Strum-Liouville (H.1) Eigenfunctions

20151230

Zill & Wright Sec 3.9

Linear Models : Boundary-Value Problem (BVP) → Ex 2

Zill & Wright Chap 13 Boundary-Value Problems in Rectangular Coordinates

IVP

$$ay'' + by' + cy = g(x) \qquad y_p$$

$$ay'' + by' + cy = 0$$

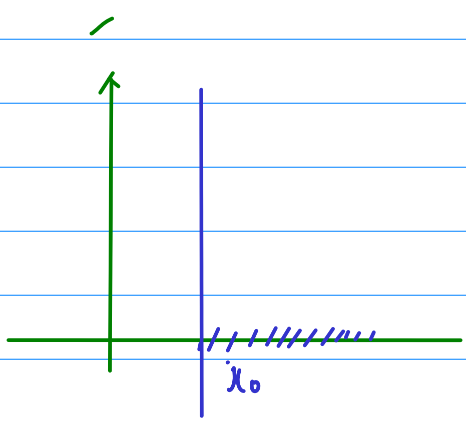
$$am^2 + bm + c = 0 \qquad m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

m_1 $\qquad$ m_2

$e^{m_1 x}$ $\qquad$ $e^{m_2 x}$

$$y_h = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

$$y = y_h + y_p = c_1 e^{m_1 x} + c_2 e^{m_2 x} + y_p$$



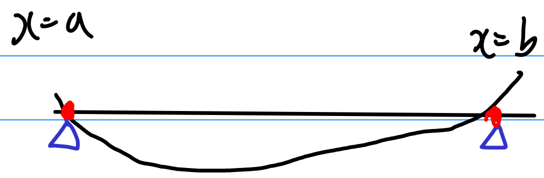
$(x \geq x_0)$

$y(x_0) = k_0$

$y'(x_0) = k_1$

IVP (Initial Value Problem)

BVP



Boundary Condition

$$ay'' + by' + cy = g(x)$$

$$y = y_h + y_p = c_1 e^{m_1 x} + c_2 e^{m_2 x} + y_p$$

$$\begin{cases} y(a) \\ y(b) \end{cases} \quad \begin{cases} y(a) \\ y'(b) \end{cases} \quad \begin{cases} y'(a) \\ y(b) \end{cases} \quad \begin{cases} y'(a) \\ y'(b) \end{cases}$$

BVP (Boundary Value Problem)

Eigenvalue & Eigen function

$$y'' + \lambda y = 0$$

$y(x)$

$\downarrow$

x's formula

$$y'' = -\lambda y$$

$$\frac{d^2}{dx^2}(y) = -\lambda y$$

$$A f = \lambda f$$

Linear Operator

eigenfunction

$$y' + \alpha y = 0$$

$$y'' + \alpha^2 y = 0$$

$$y'' - \alpha^2 y = 0$$

$$a y'' + b y' + c y = 0$$

$$a y'' + c y = 0$$

$$\frac{d^2}{dx^2}(y) = -\frac{c}{a} y$$

$$A f = \lambda f$$

$$\lambda_1 \longrightarrow f_1$$

$$\lambda_2 \longrightarrow f_2$$

$$\lambda_3 \longrightarrow f_3$$

$$\vdots$$

$$\int_a^b f_n \cdot f_m \, dx = 0$$

Regular Stram-Liouville Problem

$$\frac{d}{dx}\left[r(x)\, y' \right] + \left(q(x) + \lambda\, p(x) \right) y = 0$$

$$A_1\, y(a) + B_1\, y'(a) = 0$$
$$A_2\, y(b) + B_2\, y'(b) = 0$$

Linear Operator

$$L(y) \equiv \frac{d}{dx}\left[r(x)\, y' \right] + q(x)\, y = -\lambda\, p(x)\, y$$

more general

$$L(y) = -\underbrace{\lambda}_{\text{eigenvalue}}\, p(x)\, \underbrace{y}_{\text{eigenfunction}}$$

$$\lambda_1 \longrightarrow y_1$$
$$\lambda_2 \longrightarrow y_2$$
$$\lambda_3 \longrightarrow y_3$$
$$\vdots \qquad \vdots$$

$$\int_a^b p(x)\, y_m\, y_n\, dx = 0, \quad \lambda_m \neq \lambda_n$$

weighted orthogonal

$$\cos(x) = \frac{1}{2}\left(e^{ix} + e^{-ix}\right)$$

$$\sin(x) = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right)$$

$$\cosh(x) = \frac{1}{2}\left(e^{x} + e^{-x}\right)$$

$$\sinh(x) = \frac{1}{2}\left(e^{x} - e^{-x}\right)$$

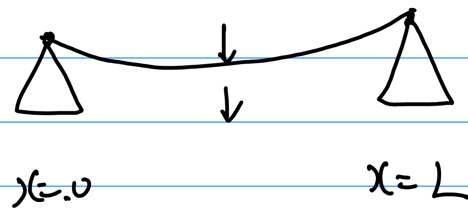
* Solve homogeneous boundary value problem

$$y'' + \lambda y = 0 \qquad y(0) = 0, \quad y(L) = 0$$

$\lambda = 0$

$\lambda < 0$

$\lambda > 0$



Case 1 $\quad \lambda = 0$

$$y'' = 0$$

$$y' = C_1$$

$$y(x) = C_1 x + C_2$$

$$y(0) = C_1 0 + C_2 \Rightarrow C_2 = 0$$

$$y(L) = C_1 . L = 0 \quad \Rightarrow \quad C_1 = 0$$

$$\boxed{y(x) = 0}$$

$$y'' + \lambda y = 0 \qquad y(0)=0, \quad y(L)=0$$

Case II $\quad \lambda < 0$

$$\lambda = -\alpha^2 \quad (\alpha > 0)$$

auxiliary eq.

