Laurent Series and z-Transform - Geometric Series
Time Shift A

20180903 Mon

Causal Signal

$$a_n = \left(\frac{1}{2}\right)^n \quad (n \geq 0)$$

$$f(z) = \left(\frac{1}{2}\right)^0 z^0 + \left(\frac{1}{2}\right)^1 z^1 + \left(\frac{1}{2}\right)^2 z^2 + \cdots = \frac{1}{1 - \left(\frac{z}{2}\right)}$$

$$X(z) = \left(\frac{1}{2}\right)^0 z^0 + \left(\frac{1}{2}\right)^1 z^{-1} + \left(\frac{1}{2}\right)^2 z^{-2} + \cdots = \frac{1}{1 - \left(\frac{1}{2z}\right)}$$

$$a_n = \left(\frac{1}{2}\right)^n \quad (n \geq 0)$$

$$f(z) = \frac{1}{1 - \left(\frac{z}{2}\right)} \rightarrow \frac{2}{2 - z} \qquad (|z| < 2)$$

$$X(z) = \frac{1}{1 - \left(\frac{1}{2z}\right)} \rightarrow \frac{z}{z - 0.5} \qquad (|z| > 0.5)$$

Anti-Causal Signal

$$a_n = \left(\frac{1}{2}\right)^n \quad (n < 0)$$

$$f(z) = \left(\frac{1}{2}\right)^{-1} z^{-1} + \left(\frac{1}{2}\right)^{-2} z^{-2} + \left(\frac{1}{2}\right)^{-3} z^{-3} + \cdots = \frac{\left(\frac{2}{z}\right)}{1 - \left(\frac{2}{z}\right)}$$

$$X(z) = \left(\frac{1}{2}\right)^{-1} z^{1} + \left(\frac{1}{2}\right)^{-2} z^{2} + \left(\frac{1}{2}\right)^{-3} z^{3} + \cdots = \frac{(2z)}{1 - (2z)}$$

$$a_n = \left(\frac{1}{2}\right)^n \quad (n < 0)$$

$$f(z) = \frac{\left(\frac{2}{z}\right)}{1 - \left(\frac{2}{z}\right)} \rightarrow \frac{2}{z - 2} \quad (|z| > 2)$$

$$X(z) = \frac{(2z)}{1 - (2z)} \rightarrow \frac{z}{0.5 - z} \quad (|z| < 0.5)$$

$$a_n = -\left(\frac{1}{2}\right)^n \quad (n < 0)$$

$$f(z) = \frac{-\left(\frac{2}{z}\right)}{1 - \left(\frac{2}{z}\right)} \rightarrow \frac{2}{2 - z} \quad (|z| > 2)$$

$$X(z) = \frac{-(2z)}{1 - (2z)} \rightarrow \frac{z}{z - 0.5} \quad (|z| < 0.5)$$

Inverse z

$z \leftarrow z^{-1}$, $ROC(z) \leftarrow ROC(z^{-1})$

Causal

$$f(z) = \frac{2}{2-z} \quad (|z| < 2)$$

$$X(z) = \frac{z}{z-0.5} \quad (|z| > 0.5)$$

anti-Causal

$$f(z^{-1}) = \frac{2}{2-z^{-1}} \quad (|z^{-1}| < 2) \Rightarrow f(z^{-1}) = X(z) = \frac{z}{z-0.5} \quad (|z| > 0.5)$$

$$X(z^{-1}) = \frac{z^{-1}}{z^{-1}-0.5} \quad (|z^{-1}| > 0.5) \Rightarrow X(z^{-1}) = f(z) = \frac{2}{2-z} \quad (|z| < 2)$$

$f(z^{-1}) = X(z)$ Laurent Series (anti-causal signal)
with the same formula as causal X(z)

$X(z^{-1}) = f(z)$ z-Transform (anti-causal signal)
with the same formula as causal f(z)

Causal | anti-Causal

$$f(z) = \frac{2}{2-z} \quad (|z| < 2) \qquad X(z^{-1}) = \frac{2}{2-z} \quad (|z| < 2)$$

$$X(z) = \frac{z}{z-0.5} \quad (|z| > 0.5) \qquad f(z^{-1}) = \frac{z}{z-0.5} \quad (|z| > 0.5)$$

Causal f(z), Anti-causal f(z $^{-1}$)

Causal | anti-Causal

$$f(z) = \frac{z}{2-z} \quad (|z| < 2) \qquad f(z^{-1}) = \frac{z}{z-0.5} \quad (|z| > 0.5)$$

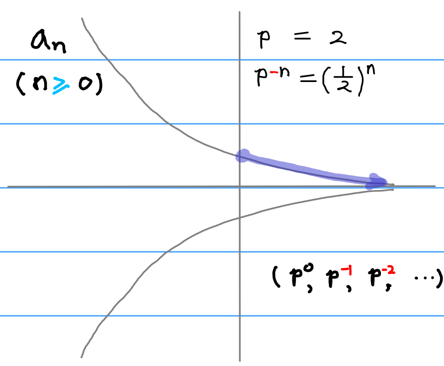
$$X(z) = \frac{z}{z-0.5} \quad (|z| > 0.5) \qquad X(z^{-1}) = \frac{z}{2-z} \quad (|z| < 2)$$

