Stationary Random Processes

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Apr 02 2022

Based on
Probability, Random Variables and Random Signal Principles,
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Outline

1. Stationary Random Processes
   - N-th Order Stationary Processes
   - Strict and Wide Sense Stationary Processes
   - Cyclostationary Process

Outline

First Order Stationary

$f_X(x; t)$

if $X(t)$ is to be a **first-order stationary**

$$f_X(x_1; t_1) = f_X(x_1; t_1 + \Delta)$$

must be true for any time t_1 and any real number Δ

the first order density function
does not change with a shift in time origin

Consequences of stationarity

$f_X(x; t)$

- $f_X(x, t_1)$ is independent of t_1
  the first order density function
  does not change with a shift in time origin

- the **process mean** value is a **constant**

$$m_X(t) = \overline{X} = constant$$

the process mean value

$$m_X(t) = \overline{X} = constant$$

$$m_X(t_1) = \int_{-\infty}^{\infty} x f_X(x; t_1) dx$$

$$m_X(t_2) = \int_{-\infty}^{\infty} x f_X(x; t_2) dx$$

let $t_2 = t_1 + \Delta$

$$m_X(t_1) = m_X(t_1 + \Delta)$$

Second-Order Stationary Process

$f_X(x_1, x_2; t_1, t_2)$

if $X(t)$ is to be a **second-order stationary**

$$f_X(x_1, x_2; t_1, t_2) = f_X(x_1, x_2; t_1 + \Delta, t_2 + \Delta)$$

must be true for any time t_1, t_2 and any real number Δ

the second order density function
does not change with a shift in time origin

Second-Order Stationary Process

$f_X(x_1, x_2; t_1, t_2)$

- $f_X(x_1, x_2; t_1, t_2)$ is independent of t_1 and t_2
  the second order density function
  does not change with a shift in time origin

- the **autocorrelation function**

$$R_{XX}(t, t+\tau) = E[X(t)X(t+\tau)] = R_{XX}(\tau)$$

N^{th}-order Stationary Processes

$f_X(x_1, \cdots, x_N; t_1, \cdots, t_N)$

if $X(t)$ is to be a **N^{th}-order stationary**

$$f_X(x_1, \cdots, x_N; t_1, \cdots, t_N) = f_X(x_1, \cdots, x_N; t_1 + \Delta, \cdots, t_N + \Delta)$$

must be true for any time $t_1, ..., t_N$
and any real number Δ

the N^{th} order density function
does not change with a shift in time origin

Stationary Process

joint probability distribution

a **stationary process** is a stochastic process
whose unconditional joint probability distribution
does not change when shifted in time.

Consequently, parameters such as **mean** and **variance** also
do not change over time.

https://en.wikipedia.org/wiki/Stationary_process

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Stationary Process - nomenclature

nomenclature

- stationary process
- strictly stationary process
- strongly stationary process
- strict sense stationary (SSS) process

https://en.wikipedia.org/wiki/Stationary_process

Strict Sense Stationary Process

for all natural number N

if $X(t)$ is to be a **strict sense stationary (SSS) process**

$$f_X(x_1, \cdots, x_N; t_1, \cdots, t_N) = f_X(x_1, \cdots, x_N; t_1 + \Delta, \cdots, t_N + \Delta)$$

must be true for any time $t_1, ..., t_N$
and any real number Δ
and for all natural number N

- **white noise** is the simplest example of a **strictly stationary process**.

https://en.wikipedia.org/wiki/Stationary_process

Wide Sense Stationary Process

1st and 2nd moments

Wide Sense Stationary (WSS) random processes
only require that
1st moment (i.e. the **mean**) and **autocovariance**
do not vary with respect to time and that
the 2nd moment is finite for all times.