$$m^2 + \lambda = m^2 - \alpha^2 = 0 \qquad m = +\alpha, -\alpha$$

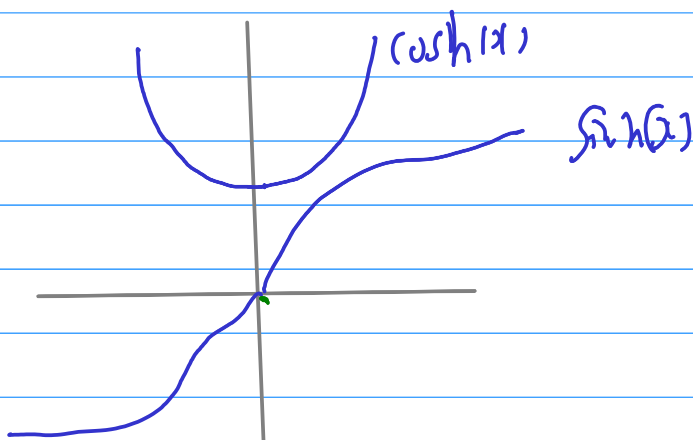
$$y(x) = c_1 e^{+\alpha x} + c_2 e^{-\alpha x}$$

$$y(x) = c_3 \cosh(\alpha x) + c_4 \sinh(\alpha x)$$

$$\left(\begin{aligned} \cosh(\alpha x) &= \tfrac{1}{2}\left(e^{\alpha x} + e^{-\alpha x}\right) \\ \sinh(\alpha x) &= \tfrac{1}{2}\left(e^{\alpha x} - e^{-\alpha x}\right) \end{aligned} \right)$$

$$y(0) = c_3 \overbrace{\cosh(0)}^{=1} + c_4 \overbrace{\sinh(0)}^{=0} = c_3 = 0$$

$$y(L) = c_4 \underbrace{\sinh(\underbrace{\alpha L}_{\neq 0})}_{\neq 0} = 0$$



$$c_4 = 0$$

$$\boxed{y(x) = 0}$$

$$y'' + \lambda y = 0 \qquad y(0)=0, \quad y(L)=0$$

Case IV $\quad \lambda > 0$

$\lambda = +\alpha^2 \quad (\alpha > 0)$

auxiliary eq.

$$m^2 + \lambda = m^2 + \alpha^2 = 0 \qquad m = +i\alpha, -i\alpha$$

$$y(x) = c_1 e^{+i\alpha} + c_2 e^{-i\alpha}$$

$$y(x) = C_3 \cos(\alpha x) + C_4 \sin(\alpha x)$$

$$y(0) = C_3 \cos(0) + C_4 \cancelto{0}{\sin(0)} = C_3 = 0$$

$$y(L) = C_4 \sin(\alpha L) = 0$$

$C_4 = 0 \quad \cdots\cdots \quad \boxed{y(x) = 0}$

$\underline{\sin(\alpha L) = 0} \qquad y(x) = C_4 \sin(\alpha x)$

$\alpha L = n\pi$

$\boxed{\alpha = \frac{n\pi}{L}} \qquad y(x) = C_4 \sin(\alpha x)$

$$y'' + \lambda y = 0 \qquad y(0)=0, \quad y(L)=0$$

Case III $\lambda > 0$

$$\lambda = +\alpha^2 \quad (\alpha > 0)$$

non-trivial solution

$$y(x) = C_4 \sin(\alpha x)$$

$$\boxed{\alpha = \frac{n\pi}{L}} \qquad \lambda = \alpha^2 = \left(\frac{n\pi}{L}\right)^2$$

$$y'' + \lambda y = 0 \qquad y(0)=0, \quad y(L)=0$$

Soution:

$\lambda = \left(\frac{n\pi}{L}\right)^2$ 일 때 $\quad y(x) = C_4 \sin\left(\frac{n\pi}{L}x\right)$

BVP $y'' + \lambda y = 0 \qquad y(0) = 0, \quad y(L) = 0$

Solution:

$\lambda = \left(\frac{n\pi}{L}\right)^2$ 일 때 $\quad y(x) = \sin\left(\frac{n\pi}{L}x\right)$

$n=1$ $\quad \lambda = \left(\frac{\pi}{L}\right)^2$

$y'' + \left(\frac{\pi}{L}\right)^2 y = 0 \Rightarrow y(x) = \sin\left(\frac{\pi}{L}x\right)$

$n=2$ $\quad \lambda = \left(\frac{2\pi}{L}\right)^2$

$y'' + \left(\frac{2\pi}{L}\right)^2 y = 0 \Rightarrow y(x) = \sin\left(\frac{2\pi}{L}x\right)$