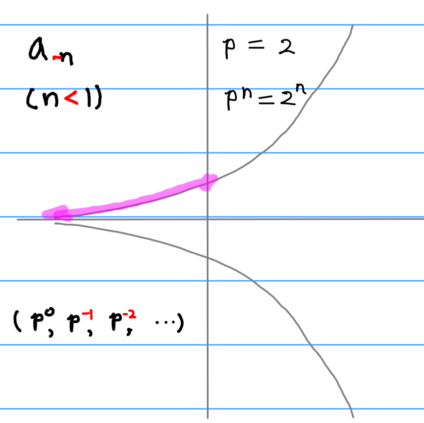
$$f(z) = \frac{z}{2-z} = \frac{1}{1-\left(\frac{z}{2}\right)} = \left(\tfrac{1}{2}\right)^0 z^0 + \left(\tfrac{1}{2}\right)^1 z^1 + \left(\tfrac{1}{2}\right)^2 z^2 + \cdots \qquad a_n = \left(\tfrac{1}{2}\right)^n$$

$$= p^0 z^0 + p^{-1} z^1 + p^{-2} z^2 + \cdots \qquad n = 0, 1, 2, \cdots$$

$$f(z^{-1}) = \frac{z}{2-z^{-1}} = \frac{1}{1-\left(\frac{1}{2z}\right)} = \left(\tfrac{1}{2}\right)^0 z^0 + \left(\tfrac{1}{2}\right)^1 z^{-1} + \left(\tfrac{1}{2}\right)^2 z^{-2} + \cdots \qquad a_{-n} = \left(\tfrac{1}{2}\right)^{-n}$$

$$= p^0 z^0 + p^{-1} z^{-1} + p^{-2} z^{-2} + \cdots \qquad n = 0, -1, -2, \cdots$$





Causal $X(z)$, Anti-causal $X(z^{-1})$

Causal | anti-Causal

$$f(z) = \frac{2}{2-z} \quad (|z| < 2) \qquad f(z^{-1}) = \frac{z}{z-0.5} \quad (|z| > 0.5)$$

$$X(z) = \frac{z}{z-0.5} \quad (|z| > 0.5) \qquad X(z^{-1}) = \frac{2}{2-z} \quad (|z| < 2)$$

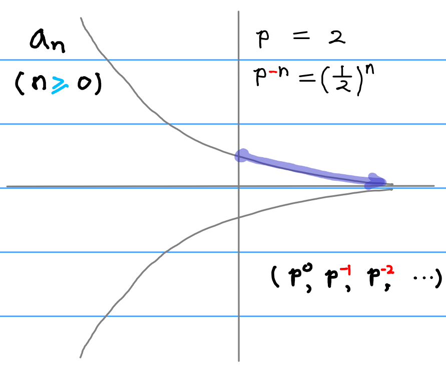
$$X(z) = \frac{z}{z-0.5} = \frac{1}{1-\left(\frac{1}{2z}\right)} = \left(\tfrac{1}{2}\right)^0 z^0 + \left(\tfrac{1}{2}\right)^1 z^{-1} + \left(\tfrac{1}{2}\right)^2 z^{-2} + \cdots \qquad a_n = \left(\tfrac{1}{2}\right)^n$$

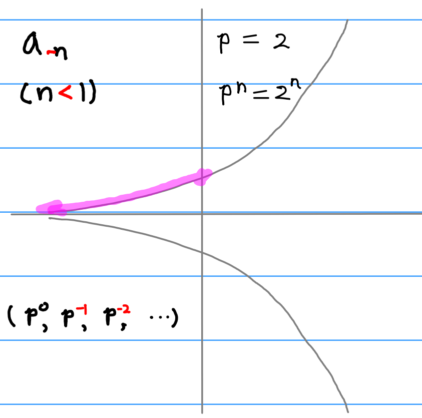
$$= p^0 z^0 + p^{-1} z^{-1} + p^{-2} z^{-2} + \cdots \qquad n = 0, 1, 2, \cdots$$

$$X(z^{-1}) = \frac{z^{-1}}{z^{-1}-0.5} = \frac{1}{1-\left(\frac{z}{2}\right)} = \left(\tfrac{1}{2}\right)^0 z^0 + \left(\tfrac{1}{2}\right)^1 z^1 + \left(\tfrac{1}{2}\right)^2 z^2 + \cdots \qquad a_{-n} = \left(\tfrac{1}{2}\right)^{-n}$$

$$= p^0 z^0 + p^{-1} z^1 + p^{-2} z^2 + \cdots \qquad n = 0, -1, -2, \cdots$$

z





Inverse ROC

$\mathrm{ROC}(z) \leftarrow \mathrm{ROC}(z^{-1})$

Causal | anti-Causal

$f(z) = \dfrac{2}{2-z} \quad (|z| < 2) \quad \Rightarrow \quad -f(z) = -\dfrac{2}{2-z} \quad (|z| > 2)$