- $E[X(t_1)] = E[X(t_2)] = \overline{X} = constant$ for all t_1 and t_2
- $C_{XX}(t_1, t_2) = C_{XX}(t_1 - t_2, 0) \triangleq C_{XX}(\tau)$ for all t_1 and t_2
- $E[|X(t)|^2] < \infty$ for all t

https://en.wikipedia.org/wiki/Stationary_process

Wide Sense Stationary Process - nomenclature

nomenclature

- weak sense stationary (WSS) process
- wide sense stationary (WSS) process

https://en.wikipedia.org/wiki/Stationary_process

WSS - auto-covariance & auto-correlation

mean, auto-covariance, auto-correlation

$$m_X(t) = \overline{X} = constant$$

$$\begin{aligned}
C_{XX}(t_1, t_2) &= E\left[\{X(t_1) - m_x(t_1)\}\{X(t_2) - m_x(t_2)\}\right] \\
&= E\left[\left\{X(t_1) - \overline{X}\right\}\left\{X(t_2) - \overline{X}\right\}\right] \\
&= E\left[X(t_1)X(t_2)\right] - \overline{X}^2 \\
&\triangleq C_{XX}(\tau) \\
&\triangleq R_{XX}(\tau) - \overline{X}^2
\end{aligned}$$

$$R_{XX}(t_1, t_2) \triangleq R_{XX}(\tau)$$

https://en.wikipedia.org/wiki/Stationary_process

Wide Sense Stationary Process

$m_X(t)$, $R_{XX}(\tau)$

WSS random processes only require that
1st moment (i.e. the **mean**) and **auto**correlation
do not vary with respect to time

$$E[X(t)] = m_X(t) = \overline{X} = constant$$

$$E[X(t)X(t+\tau)] = R_{XX}(\tau)$$

Wide Sense Stationary Process

$m_X(t)$, $R_{XX}(\tau)$

- the 2nd order stationarity is sufficient
  for wide sense stationarity
- if $f_X(x_1; t_1)$ is independent of t_1
  then $E[X(t)] = constant$
- if $f_X(x_1, x_2; t_1, t_2)$ is independent of t_1 and t_2
  then $E[X(t)X(t+\tau)] = R_{XX}(\tau)$

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Cyclostationary process (1)

- A **cyclostationary process** is a signal having *statistical properties* that vary cyclically with time. A cyclostationary process can be viewed as multiple interleaved **stationary processes**.

https://en.wikipedia.org/wiki/Cyclostationary_process

Cyclostationary process (2)

- For example, the *maximum daily temperature* in New York City can be modeled as a **cyclostationary process**:
  - the *maximum temperature* on *July 21* is statistically different from the *temperature* on *December 20*;
  - however, it is a reasonable approximation that the *temperature* on *December* 20 of *different years* has identical statistics.
  - Thus, we can view the random process composed of *daily maximum temperatures* as 365 interleaved stationary processes, each of which takes on a *new value once per year*.

https://en.wikipedia.org/wiki/Cyclostationary_process

Cyclostationary process (3)

- **wide-sense cyclostationary** signals
  - **cyclostationarity** in second-order statistics (e.g., the autocorrelation function).
- The exact definition differs depending on whether the signal is treated as
  - a stochastic process
  - a deterministic time series.

https://en.wikipedia.org/wiki/Cyclostationary_process

Cyclostationary process (4)

A **stochastic process** $x(t)$ of **mean** $\mathrm{E}[x(t)]$
and **autocorrelation function**:

$$R_x(t,\tau) = \mathrm{E}\{x(t+\tau)x^*(t)\},$$

where the star denotes complex conjugation,
is said to be **wide-sense cyclostationary** with period T_0
if both $\mathrm{E}[x(t)]$ and $R_x(t,\tau)$ are **cyclic** in t with period T_0, i.e.:

$$\mathrm{E}[x(t)] = \mathrm{E}[x(t+T_0)] \text{ for all } t$$

$$R_x(t,\tau) = R_x(t+T_0;\tau) \text{ for all } t,\tau.$$

https://en.wikipedia.org/wiki/Cyclostationary_process

Cyclostationary process (5)

The **autocorrelation function** is thus periodic in t and can be expanded in Fourier series:

$$R_x(t,\tau) = \sum_{n=-\infty}^{\infty} R_x^{n/T_0}(\tau) e^{j2\pi \frac{n}{T_0} t}$$

where $R_x^{n/T_0}(\tau)$ is called **cyclic autocorrelation function** and equal to:

$$R_x^{n/T_0}(\tau) = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} R_x(t,\tau) e^{-j2\pi \frac{n}{T_0} t} \mathrm{d}t.$$

The frequencies n/T_0, $n \in \mathbb{Z}$, are called **cyclic frequencies**.
Wide-sense stationary processes are a special case of cyclostationary processes with only $R_x^0(\tau) \neq 0$

https://en.wikipedia.org/wiki/Cyclostationary_process