$n=3$ $\quad \lambda = \left(\frac{3\pi}{L}\right)^2$

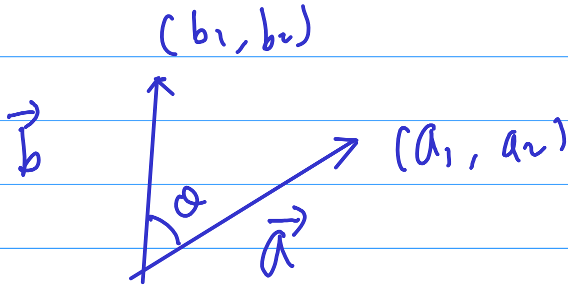
$y'' + \left(\frac{3\pi}{L}\right)^2 y = 0 \Rightarrow y(x) = \sin\left(\frac{3\pi}{L}x\right)$

$n=4$ $\quad \lambda = \left(\frac{4\pi}{L}\right)^2$

$y'' + \left(\frac{4\pi}{L}\right)^2 y = 0 \Rightarrow y(x) = \sin\left(\frac{4\pi}{L}x\right)$

eigenvalue $\qquad$ eigenfunction

orthogonal



Inner product $\quad \vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}| \cos\theta$

$$= a_1 b_1 + a_2 b_2$$

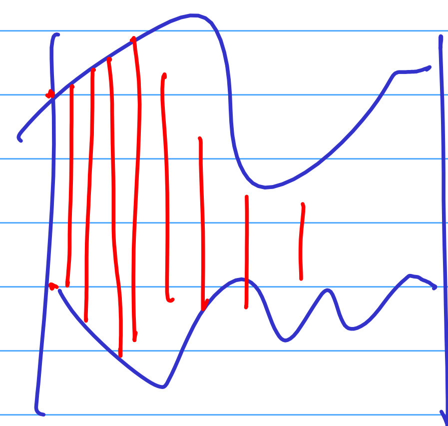
$$\vec{a} = (a_1, \; a_2 \; \cdots\cdots \; a_n)$$

$$\vec{b} = (b_1, \; b_2, \; \cdots \; b_n)$$

Σ

$\vec{a} \cdot \vec{b} = \quad f(x)$

$g(x)$



$$\int_0^b f(x)\, g(x)\, dx$$

$$\int \underbrace{\cos mx \, \cos nx \, dx}_{f \qquad g}$$

Eigenfunction

a linear second order differential equation
a two point boundary-value problem
a parameter λ

* generates an orthogonal set of functions

12.5 (1)식
→ 3.9 예제 2)
200p

$$\boxed{y'' + \lambda y = 0 \qquad y(0)=0 \qquad y(L)=0}$$

$\lambda = \lambda_n = \dfrac{n^2\pi^2}{L^2}$ eigenvalues

$y = C_2 \sin\left(\dfrac{n\pi}{L}x\right)$ eigenfunctions

$n=1,2,3,\ldots$ $\left\{\sin\left(\dfrac{n\pi}{L}x\right)\right\}$ an orthogonal set on the interval $[0, L]$

12.5 (2)식
예제 1 p29

$$\boxed{y'' + \lambda y = 0 \qquad y'(0)=0 \qquad y'(L)=0}$$

$\lambda = \lambda_n = \dfrac{n^2\pi^2}{L^2}$ eigenvalues

$y = C_1 \cos\left(\dfrac{n\pi}{L}x\right)$ eigenfunctions

$\left\{\cos\left(\dfrac{n\pi}{L}x\right)\right\}$ an orthogonal set on the interval $[0, L]$

12.5 (3)식
예제 2 p31

$$\boxed{y'' + \lambda y = 0 \quad 1 \cdot y(0) + 0 \cdot y'(0) = 0 \quad 1 \cdot y(1) + 1 \cdot y'(1) = 0}$$

Regular Stram-Liouville Problem

$$\boxed{\frac{d}{dx}\left[r(x)\, y' \right] + \left(q(x) + \lambda\, p(x) \right) y = 0 \quad \boxed{\begin{array}{l} A_1\, y(a) + B_1\, y'(a) = 0 \\ A_2\, y(b) + B_2\, y'(b) = 0 \end{array}}}$$

$\lambda = \alpha^2 > 0$

$\tan\alpha = \alpha \longrightarrow$ 2.0288 $\longrightarrow \lambda = 2.0288^2$

4.9132 $\quad \lambda = 4.\cdots^2$

7.9787 $\quad \lambda =$

$\vdots$

$\lambda_1 \longrightarrow y_1 = \sin \alpha_1 x$

$\lambda_2 \longrightarrow y_2 = \sin \alpha_2 x$

$A\vec{v} = \lambda\vec{v}$

en.wikipedia.org

In mathematics, an **eigenfunction** of a linear operator, A, defined on some function space, is any non-zero function f in that space that returns from the operator exactly as is, except for a multiplicative scaling factor. More precisely, one has

$$Af = \lambda f$$

for some scalar, λ, the corresponding eigenvalue. The solution of the differential eigenvalue problem also depends on any boundary conditions required of f. In each case there are only certain eigenvalues $\lambda = \lambda_n$ $(n = 1, 2, 3, \ldots)$ that admit a corresponding solution for $f = f_n$ (with each f_n belonging to the eigenvalue λ_n) when combined with the boundary conditions. Eigenfunctions are used to analyze A.

In mathematics, a **function space** is a set of functions of a given kind from a set X to a set Y. It is called a space because in many applications it is a topological space (including metric spaces), a vector space, or both. Namely, if Y is a field, functions have inherent vector structure with two operations of pointwise addition and multiplication to a scalar. Topological and metrical structures of function spaces are more diverse.