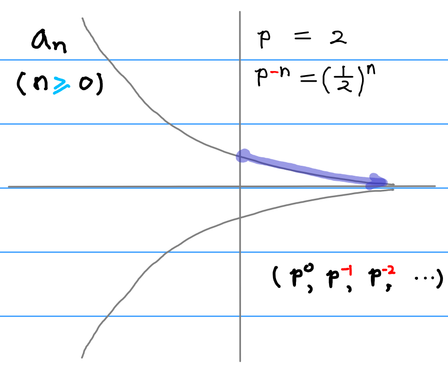
$X(z) = \dfrac{z}{z-0.5} \quad (|z| > 0.5) \quad \Rightarrow \quad -X(z) = -\dfrac{z}{z-0.5} \quad (|z| < 0.5)$

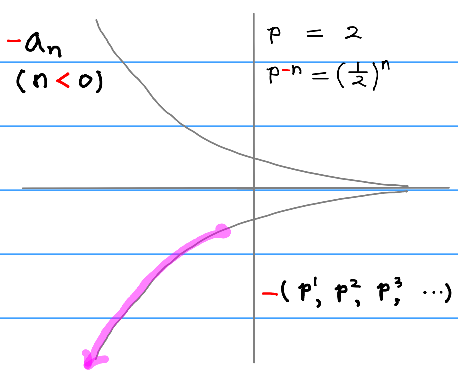
$$f(z) = \frac{2}{2-z} = \frac{1}{1-\left(\frac{z}{2}\right)} = \left(\tfrac{1}{2}\right)^0 z^0 + \left(\tfrac{1}{2}\right)^1 z^1 + \left(\tfrac{1}{2}\right)^2 z^2 + \cdots \qquad a_n = \left(\tfrac{1}{2}\right)^n$$

$$= p^0 z^0 + p^{-1} z^1 + p^{-2} z^2 + \cdots \qquad n = 0, 1, 2, \cdots$$

$$-f(z) = \frac{2}{z-2} = \frac{\left(\frac{2}{z}\right)}{1-\left(\frac{2}{z}\right)} = \left(\tfrac{1}{2}\right)^{-1} z^{-1} + \left(\tfrac{1}{2}\right)^{-2} z^{-2} + \left(\tfrac{1}{2}\right)^{-3} z^{-3} + \cdots \qquad a_n = -\left(\tfrac{1}{2}\right)^n$$

$$= p^1 z^{-1} + p^2 z^{-2} + p^3 z^{-3} + \cdots \qquad n = -1, -2, -3, \cdots$$





Causal f(z), Anti-causal f(z)

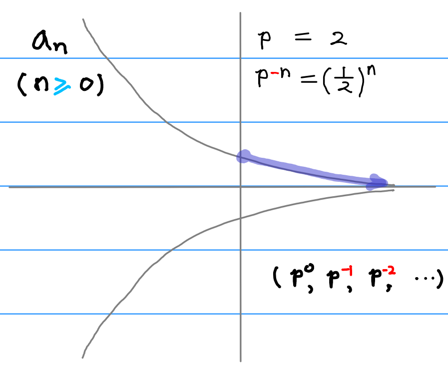
Causal	anti-Causal
$f(z) = \dfrac{2}{2-z} \quad (\lvert z\rvert < 2)$	$f(z) = \dfrac{2}{2-z} \quad (\lvert z\rvert > 0.5)$
$X(z) = \dfrac{z}{z-0.5} \quad (\lvert z\rvert > 0.5)$	$X(z) = \dfrac{z}{z-0.5} \quad (\lvert z\rvert < 2)$

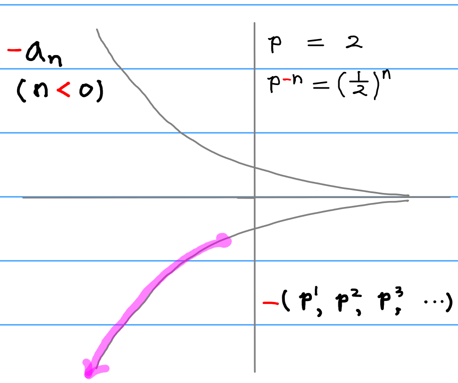
$$f(z) = \frac{2}{2-z} = \frac{1}{1-\left(\frac{z}{2}\right)} = \left(\tfrac{1}{2}\right)^0 z^0 + \left(\tfrac{1}{2}\right)^1 z^1 + \left(\tfrac{1}{2}\right)^2 z^2 + \cdots \qquad a_n = \left(\tfrac{1}{2}\right)^n$$

$$= p^0 z^0 + p^{-1} z^1 + p^{-2} z^2 + \cdots \qquad n = 0, 1, 2, \cdots$$

$$f(z) = \frac{2}{2-z} = \frac{-\left(\frac{2}{z}\right)}{1-\left(\frac{2}{z}\right)} = -\left[(2)^1 z^{-1} + (2)^2 z^{-2} + \left(\tfrac{1}{2}\right)^3 z^{-3} + \cdots\right] \qquad -a_n = -\left(\tfrac{1}{2}\right)^n$$

$$= -\left[p^1 z^{-1} + p^2 z^{-2} + p^3 z^{-3} + \cdots\right] \qquad n = -1, -2, -3, \cdots$$





Causal X(z), Anti-causal X(z)

