Moment Functions

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May 9, 2020

Based on
Probability, Random Variables and Random Signal Principles,
P.Z. Peebles,Jr. and B. Shi

Outline

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Characteristic Function

Definition

the characteristic function of a random varialbe X

$$\Phi_X(\omega) = E[e^{j\omega X}] = \int_{-\infty}^{+\infty} e^{j\omega x} f_X(x) dx \qquad (-\infty < \omega < +\infty)$$

can be considered as the Fourier transform of $f_X(x)$

Characteristic Function as a Fourier Transform

Definitions

the forward Fourier transform of $f_X(x)$

$$\Phi_X(\omega) = E[e^{j\omega X}] = \int_{-\infty}^{+\infty} e^{j\omega x} f_X(x)dx \qquad (-\infty < \omega + \infty)$$

the inverse Fourier transform of $\Phi_X(\omega)$

$$f_X(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \Phi(\omega) e^{-j\omega x} d\omega$$

Moments from the Characteristic Function

Definition

differentiating $\Phi_X(\omega)$ n times and setting to $\omega = 0$

$$m_n = (-j)^n \frac{d^n \Phi_X(\omega)}{d\omega^n}\bigg|_{\omega=0}$$

$$|\Phi_X(\omega)| \leq \Phi_X(0) = 1$$

Derivatives of $\Phi_X(\omega)$

differentiating $\Phi_X(\omega)$ n times

$$\Phi_X(\omega) = E[e^{j\omega X}] = \int_{-\infty}^{+\infty} e^{j\omega x} f_X(x) dx$$

$$\frac{d}{d\omega}\Phi_X(\omega) = E[jXe^{j\omega X}] = \int_{-\infty}^{+\infty} je^{j\omega x} x f_X(x) dx$$

$$\frac{d^2}{d\omega^2}\Phi_X(\omega) = E[j^2 X^2 e^{j\omega X}] = \int_{-\infty}^{+\infty} j^2 e^{j\omega x} x^2 f_X(x) dx$$

Derivatives of $\Phi_X(\omega)|_{\omega=0}$

differentiating $\Phi_X(\omega)$ n times and setting to $\omega = 0$

$$\Phi_X(\omega)|_{\omega=0} = E[1] = 1$$

$$\frac{d}{d\omega}\Phi_X(\omega)\Big|_{\omega=0} = jE[X] = j\int_{-\infty}^{+\infty} x f_X(x) dx$$

$$\frac{d^2}{d\omega^2}\Phi_X(\omega)\Big|_{\omega=0} = j^2 E[X^2] = j^2 \int_{-\infty}^{+\infty} x^2 f_X(x) dx$$

Moment Generating Function

Definitions

Moment Generating Function

$$M_X(v) = E[e^{vX}] = \int_{-\infty}^{+\infty} e^{vx} f_X(x) dx$$

Characteristic Function

$$\Phi_X(\omega) = E[e^{j\omega X}] = \int_{-\infty}^{+\infty} e^{j\omega x} f_X(x) dx$$

Moment from the generating function

Definitions

Moment from the Generating Function

$$m_n = \left. \frac{d^n M_X(v)}{dv^n} \right|_{v=0}$$

Moment from the Characteristic Function

$$m_n = (-j)^n \left. \frac{d^n \Phi_X(\omega)}{d\omega^n} \right|_{\omega=0}$$