Linear Operator

en.wikipedia.org

In mathematics, a **linear map** (also called a **linear mapping**, **linear transformation** or, in some contexts, **linear function**) is a mapping $V \to W$ between two modules (including vector spaces) that preserves (in the sense defined below) the operations of addition and scalar multiplication. Linear maps can generally be represented as matrices, and simple examples include rotation and reflection linear transformations.

An important special case is when $V = W$ in which case the map is called a **linear operator**, or an endomorphism of V. Sometimes the term *linear function* has the same meaning as *linear map*, while in analytic geometry it does not.

A linear map always maps linear subspaces onto linear subspaces (possibly of a lower dimension); for instance it maps a plane through the origin to a plane, straight line or point.

In the language of abstract algebra, a linear map is a module homomorphism. In the language of category theory it is a morphism in the category of modules over a given ring.

Let V and W be vector spaces over the same field K. A function $f : V \to W$ is said to be a *linear map* if for any two vectors $\mathbf{x}$ and $\mathbf{y}$ in V and any scalar α in K, the following two conditions are satisfied:

$$f(\mathbf{x}+\mathbf{y}) = f(\mathbf{x}) + f(\mathbf{y}) \qquad \text{additivity}$$
$$f(\alpha\mathbf{x}) = \alpha f(\mathbf{x}) \qquad \text{homogeneity of degree 1}$$

This is equivalent to requiring the same for any linear combination of vectors, i.e. that for any vectors $x_1, \ldots, x_m \in V$ and scalars $a_1, \ldots, a_m \in K$, the following equality holds:

$$f(a_1\mathbf{x}_1 + \cdots + a_m\mathbf{x}_m) = a_1 f(\mathbf{x}_1) + \cdots + a_m f(\mathbf{x}_m).$$

Denoting the zero elements of the vector spaces V and W by $\mathbf{0}_V$ and $\mathbf{0}_W$ respectively, it follows that $f(\mathbf{0}_V) = \mathbf{0}_W$ because letting $\alpha = 0$ in the equation for homogeneity of degree 1,

$$f(\mathbf{0}_V) = f(0 \cdot \mathbf{0}_V) = 0 \cdot f(\mathbf{0}_V) = \mathbf{0}_W.$$

Occasionally, V and W can be considered to be vector spaces over different fields. It is then necessary to specify which of these ground fields is being used in the definition of "linear". If V and W are considered as spaces over the field K as above, we talk about K-linear maps. For example, the conjugation of complex numbers is an $\mathbf{R}$-linear map $\mathbf{C} \to \mathbf{C}$, but it is not $\mathbf{C}$-linear.

A linear map from V to K (with K viewed as a vector space over itself) is called a linear functional.

These statements generalize to any left-module $_R M$ over a ring R without modification, and to any right-module upon reversing of the scalar multiplication.

$$g_1(x) \longrightarrow \boxed{\frac{d}{dx}} \longrightarrow g_1'(x)$$

additivity

$$g_2(x) \longrightarrow \boxed{\frac{d}{dx}} \longrightarrow g_2'(x)$$

$$g_1(x) + g_2(x) \longrightarrow \boxed{\frac{d}{dx}} \longrightarrow \left(g_1(x) + g_2(x) \right)'$$

$$= g_1'(x) + g_2'(x)$$

$$k\, g_1(x) \longrightarrow \boxed{\frac{d}{dx}} \longrightarrow k\, g_1'(x)$$

homogeneity

Eigenfunction & Eigenvector

$$\boxed{y'' + \lambda y = 0 \qquad y(0)=0 \qquad y(L)=0}$$

$$\lambda = \lambda_n = \frac{n^2\pi^2}{L^2}$$ eigenvalues

$$y = C_2 \sin\left(\frac{n\pi}{L}x\right)$$ eigenfunctions

$$\left\{ \sin\left(\frac{n\pi}{L}x\right) \right\}$$ an orthogonal set on the interval $[0, L]$

eigenvalue

$$\frac{d^2}{dx^2} y = \boxed{-\lambda}\, y$$

Linear Operator

eigenfunction

eigenvalue

$$A x = \boxed{\lambda}\, x$$

Linear Operator

eigenvector

Regular Strum-Liouville Problem

① $y'' + \lambda y = 0 \qquad y(0) = 0 \qquad y(L) = 0$

② $y'' + \lambda y = 0 \qquad y'(0) = 0 \qquad y'(L) = 0$

$$\boxed{\begin{gathered} \frac{d}{dx}\left[r(x)\, y' \right] + \left(q(x) + \lambda\, p(x) \right) y = 0 \\ \boxed{\begin{aligned} A_1\, y(a) + B_1\, y'(a) = 0 \\ A_2\, y(b) + B_2\, y'(b) = 0 \end{aligned}} \end{gathered}}$$

① $r(x) = 1 \quad q(x) = 0 \quad p(x) = 1 \quad A_1 = 1 \quad B_1 = 0 \quad a = 0$

$A_2 = 1 \quad B_2 = 0 \quad b = L$

② $r(x) = 1 \quad q(x) = 0 \quad p(x) = 1 \quad A_1 = 0 \quad B_1 = 1 \quad a = 0$