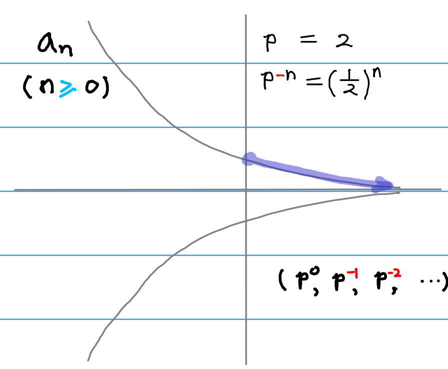
Causal	anti-Causal
$f(z) = \dfrac{2}{2-z} \quad (\lvert z\rvert < 2)$	$f(z) = \dfrac{2}{2-z} \quad (\lvert z\rvert > 0.5)$
$X(z) = \dfrac{z}{z-0.5} \quad (\lvert z\rvert > 0.5)$	$X(z) = \dfrac{z}{z-0.5} \quad (\lvert z\rvert < 2)$

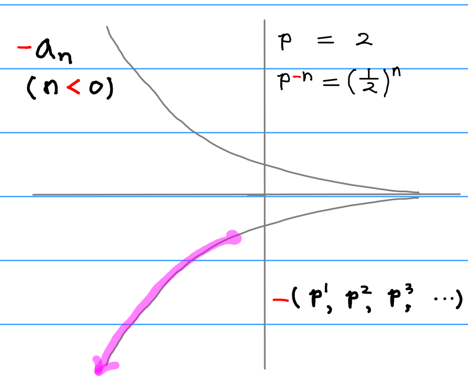
$$X(z) = \frac{z}{z-0.5} = \frac{1}{1-\left(\frac{1}{2z}\right)} = \left(\tfrac{1}{2}\right)^0 z^0 + \left(\tfrac{1}{2}\right)^1 z^{-1} + \left(\tfrac{1}{2}\right)^2 z^{-2} + \cdots \qquad a_n = \left(\tfrac{1}{2}\right)^n$$

$$= p^0 z^0 + p^{-1} z^{-1} + p^{-2} z^{-2} + \cdots \qquad n = 0, 1, 2, \cdots$$

$$X(z) = \frac{z}{z-0.5} = \frac{-(2z)}{1-(2z)} = -\left[(2)^1 z^1 + (2)^2 z^2 + \left(\tfrac{1}{2}\right)^3 z^3 + \cdots\right] \qquad -a_n = -\left(\tfrac{1}{2}\right)^n$$

$$= -\left[p^1 z^1 + p^2 z^2 + p^3 z^3 + \cdots\right] \qquad n = -1, -2, -3, \cdots$$





p=2

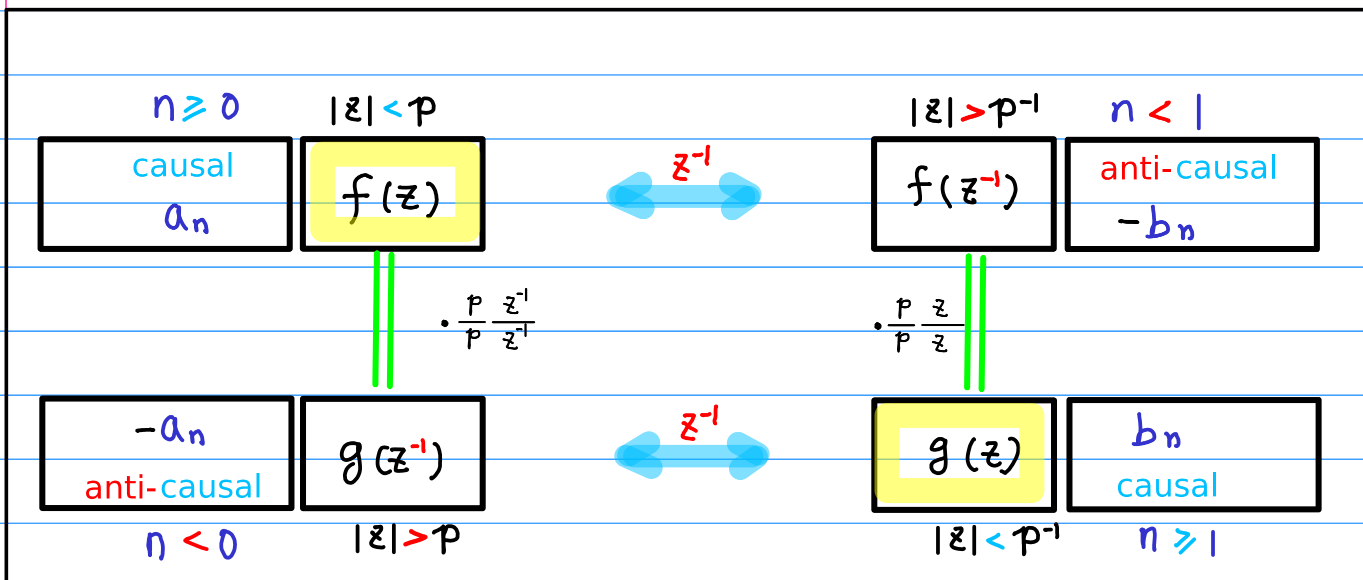
	causal	anti-causal
	$a_n = \left(\frac{1}{2}\right)^n \quad (n \geq 0)$	$a_n = \left(\frac{1}{2}\right)^n \quad (n < 0)$
	$f(z) = \dfrac{1}{1-\left(\frac{z}{2}\right)} \quad (\lvert z\rvert < 2)$ $X(z) = \dfrac{1}{1-\left(\frac{1}{2z}\right)} \quad (\lvert z\rvert > 0.5)$	$f(z) = \dfrac{\left(\frac{2}{z}\right)}{1-\left(\frac{2}{z}\right)} \quad (\lvert z\rvert > 2)$ $X(z) = \dfrac{(2z)}{1-(2z)} \quad (\lvert z\rvert < 0.5)$
	$f(z) = \dfrac{2}{2-z} \quad (\lvert z\rvert < 2)$ $X(z) = \dfrac{z}{z-0.5} \quad (\lvert z\rvert > 0.5)$	$f(z) = \dfrac{2}{z-2} \quad (\lvert z\rvert > 2)$ $X(z) = \dfrac{z}{0.5-z} \quad (\lvert z\rvert < 0.5)$

