}{(1 + s z_1)} + \frac{B}{(1 + s z_2)} \right]$$

$$A = \frac{1}{(1 + s z_2)} \Big|_{s = -\frac{1}{z_1}} = \frac{1}{(1 - \frac{z_2}{z_1})} = \frac{z_1}{z_1 - z_2}$$
$$B = \frac{1}{(1 + s z_1)} \Big|_{s = -\frac{1}{z_2}} = \frac{1}{(1 - \frac{z_1}{z_2})} = \frac{z_2}{z_2 - z_1}$$

$$H(s) = \frac{1}{z_1 - z_2} \left[ \frac{z_1}{(1 + s z_1)} - \frac{z_2}{(1 + s z_2)} \right]$$

$$h(t) = \frac{1}{z_1 - z_2} \left[ z_1 e^{-t/z_1} - z_2 e^{-t/z_2} \right]$$

step response to  $\downarrow$

$$V_{out}(t) = \frac{1}{z_1 - z_2} \left[ z_1 e^{-t/z_1} - z_2 e^{-t/z_2} \right] V_{DD}$$

$$\tau_1, \tau_2 = \frac{1}{2} [R_1 C_1 + (R_1 + R_2) C_2] \left[ 1 \pm \sqrt{1 - \frac{4 R' C'}{[1 + (1 + R') C']^2}} \right]$$