$A_2 = 0 \quad B_2 = 1 \quad b = L$

Boundary Conditions

Homogeneous Boundary Conditions

$$A_1 y(a) + B_1 y'(a) = 0$$
$$A_2 y(b) + B_2 y'(b) = 0$$

Non-homogeneous Boundary Conditions

$$A_1 y(a) + B_1 y'(a) = c_1 \neq 0$$
$$A_2 y(b) + B_2 y'(b) = c_2 \neq 0$$

Homogeneous BVP

- homogeneous linear differential equation
- homogeneous boundary conditions

Separated Boundary Conditions

$$A_1 y(a) + B_1 y'(a) = 0 \qquad x = a$$
$$A_2 y(b) + B_2 y'(b) = 0 \qquad x = b$$

separated

Mixed Boundary Conditions

$$y(a) = y(b)$$
$$y'(a) = y'(b)$$

Properties

(a) an infinite number of real eigenvalues

that can be in increasing order

$$\lambda_1 < \lambda_2 < \lambda_3 \cdots \qquad n \to \infty \quad \lambda_n \to \infty$$

(b) one eigenvalue $\to$ one eigenfunction

$$\lambda_i \to y_i(x)$$

(c)

$$\boxed{\begin{matrix} \lambda_i & \to & y_i(x) \\ \lambda_j & \to & y_j(x) \end{matrix}}$$

$$\lambda_i \neq \lambda_j \Rightarrow y_i(x), y_j(x) : \boxed{\text{linearly independent}}$$

(d)

$$\boxed{\begin{matrix} \lambda_i & \to & y_i(x) \\ \lambda_j & \to & y_j(x) \end{matrix}}$$

$$\lambda_i \neq \lambda_j \Rightarrow y_i(x), y_j(x) : \boxed{\text{orthogonal}}$$

w.r.t the weigh function $p(x)$

$$\int_a^b p(x)\, y_i(x)\, y_j(x)\, dx = 0$$

$$\boxed{\begin{array}{lcl} \lambda_m & \longrightarrow & y_m(x) \\ \lambda_n & \longrightarrow & y_n(x) \end{array}}$$

$$\frac{d}{dx}\left[r(x)\, y_m' \right] + \left(q(x) + \lambda_m p(x) \right) y_m = 0 \qquad \times\, y_n$$

$$\frac{d}{dx}\left[r(x)\, y_n' \right] + \left(q(x) + \lambda_n p(x) \right) y_n = 0 \qquad \times\, y_m$$

$$-)\quad\overline{\hphantom{xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx}}$$

$$y_n \frac{d}{dx}\left[r(x)\, y_m' \right] - y_m \frac{d}{dx}\left[r(x)\, y_n' \right]$$

$$+ \left(\lambda_m - \lambda_n \right) p(x)\, y_m\, y_n = 0$$

$$\left(\lambda_m - \lambda_n \right) p(x)\, y_m\, y_n = y_m \frac{d}{dx}\left[r(x)\, y_n' \right] - y_n \frac{d}{dx}\left[r(x)\, y_m' \right]$$

$$\left(\lambda_m - \lambda_n \right) \int_a^b p(x)\, y_m\, y_n \, dx = \Big[r(x)\, y_m\, y_n' - r(x)\, y_n\, y_m' \Big]_a^b$$

$$\int y_m \frac{d}{dx}\left[r(x)\, y_n' \right] dx = y_m\, r(x)\, y_n' - \int y_m'\, r(x)\, y_n' \, dx$$

$$\int y_n \frac{d}{dx}\left[r(x)\, y_m' \right] dx = y_n\, r(x)\, y_m' - \int y_m'\, r(x)\, y_n' \, dx$$

$$(\lambda_m - \lambda_n)\int_a^b p(x)\, y_m\, y_n\, dx$$

$$= \Big[r(x)\big(y_m(x)\, y_n'(x) - y_n(x)\, y_m'(x) \big) \Big]_a^b$$

$$\left.\begin{aligned} &= r(b)\big(y_m(b)\, y_n'(b) - y_n(b)\, y_m'(b) \big) \\ &\quad - r(a)\big(y_m(a)\, y_n'(a) - y_n(a)\, y_m'(a) \big) \end{aligned}\right\} \to 0$$

$$A_1\, y(a) + B_1\, y'(a) = 0$$
$$A_2\, y(b) + B_2\, y'(b) = 0$$

$$\begin{aligned} A_1\, y_m(a) + B_1\, y_m'(a) &= 0 \\ A_1\, y_n(a) + B_1\, y_n'(a) &= 0 \end{aligned} \qquad \begin{bmatrix} y_m(a) & y_m'(a) \\ y_n(a) & y_n'(a) \end{bmatrix} \begin{bmatrix} A_1 \\ B_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\cancel{A_1 = B_1 = 0} \Rightarrow \begin{vmatrix} y_m(a) & y_m'(a) \\ y_n(a) & y_n'(a) \end{vmatrix} = 0 \qquad y_m(a)\, y_n'(a) - y_m'(a)\, y_n(a) \simeq 0$$

$$\begin{aligned} A_2\, y_m(b) + B_2\, y_m'(b) &= 0 \\ A_2\, y_n(b) + B_2\, y_n'(b) &= 0 \end{aligned} \qquad \begin{bmatrix} y_m(b) & y_m'(b) \\ y_n(b) & y_n'(b) \end{bmatrix} \begin{bmatrix} A_2 \\ B_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\cancel{A_2 = B_2 = 0} \Rightarrow \begin{vmatrix} y_m(b) & y_m'(b) \\ y_n(b) & y_n'(b) \end{vmatrix} = 0 \qquad y_m(b)\, y_n'(b) - y_m'(b)\, y_n(b) \simeq 0$$