p=1/2

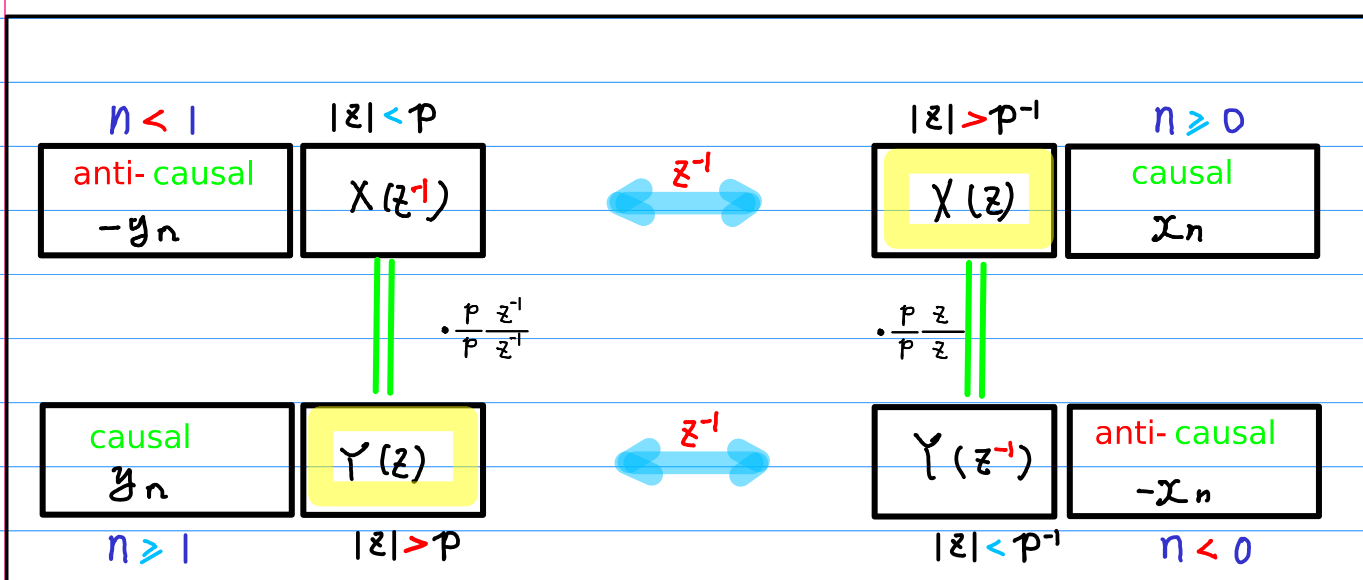
	causal	anti-causal
	$a_n = (2)^n \quad (n \geq 0)$	$a_n = (2)^n \quad (n < 0)$
	$f(z) = \dfrac{1}{1-(2z)} \quad (\lvert z\rvert < 0.5)$ $X(z) = \dfrac{1}{1-\left(\frac{2}{z}\right)} \quad (\lvert z\rvert > 2)$	$f(z) = \dfrac{\left(\frac{1}{2z}\right)}{1-\left(\frac{1}{2z}\right)} \quad (\lvert z\rvert > 0.5)$ $X(z) = \dfrac{\left(\frac{z}{2}\right)}{1-\left(\frac{z}{2}\right)} \quad (\lvert z\rvert < 2)$
	$f(z) = \dfrac{0.5}{0.5-z} \quad (\lvert z\rvert < 0.5)$ $X(z) = \dfrac{z}{z-2} \quad (\lvert z\rvert > 2)$	$f(z) = \dfrac{0.5}{z-0.5} \quad (\lvert z\rvert > 0.5)$ $X(z) = \dfrac{z}{2-z} \quad (\lvert z\rvert < 2)$