$$\tau = \tau_1 + \tau_2 = [R_1 C_1 + (R_1 + R_2) C_2]$$

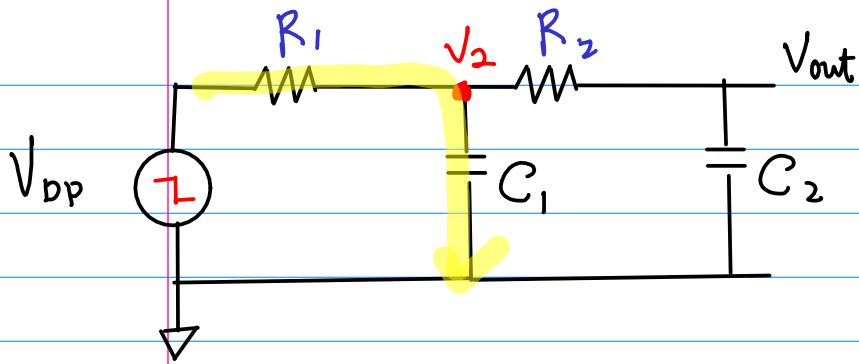
$$R = R_1 = R_2$$

$$C = C_1 = C_2$$

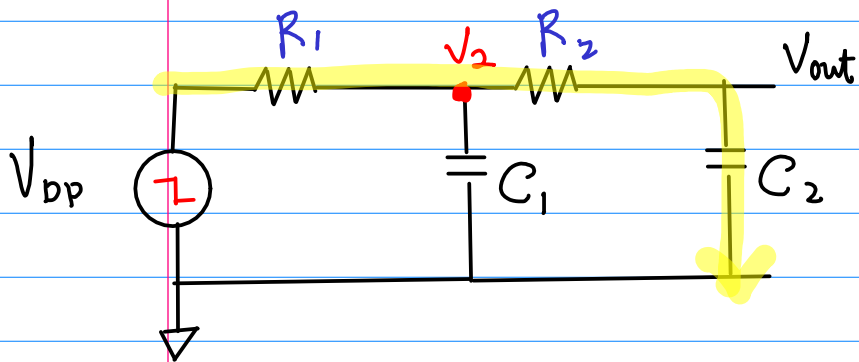
$$\tau_1 = 2.6 RC$$

$$\tau_2 = 0.4 RC$$

$$\tau = 3.0 RC$$

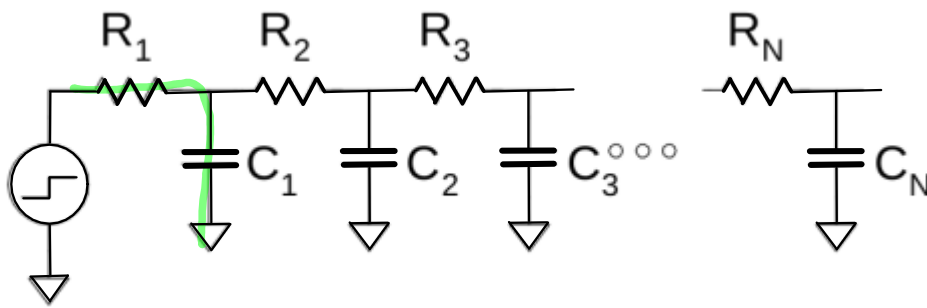


$$Z_1 = R_1 C_1$$

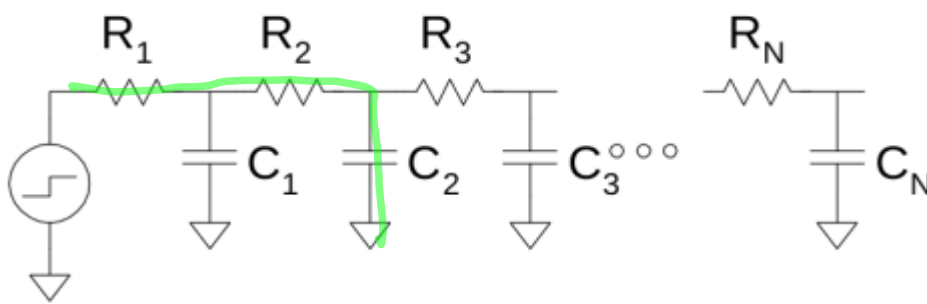


$$Z_2 = (R_1 + R_2) C_2$$

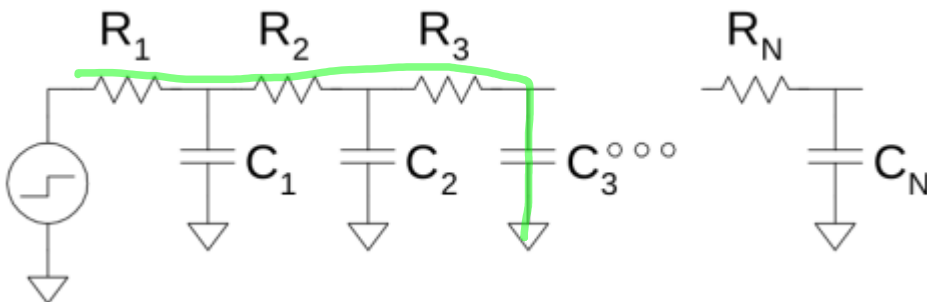
$$Z = Z_1 + Z_2 = [R_1 C_1 + (R_1 + R_2) C_2]$$



