Temporal Characteristics of Random Processes

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Based on
Probability, Random Variables and Random Signal Principles,
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Outline

Random Variable Definition

A random variable

a **function** over a **sample space** $S = \{s_1, s_2, s_3, ..., s_n\}$

$$s \rightarrow X(s)$$

$$x = X(s)$$

a **function** of a possible **outcome** s of an **experiment**

Random Variable Definition

A random variable

- a **random variable** : a capital letter X
- a particular value : a lowercase letter x
- a **sample space** $S = \{s_1, s_2, s_3, ..., s_n\}$
- an **outcome** (an element of S) : s

$$s \rightarrow X(s)$$

$$x = X(s)$$

$$s \rightarrow x$$

Understanding Random Variables (1)

random variables are used to quantify
outcomes of a random occurrence, and therefore,
can take on many values.
Random variables are required to be measurable and
are typically real numbers.
For example, the letter X may be designated
to represent the sum of the resulting numbers
after three dice are rolled.
In this case, X could be 3 $(1 + 1 + 1)$, 18 $(6 + 6 + 6)$,
or somewhere between 3 and 18,
since the highest number of a die is 6 and
the lowest number is 1.

https://www.investopedia.com/terms/r/random-variable.asp

Understanding Random Variables (2)

A random variable is different from an algebraic variable.
The variable in an algebraic equation is an unknown value that can be calculated.
The equation $10 + x = 13$ shows that we can calculate the specific value for x which is 3.
On the other hand, a random variable has a set of values, and any of those values could be the resulting outcome as seen in the example of the dice above.

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Understanding Random Variables (3)

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Understanding Random Variables (4)

A typical example of a random variable is the outcome of a coin toss. Consider a probability distribution in which the outcomes of a random event are not equally likely to happen. If the random variable Y is the number of heads we get from tossing two coins, then Y could be 0, 1, or 2. This means that we could have no heads, one head, or both heads on a two-coin toss.

https://www.investopedia.com/terms/r/random-variable.asp

Formal definition of a random variable

A random variable X is a measurable function $X\colon \Omega \to E$ from a set of possible outcomes Ω to a measurable space E.

The technical axiomatic definition requires Ω to be a sample space of a probability triple $(\Omega, \mathscr{F}, \mathrm{P})$

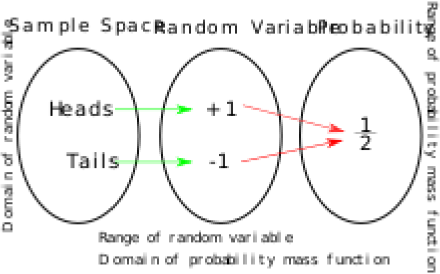
A random variable is often denoted by capital roman letters such as X, Y, Z, T.

The probability that X takes on a value in a measurable set $S \subseteq E$ is written as

$\mathrm{P}(X \in S) = \mathrm{P}(\{\omega \in \Omega \mid X(\omega) \in S\})$

https://en.wikipedia.org/wiki/Random_variable

Random variable example



This graph shows how random variable is a function from all possible outcomes to real values. It also shows how random variable is used for defining probability mass functions.

https://en.wikipedia.org/wiki/Random_variable

Probability Space (1)

In probability theory, a probability space or a probability triple $(\Omega, \mathscr{F}, P)$ is a mathematical construct

that provides a formal model of a random process or "experiment".

For example, one can define a probability space which models the throwing of a die

https://en.wikipedia.org/wiki/Probability_space

Probability Space (2)

A probability space consists of three elements
A sample space, Ω , which is the set of all possible outcomes.
An event space, which is a set of events $\mathscr{F}$,
an event being a set of outcomes in the sample space.
A probability function, which assigns each event in the event space a probability, which is a number between 0 and 1.

https://en.wikipedia.org/wiki/Probability_space

Probability Space (3)

In the example of the throw of a standard die, we would take the sample space to be $\{1, 2, 3, 4, 5, 6\}$. For the event space, we could simply use the set of all subsets of the sample space, which would then contain simple events such as $\{5\}$ ("the die lands on 5"), as well as complex events such as $\{2, 4, 6\}$ ("the die lands on an even number"). Finally, for the probability function, we would map each event to the number of outcomes in that event divided by 6 — so for example, $\{5\}$ would be mapped to 1 / 6 1/6, and $\{2, 4, 6\}$ would be mapped to $3/6 = 1/2$.

https://en.wikipedia.org/wiki/Probability_space

Random Process (1)

A random process

a function of both **outcome** s and **time** t

$$X(t, s)$$

assigning a time function to every **outcome** s_i

$$s_i \rightarrow x(t, s_i)$$

Random Process (2)

A random process

the family of such time functions is called a random process

$$x(t, s_i) = X(t, s_i)$$

$$x(t, s) = X(t, s)$$

Random Process (3)

We have seen that a random variable X is
a rule which assigns a number
to every outcome e of an experiment.
The random variable is a function X(e)
that maps the set of experiment outcomes
to the set of numbers.
A random process is a rule
that maps every outcome e of an experiment
to a function X(t, e).
A random process is usually conceived of
as a function of time,
but there is no reason to not consider random processes
that are functions of other independent variables,
such as spatial coordinates.
The function X(u, v, e) would be a function
whose value depended on the location (u, v) and the outcome e,