$f(z)$ $g(z^{-1})$	$f(z^{-1})$ $g(z)$	$X(z^{-1})$ $Y(z)$	$X(z)$ $Y(z^{-1})$

Laurent Series $a_n \leftrightarrow f(z)$ $b_n \leftrightarrow g(z)$



Z- Transform $X(z) \leftrightarrow x_n$ $Y(z) \leftrightarrow y_n$



$$f(z),\ X(z) = \frac{2}{2-z},\ \frac{z}{z-0.5}$$

f(z) Laurent Series

$n \in [0, \infty)$ causal $a_n = 0.5^n$

$|z| < 2$ $\boxed{\frac{2}{2-z}}$ $\xleftrightarrow{z^{-1}}$ $|z| > 0.5$ $\boxed{\frac{z}{z-0.5}}$

$n \in (-\infty, 0]$ anti-causal $a_{-n} = 0.5^{-n}$

$\cdot \frac{p}{p} \frac{z^{-1}}{z^{-1}}$ $\quad$ $\cdot \frac{p}{p} \frac{z}{z}$

$-a_n = -0.5^n$ anti-causal $n \in (-\infty, -1]$

$\boxed{\frac{2}{2-z}}$ $|z| > 2$ $\xleftrightarrow{z^{-1}}$ $\boxed{\frac{z}{z-0.5}}$ $|z| < 0.5$

$-a_{-n} = -0.5^{-n}$ causal $n \in [1, \infty)$

X(z) z-Transform

$n \in (-\infty, 0]$ anti-causal X(z) $a_{-n} = 0.5^{-n}$

$|z| < p$ $\boxed{\frac{2}{2-z}}$ $\xleftrightarrow{z^{-1}}$ $|z| > p^{-1}$ $\boxed{\frac{z}{z-0.5}}$

$n \in [0, \infty)$ causal X(z) $a_n = 0.5^n$

$\cdot \frac{p}{p} \frac{z^{-1}}{z^{-1}}$ $\quad$ $\cdot \frac{p}{p} \frac{z}{z}$

$-a_{-n} = 0.5^{-n}$ causal X(z) $n \in [1, \infty)$

$\boxed{\frac{2}{2-z}}$ $|z| > p$ $\xleftrightarrow{z^{-1}}$ $\boxed{\frac{z}{z-0.5}}$ $|z| < p^{-1}$

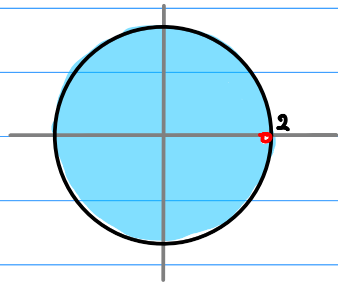
$-a_n = -0.5^n$ anti-causal X(z) $n \in (-\infty, -1]$