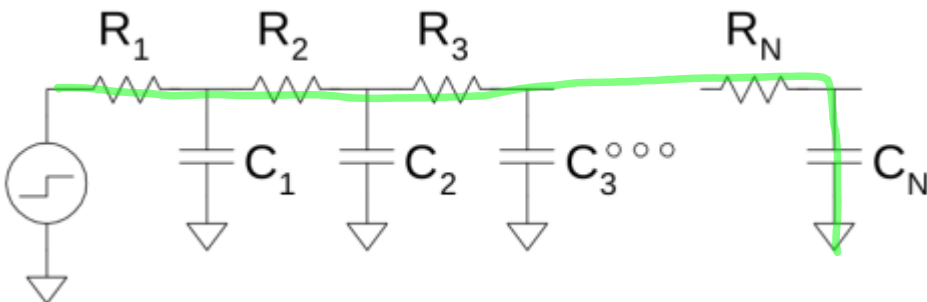
$$R_1 C_1$$



$$(R_1 + R_2) C_2$$



$$(R_1 + R_2 + R_3) C_3$$

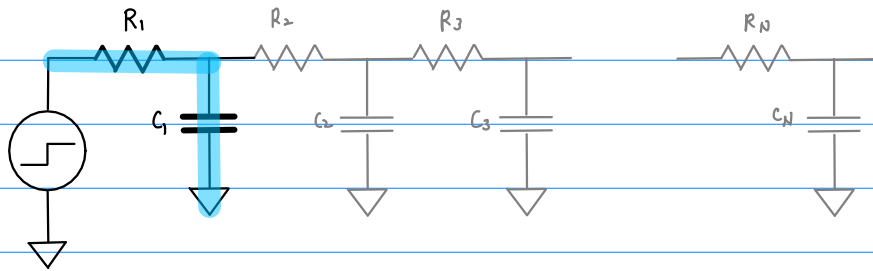


$$(R_1 + R_2 + \dots + R_N) C_N$$

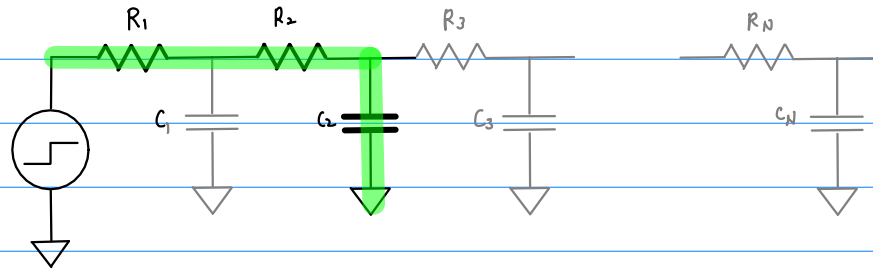
$$t_{pd} = R_1 C_1 + (R_1 + R_2) C_2 + \dots + (R_1 + R_2 + \dots + R_N) C_N$$