Ensemble of time functions

Time functions

A random process $X(t, s)$ represents
a family or ensemble of time functions

$X(t, s)$ represents

- a single time function $x(t, s)$
- when t is a variable and s is fixed at an outcome

$x(t, s)$ represents

- a sample function,
- an ensemble member,
- a realization of the process

Short-form notation for time functions

The short-form notation $x(t)$

to represent a specific waveform of a random process $X(t)$
for a given **outcome** s_i

$$x(t) = x(t, s)$$

$$X(t) = X(t, s)$$

Random Process Example

Example

$$X(t, s_1) = x_1(t) \qquad s_1 \longrightarrow x_1(t)$$
$$X(t, s_2) = x_2(t) \qquad s_2 \longrightarrow x_2(t)$$
$$\cdots \qquad \cdots$$
$$X(t, s_n) = x_n(t) \qquad s_n \longrightarrow x_n(t)$$

$$S = \{s_1, s_2, s_3, \ldots, s_n\} \quad \text{a sample space}$$
$$X(t) = \{x_1(t), x_2(t), x_3(t), \ldots, x_n(t)\} \quad \text{a random process}$$

Random variables with time

a **random process** $X(t,s)$ represents a single time function when t is a variable and s is fixed at an outcome

a random process $X(t,s)$ represents a single **random variable** when both t and s are fixed at a time and an outcome, respectively

$$X_i = X(t_i, s) = X(t_i) \qquad \textit{\textbf{random variable}}$$

$$X(t,s) = X(t) \qquad \textit{\textbf{random process}}$$

An alphabet

the alphabet of $X(t)$

the set of its possible values

- the values of **time** t for which a random process is defined
- the **alphabet** of the random variable $X = X(t)$ at time t

Classification of Random Processes

(1) Types of time and alphabet

- the values of **time** t for which a random process is defined
  - continuous time
  - discrete time
- the **alphabet** of the random variable $X = X(t)$ at time t
  - continuous alphabet
  - discrete alphabet

Classification of Random Processes

(2) types of the random variable $X(t)$ and the time t

- a continuous **alphabet** continuous **time** random process
  - $X(t)$ has continuous values and t has continuous values
- a discrete **alphabet** continuous **time** random process
  - $X(t)$ has discrete values and t has continuous values
- a continuous **alphabet** discrete **time** random process
  - $X(t)$ has continuous values and t has discrete values
- a discrete **alphabet** discrete **time** random process
  - $X(t)$ has discrete values and t has discrete values

Deterministic and Non-deterministic Random Processes

- A process is **non-deterministic**
  if future values of any sample function
  cannot be predicted exactly from observed past values
- A process is **deterministic**
  if future values of any sample function
  can be predicted from observed past values

Deterministic Random Process Example (1)

$$X(t) = A\cos(\omega_0 t + \Theta)$$

A, Θ, or ω_0(or all) can be random variables.

a sample function corresponds to the above equation with particular values of these random variables.

$$x_i(t) = A_i\cos(\omega_{0,i} t + \Theta_i)$$

Deterministic Random Process Example (2)

$$x_i(t) = A_i \cos(\omega_{0,i} t + \Theta_i)$$

the knowledge of the sample function
prior to any time instance fully allows
the prediction of the sample function's future values
because all the necessary information is known

$$x_i(t) \quad t \leq 0 \qquad \Longrightarrow \qquad x_i(t) \quad t > 0$$

Functions and variables of a random process $X(t, \theta)$ (1)

$X(t, \theta)$	a family of functions, an ensemble
$X(t, \theta_k)$	a single time function selected by the outcome θ_k
$X(t_1, \theta)$	a random variable at the time $t = t_1$
$X(t_1, \theta_k)$	a number at the time $t = t_1$, of the outcome θ_k

https://www.cis.rit.edu/class/simg713/notes/chap7-random-process.pdf

Functions and variables of a random process $X(t, \theta)$ (2)

- $X(t, \theta)$ is a family of functions. Imagine a giant strip chart recording in which each pen is identified with a different θ. This family of functions is traditionally called an ensemble.
- A single function $X(t, \theta_k)$ is selected by the outcome θ_k. This is just a time function that we could call $X_k(t)$. Different outcomes give us different time functions.
- If t is fixed, say $t = t_1$, then $X(t_1, \theta)$ is a random variable. Its value depends on the outcome θ.
- If both t_1 and θ_k are given then $X(t_1, \theta_k)$ is just a number.

https://www.cis.rit.edu/class/simg713/notes/chap7-random-process.pdf

Stochastic Process (1)

In probability theory and related fields, a stochastic (/stoʊˈkæstɪk/) or random process is a mathematical object usually defined as a family of random variables.

https://www.cis.rit.edu/class/simg713/notes/chap7-random-process.pdf

Stochastic Process (2)

The definition of a stochastic process varies, but a stochastic process is traditionally defined as a collection of random variables indexed by some set. The terms random process and stochastic process are considered synonyms and are used interchangeably, without the index set being precisely specified. Both "collection", or "family" are used[74] while instead of "index set", sometimes the terms "parameter set"[29] or "parameter space" are used.

https://www.cis.rit.edu/class/simg713/notes/chap7-random-process.pdf