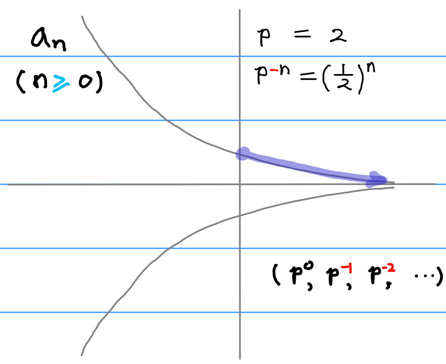
$n \in [0, \infty)$ $|z| < 2$

causal

$a_n = 0.5^n$

$\frac{2}{2 - z}$



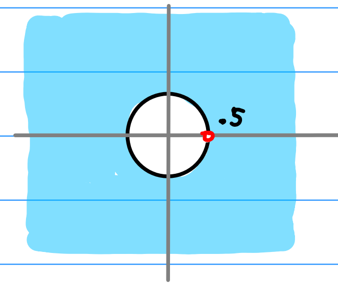


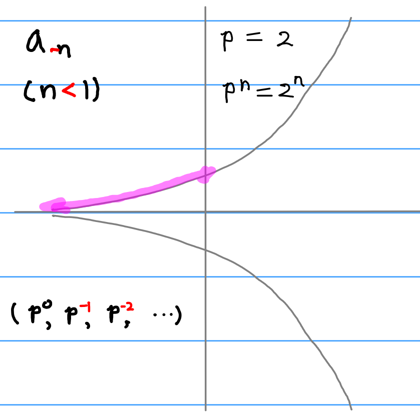
$|z| > 0.5$ $n \in (-\infty, 0]$

$\frac{z}{z - 0.5}$

anti-causal

$a_{-n} = 2^n$



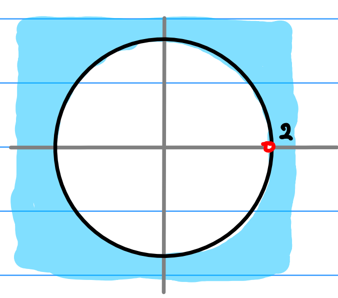


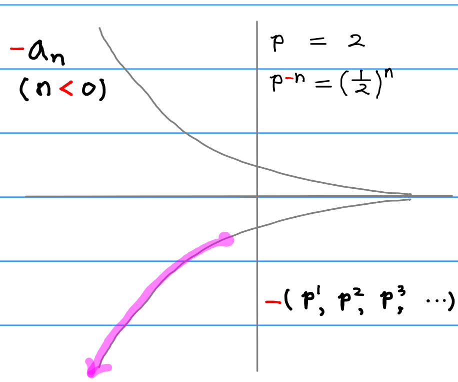
$-a_n = -0.5^n$

anti-causal

$\frac{2}{2 - z}$

$n \in (-\infty, -1]$ $|z| > 2$



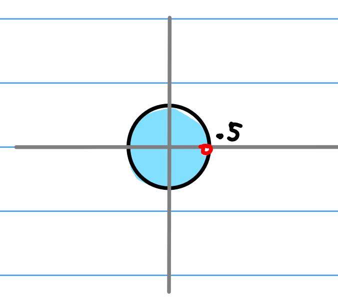


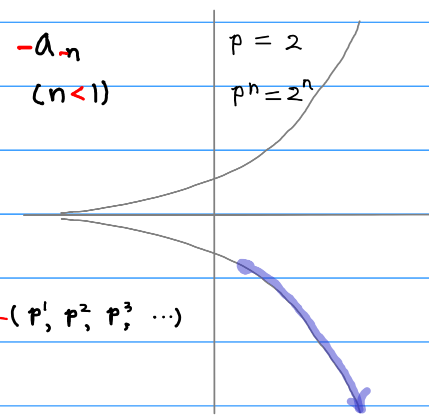
$\frac{z}{z - 0.5}$

$-a_{-n} = -2^n$

causal

$|z| < 0.5$ $n \in [1, \infty)$

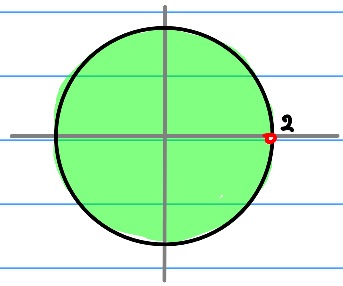


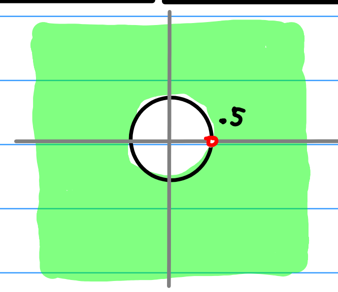


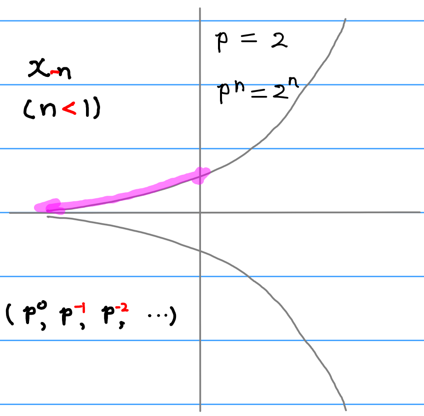
	$n < 1$	$\lvert z\rvert < 2$
x_{-n}	anti-causal $-(p^{-1}, p^{-2}, p^{-3}, \cdots)$	$-\dfrac{0.5}{1-0.5z}$

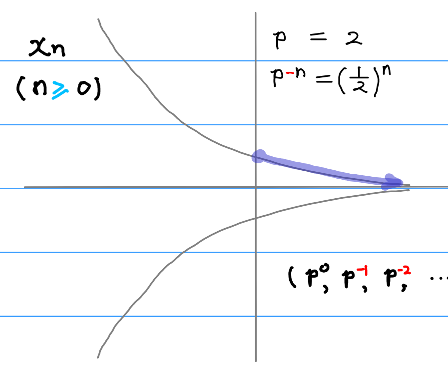
$\xleftrightarrow{z^{-1}}$

$\lvert z\rvert > 0.5$	$n \geq 0$	
$-\dfrac{0.5}{1-0.5z^{-1}}$	causal $-(p^{-1}, p^{-2}, p^{-3}, \cdots)$	x_n





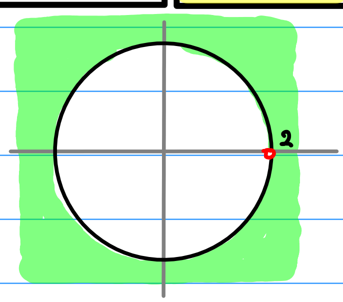


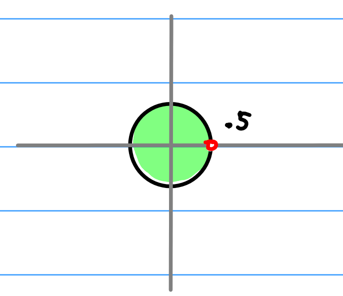


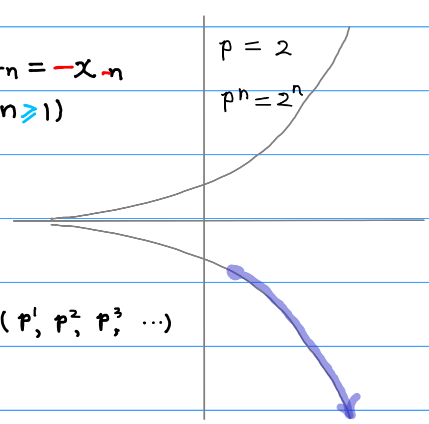
	$n \geq 1$	$\lvert z\rvert > 2$
y_n	$(p^0, p^1, p^2, \cdots)$ causal	$\dfrac{z^{-1}}{1-2z^{-1}}$