$$\therefore\ (\lambda_m - \lambda_n)\int_a^b p(x)\, y_m\, y_n\, dx = 0$$

$$\begin{array}{ccccc} \lambda_1, & \lambda_2, & \lambda_3, & \lambda_4, & \dots \\ \downarrow & \downarrow & \downarrow & \downarrow & \\ \{\, y_1(x), & y_2(x), & y_3(x), & y_4(x), & \dots \,\} \end{array}$$

orthogonal set of eigenfunctions

of a regular strum-Liouville problem

complete on $[a, b]$

Orthogonal Set

A set of real valued functions

$$\{\phi_0(x), \phi_1(x), \phi_2(x), \dots\}$$

orthogonal w.r.t $w(x)$ on $[a, b]$

$$\int_a^b w(x)\, \phi_m(x)\, \phi_n(x)\, dx = 0 \qquad m \neq n$$

a weight function $w(x) > 0$

on an interval of orthogonality $[a, b]$

$\{1, \cos x, \cos 2x, \dots\}$ $\quad w(x) = 1 \quad [-\pi, \pi]$

* Orthogonal Series Expansion

$$\{\phi_n(x)\} = \{\phi_0(x), \phi_1(x), \phi_2(x), \dots\}$$

an infinite orthogonal set of functions on a interval $[a, b]$

$\Rightarrow$ can determine a set of coefficients c_n

$$\boxed{f(x) = c_0\phi_0(x) + c_1\phi_1(x) + \cdots + c_n\phi_n(x) + \cdots}$$

$$\int_a^b f(x)\phi_m(x)\,dx$$

$$= \int_a^b c_0\phi_0(x)\phi_m(x)\,dx + \int_a^b c_1\phi_1(x)\phi_m(x)\,dx + \cdots + \int_a^b c_n\phi_n(x)\phi_m(x)\,dx + \cdots$$

$$= c_0(\phi_0(x), \phi_m(x)) + c_1(\phi_1(x), \phi_m(x)) + \cdots + c_n(\phi_n(x), \phi_m(x)) + \cdots$$

$$\int_a^b f(x)\phi_m(x)\,dx = c_m\int_a^b \phi_m^2(x)\,dx$$

$$\boxed{c_m = \frac{\int_a^b f(x)\phi_m(x)\,dx}{\int_a^b \phi_m^2(x)\,dx}}$$

$$\{\phi_n(x)\} = \{\phi_0(x), \phi_1(x), \phi_2(x), \cdots\}$$

$$\boxed{f(x) = c_0\phi_0(x) + c_1\phi_1(x) + \cdots + c_n\phi_n(x) + \cdots}$$

$$\boxed{c_m = \frac{\int_a^b f(x)\phi_m(x)\,dx}{\int_a^b \phi_m^2(x)\,dx}}$$

$$\left\{\cos\left(\frac{0\pi}{p}x\right), \cos\left(\frac{1\pi}{p}x\right), \cos\left(\frac{2\pi}{p}x\right), \cos\left(\frac{3\pi}{p}x\right), \cdots\cdots \right\}$$

$$f(x) = a_0\cos\left(\frac{0\pi}{p}x\right) + a_1\cos\left(\frac{1\pi}{p}x\right) + \cdots$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi}{p}x\right)$$

$$a_0 = \frac{2}{p}\int_0^p f(x)\,dx$$

$$a_n = \frac{2}{p}\int_0^p f(x)\cos\left(\frac{n\pi}{p}x\right)dx$$

* $f(x)$: even $[-p, +p]$ Cosine Series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{p} x\right)$$

$$a_0 = \frac{2}{p} \int_0^p f(x)\, dx$$

$$a_n = \frac{2}{p} \int_0^p f(x) \cos\left(\frac{n\pi}{p} x\right) dx$$

$f(x)$: odd $[-p, +p]$ Sine Series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{p} x\right)$$

$$b_n = \frac{2}{p} \int_0^p f(x) \sin\left(\frac{n\pi}{p} x\right) dx$$

$$\{\phi_n(x)\} = \{\phi_0(x), \phi_1(x), \phi_2(x), \cdots\} \quad \text{Orthogonal}$$

$$\boxed{(\phi_n(x), \phi_m(x)) = \int_a^b \phi_n(x)\phi_m(x)\,dx = 0 \qquad n \neq m}$$