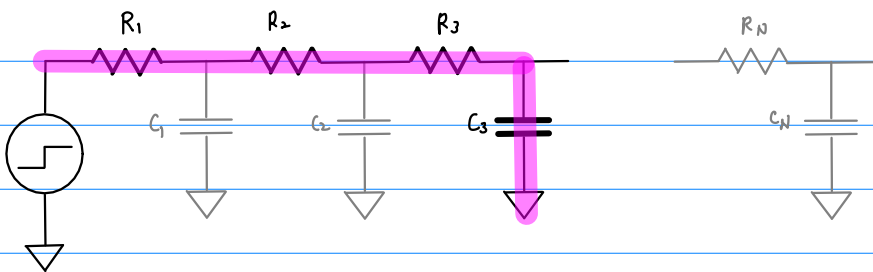




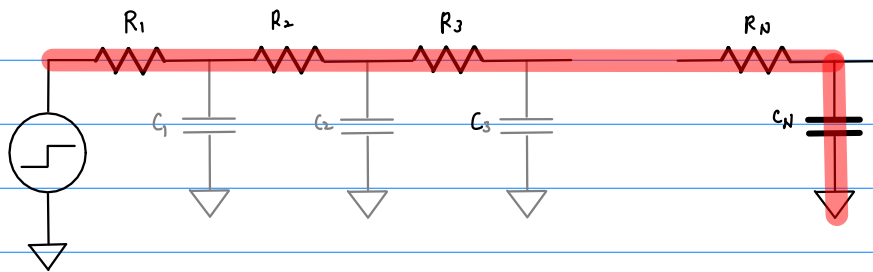
$$(R_1) C_1$$



$$(R_1 + R_2) C_2$$



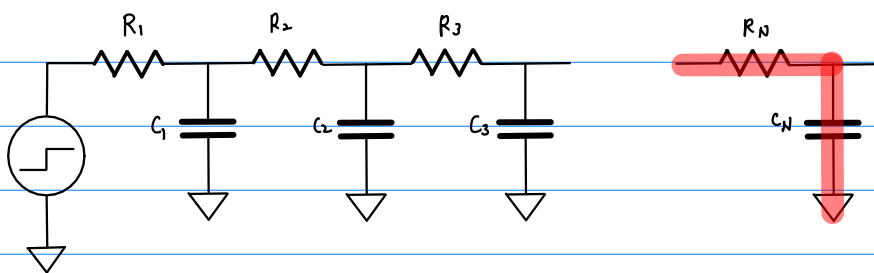
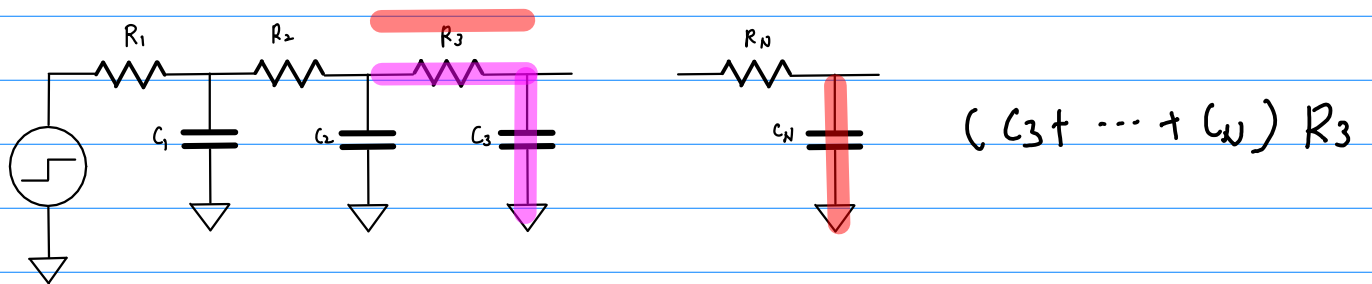
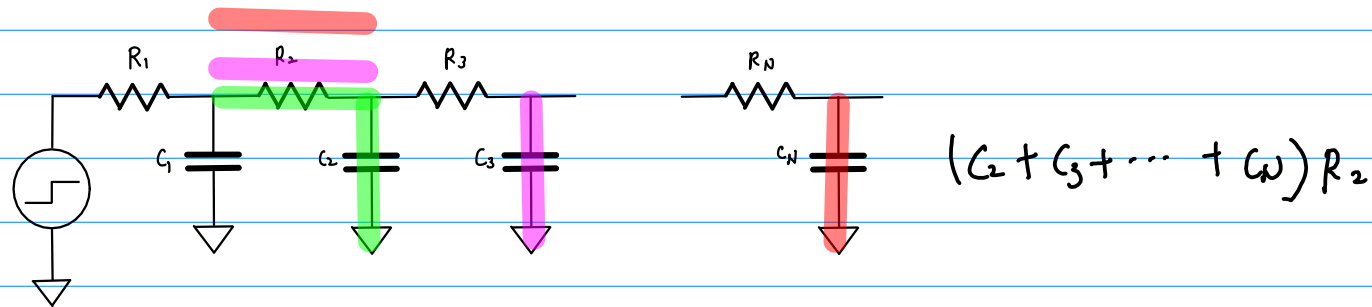
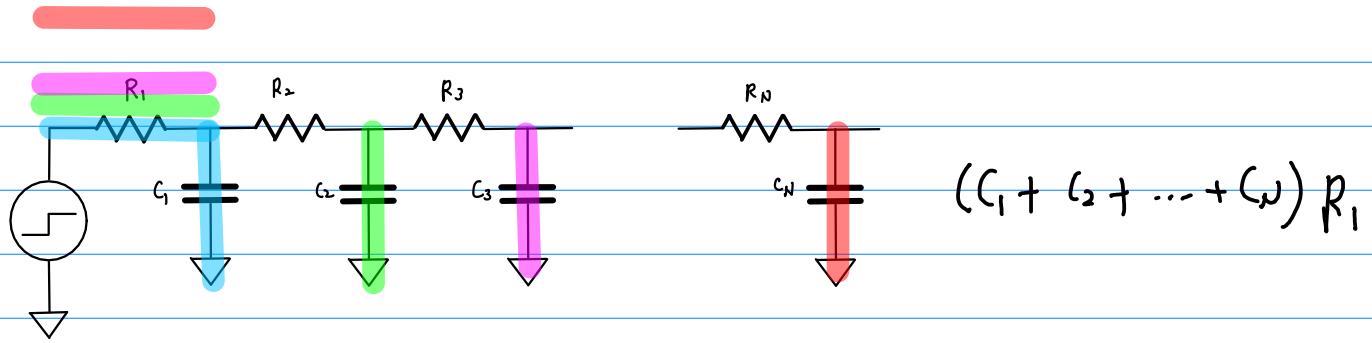
$$(R_1 + R_2 + R_3) C_3$$



$$(R_1 + R_2 + \dots + R_N) C_N$$

$$t_{pd} = (R_1) C_1 + (R_1 + R_2) C_2 + (R_1 + R_2 + R_3) C_3 + \dots + (R_1 + R_2 + \dots + R_N) C_N$$

$$t_{pd} = \sum_{i=1}^n \left( \sum_{j=1}^i R_j \right) C_i$$



$$t_{pd} = (C_1 + C_2 + \dots + C_N) R_1 + (C_2 + C_3 + \dots + C_N) R_2 + (C_3 + C_4 + \dots + C_N) R_3 + \dots + C_N R_N$$

$$t_{pd} = \sum_{i=1}^n \left( \sum_{j=i}^n C_j \right) R_i$$

