

# Moment Functions

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Based on  
Probability, Random Variables and Random Signal Principles,  
P.Z. Peebles,Jr. and B. Shi

# Outline

## 1 Moment Related Functions

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# Characteristic Function

## Definition

the characteristic function of a random variable  $X$

$$\Phi_X(\omega) = E[e^{j\omega X}] = \int_{-\infty}^{+\infty} e^{j\omega x} f_X(x) dx \quad (-\infty < \omega < +\infty)$$

can be considered as the Fourier transform of  $f_X(x)$

## Characteristic Function as a Fourier Transform

## Definitions

the forward Fourier transform of  $f_X(x)$

$$\Phi_X(\omega) = E[e^{j\omega X}] = \int_{-\infty}^{+\infty} e^{j\omega x} f_X(x) dx \quad (-\infty < \omega < +\infty)$$

the inverse Fourier transform of  $\Phi_X(\omega)$

$$f_X(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \Phi(\omega) e^{-j\omega x} d\omega$$

## Moments from the Characteristic Function

## Definition

differentiating  $\Phi_X(\omega)$  n times and setting to  $\omega = 0$

$$m_n = (-j)^n \left. \frac{d^n \Phi_X(\omega)}{d\omega^n} \right|_{\omega=0}$$

$$|\Phi_X(\omega)| \leq \Phi_X(0) = 1$$

Derivatives of  $\Phi_X(\omega)$ 

differentiating  $\Phi_X(\omega)$  n times

$$\Phi_X(\omega) = E[e^{j\omega X}] = \int_{-\infty}^{+\infty} e^{j\omega x} f_X(x) dx$$

$$\frac{d}{d\omega} \Phi_X(\omega) = E[jX e^{j\omega X}] = \int_{-\infty}^{+\infty} j e^{j\omega x} x f_X(x) dx$$

$$\frac{d^2}{d\omega^2} \Phi_X(\omega) = E[j^2 X^2 e^{j\omega X}] = \int_{-\infty}^{+\infty} j^2 e^{j\omega x} x^2 f_X(x) dx$$



Derivatives of  $\Phi_X(\omega)|_{\omega=0}$ 

differentiating  $\Phi_X(\omega)$  n times and setting to  $\omega = 0$

$$\Phi_X(\omega)|_{\omega=0} = E[1] = 1$$

$$\left. \frac{d}{d\omega} \Phi_X(\omega) \right|_{\omega=0} = jE[X] = j \int_{-\infty}^{+\infty} x f_X(x) dx$$

$$\left. \frac{d^2}{d\omega^2} \Phi_X(\omega) \right|_{\omega=0} = j^2 E[X^2] = j^2 \int_{-\infty}^{+\infty} x^2 f_X(x) dx$$

# Moment Generating Function

## Definitions

Moment Generating Function

$$M_X(\mathbf{v}) = E[e^{\mathbf{v}X}] = \int_{-\infty}^{+\infty} e^{\mathbf{v}x} f_X(x) dx$$

Characteristic Function

$$\Phi_X(\omega) = E[e^{j\omega X}] = \int_{-\infty}^{+\infty} e^{j\omega x} f_X(x) dx$$

# Moment from the generating function

## Definitions

Moment from the Generating Function

$$m_n = \left. \frac{d^n M_X(\mathbf{v})}{d\mathbf{v}^n} \right|_{\mathbf{v}=0}$$

Moment from the Characteristic Function

$$m_n = (-j)^n \left. \frac{d^n \Phi_X(\omega)}{d\omega^n} \right|_{\omega=0}$$