$$\boxed{f(x) = c_0\phi_0(x) + c_1\phi_1(x) + \cdots + c_n\phi_n(x) + \cdots}$$

$$\boxed{c_m = \frac{\int_a^b f(x)\phi_m(x)\,dx}{\int_a^b \phi_m^2(x)\,dx}}$$

$$\boxed{\int_a^b w(x)\phi_n(x)\phi_m(x)\,dx = 0 \qquad n \neq m}$$

$$\boxed{f(x) = c_0\phi_0(x) + c_1\phi_1(x) + \cdots + c_n\phi_n(x) + \cdots}$$

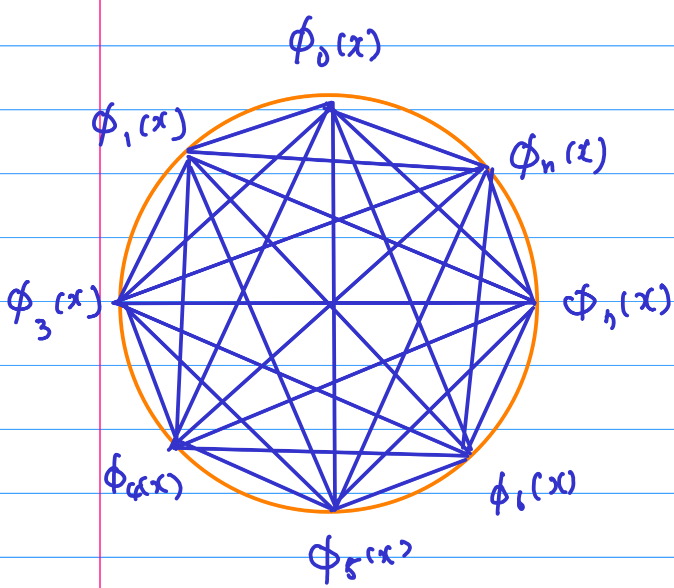
$$\boxed{c_m = \frac{\int_a^b w(x)f(x)\phi_m(x)\,dx}{\int_a^b w(x)\phi_m^2(x)\,dx}}$$

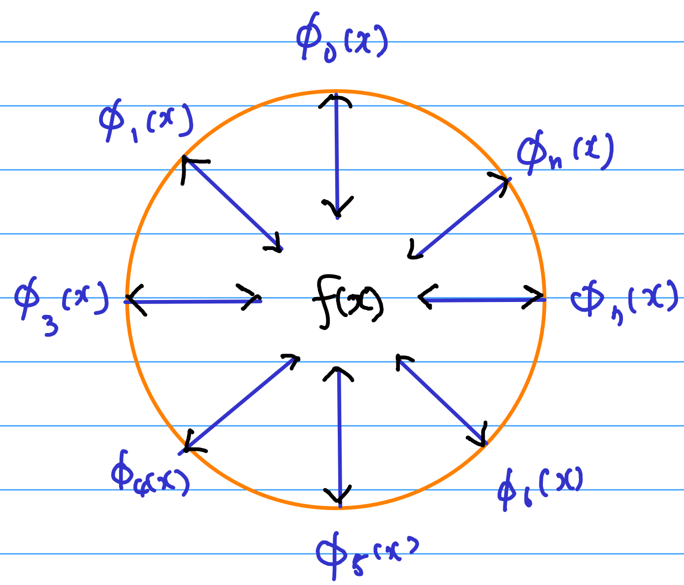
Complete Set

$$\boxed{f(x) = c_0\,\phi_0(x) + c_1\,\phi_1(x) + \cdots + c_n\,\phi_n(x) + \cdots}$$

condition

$f(x)$





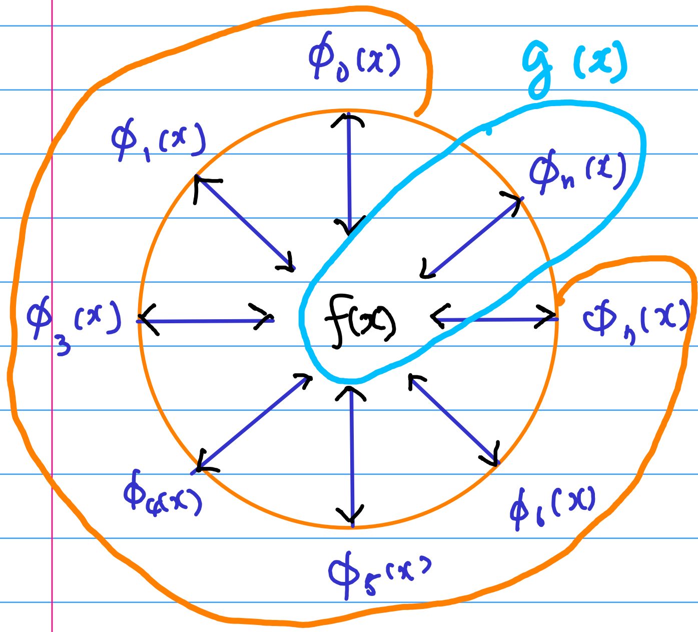
Orthogonal set

the only $f(x)$ that is orthogonal to each of $\phi_i(x)$

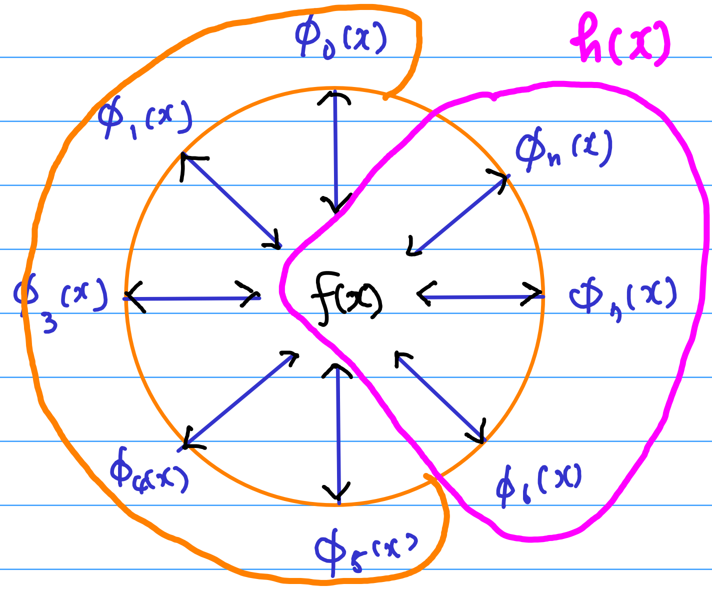
$\Rightarrow 0 \Rightarrow \boxed{c_i = 0}$

nonzero $f(x)$ must not be orthogonal to each of $\phi_i(x)$

Complete Set



incomplete set



incomplete set

$g(x)$ has a component that are orthogonal to each ϕ_i in the orthogonal set

$h(x)$ has a component that are orthogonal to each ϕ_i in the orthogonal set

$\{ f_1(x), f_2(x), f_3(x), \dots \}$

infinite set of real valued function
that are continuous on $[a, b]$
and linearly independent on $[a, b]$

$\Longrightarrow$ always made into an orthogonal set

infinitely linearly independent set $S = \{ x_1, x_2, \dots \}$

$\triangleq$ if every subset of S is linearly independent

Regular Strum-Liouville Problem

$$\frac{d}{dx}\left[r(x)\, y' \right] + \left(q(x) + \lambda\, p(x) \right) y = 0$$

$$\Rightarrow \int_a^b p(x)\, y_m\, y_n\, dx = 0, \quad \lambda_m \neq \lambda_n$$

orthogonal

Regular Strum-Liouville Problem Example

$$\frac{d}{dx}\left[r(x)\, y' \right] + \left(q(x) + \lambda\, p(x) \right) y = 0$$

$$A_1\, y(a) + B_1\, y'(a) = 0$$
$$A_2\, y(b) + B_2\, y'(b) = 0$$

$$y'' + \lambda y = 0$$

$$y(0) = 0$$
$$y(1) + y(1) = 0$$

$$y'' + \lambda y = 0 \qquad y(0)=0, \quad y(L)=0$$

$$y'' + \lambda y = 0 \qquad y(0)=0, \quad y(1)+y'(1)=0$$

- $\lambda = 0$
- $\lambda > 0 \qquad \lambda = +\alpha^2 \qquad (\alpha > 0)$
- $\lambda < 0 \qquad \lambda = -\alpha^2 \qquad (\alpha > 0)$

$\left\{\begin{array}{l}\lambda = 0\\ \lambda < 0\end{array}\right.$ trivial solution $y = 0$

$\lambda > 0$ general solution $y = C_1 \cos\alpha x + C_2 \sin\alpha x$

$y(0)=0 \qquad y = C_1 = 0$

$y(1)+y'(1)=0 \qquad C_2 \sin\alpha x + \alpha C_2 \cos\alpha x = 0$

$C_2 \sin\alpha + \alpha C_2 \cos\alpha = 0$

If $C_2 \neq 0 \qquad -\alpha = \tan\alpha$

Singular Strum-Liouville Problem

$$\frac{d}{dx}\left[r(x)\, y' \right] + \left(q(x) + \lambda\, p(x) \right) y = 0$$

$$\begin{aligned} A_1\, y(a) + B_1\, y'(a) &= 0 \\ A_2\, y(b) + B_2\, y'(b) &= 0 \end{aligned}$$

singular BVP

$$\frac{d}{dx}\left[r(x)\, y' \right] + \left(q(x) + \lambda\, p(x) \right) y = 0$$

$$\begin{aligned} r(a) &= 0 \\ A_2\, y(b) + B_2\, y'(b) &= 0 \end{aligned}$$

singular BVP

$$\frac{d}{dx}\left[r(x)\, y' \right] + \left(q(x) + \lambda\, p(x) \right) y = 0$$

$$\begin{aligned} A_1\, y(a) + B_1\, y'(a) &= 0 \\ r(b) &= 0 \end{aligned}$$

Singular Strum-Liouville Problem

$$\frac{d}{dx}\left[r(x)\, y'\right] + \left(q(x) + \lambda\, p(x)\right) y = 0$$

$$A_1 y(a) + B_1 y'(a) = 0$$
$$A_2 y(b) + B_2 y'(b) = 0$$

$$\frac{d}{dx}\left[r(x)\, y'\right] + \left(q(x) + \lambda\, p(x)\right) y = 0$$

$$r(a) = 0$$
$$r(b) = 0$$

periodic BVP

$$\frac{d}{dx}\left[r(x)\, y'\right] + \left(q(x) + \lambda\, p(x)\right) y = 0$$

$$r(a) = r(b) \quad y(a) = y(b)$$
$$y'(a) = y'(b)$$

a set of Bessel functions

$\{ J_n(\alpha_i x) \}$ for a fixed n

$i = 1, 2, 3, 4 \cdots$

$\{ J_n(\alpha_1 x), J_n(\alpha_2 x), J_n(\alpha_3 x), \cdots \}$

orthogonal set

12.3 예제 3.

Bessel Eq

$$x^2 y'' + x y' + (\alpha^2 x^2 - n^2) y = 0$$

$n = 0, 1, 2, \cdots$

general solution $y = c_1 J_n(\alpha x) + c_2 Y_n(\alpha x)$

$$\lambda_i = \alpha_i^2 = \left(\frac{x_i}{b} \right)^2$$

$[a, b]$

$$f(x) = \sum_{i=1}^{\infty} c_i \boxed{J_n(\alpha_i x)}$$

$$c_i = \frac{2}{b^2 J_{n+1}^2(\alpha_i b)} \int_a^b x \, J_n(\alpha_i x) \, f(x) \, dx$$

$$f(x) = \sum_{n=0}^{\infty} C_n \, P_n(x)$$

$$C_n = \frac{2n+1}{2} \int_{-1}^{+1} f(x) \, P_n(x) \, dx$$