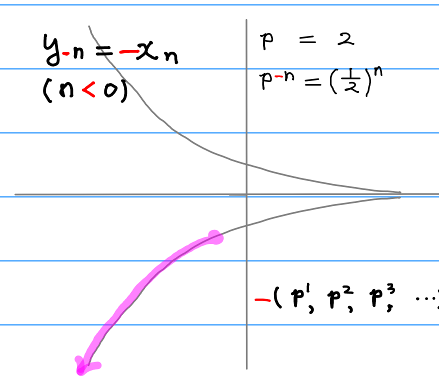
$\xleftrightarrow{z^{-1}}$

$\lvert z\rvert < 0.5$	$n < 0$	
$\dfrac{z}{1-2z}$	$(p^0, p^1, p^2, \cdots)$ anti-causal	y_{-n}









2 formulas (A)

$$\boxed{\frac{1}{z-p}} \qquad \boxed{\frac{1}{z^{-1}-p}}$$

2 representations each

$$\boxed{\frac{1}{z-p}} \;\rightarrow\; \boxed{-\frac{p^{-1}}{1-p^{-1}z}}, \quad \boxed{\frac{z^{-1}}{1-pz^{-1}}}$$

$$\boxed{\frac{1}{z^{-1}-p}} \;\rightarrow\; \boxed{\frac{z}{1-pz}}, \quad \boxed{-\frac{p^{-1}}{1-p^{-1}z^{-1}}}$$

2 formulas (B)

$$\boxed{\frac{p}{z-p}} \qquad \boxed{\frac{p}{z^{-1}-p}}$$

2 representations each

$$\boxed{\frac{p}{z-p}} \;\rightarrow\; \boxed{-\frac{1}{1-p^{-1}z}}, \quad \boxed{\frac{pz^{-1}}{1-pz^{-1}}}$$

$$\boxed{\frac{p}{z^{-1}-p}} \;\rightarrow\; \boxed{\frac{pz}{1-pz}}, \quad \boxed{-\frac{1}{1-p^{-1}z^{-1}}}$$

$$\xrightarrow{\;p\;}$$

$$\xleftarrow[\;1/p\;]{}$$

Ⓐ $f(z) \longrightarrow a_n$

2 formulas Ⓐ

$\boxed{\frac{1}{z-p}}$ $\boxed{\frac{1}{z^{-1}-p}}$

2 representations each

$\boxed{\frac{1}{z-p}}$

- $\boxed{-\frac{p^{-1}}{1-p^{-1}z}} \triangleq f(z)$ causal $\quad -p^{-(n+1)} \quad (n \geq 0)$
- $\|$
- $\boxed{\frac{z^{-1}}{1-pz^{-1}}} \triangleq g(z^{-1})$ anti- causal $\quad p^{-(n+1)} \quad (n < 0)$

$\boxed{\frac{1}{z^{-1}-p}}$

- $\boxed{\frac{z}{1-pz}} \triangleq g(z)$ causal $\quad p^{n-1} \quad (n \geq 1)$
- $\|$
- $\boxed{-\frac{p^{-1}}{1-p^{-1}z^{-1}}} \triangleq f(z^{-1})$ anti- causal $\quad -p^{n-1} \quad (n < 1)$

$\boxed{-\frac{1}{2-z}}$

- $\boxed{\frac{-0.5}{1-0.5z}}$ causal -0.5^{n+1} $(n \geq 0)$
- $\boxed{\frac{z^{-1}}{1-2z^{-1}}}$ anti 2^{-n-1} $(n<1)$

$\boxed{-\frac{0.5z}{z-0.5}}$

- $\boxed{\frac{z}{1-2z}}$ causal 2^{n-1} $(n \geq 1)$
- $\boxed{\frac{-0.5}{1-0.5z^{-1}}}$ anti -0.5^{-n+1} $(n<0)$

A $f(z) \longrightarrow a_n$ method 1

p^{-1} $\boxed{-\dfrac{p^{-1}}{1-p^{-1}z}}$ $-(p^{-1}) \Rightarrow -(p^{-1})^{n+1} \Rightarrow -p^{-(n+1)}$

$n = 0, 1, 2, \cdots$

$-(p^{-1}z^{0} + p^{-2}z^{1} + p^{-3}z^{2} + \cdots)$

p $\boxed{\dfrac{z^{-1}}{1-pz^{-1}}}$ $(p) \Rightarrow (p)^{-n-1} \Rightarrow p^{-(n+1)}$

$n = -1, -2, -3, \cdots$

$(p^{0}z^{-1} + p^{1}z^{-2} + p^{2}z^{-3} + \cdots)$

p $\boxed{\dfrac{z}{1-pz}}$ $(p) \Rightarrow (p)^{n-1} \Rightarrow p^{n-1}$

$n = 1, 2, 3, \cdots$

$(p^{0}z^{1} + p^{1}z^{2} + p^{2}z^{3} + \cdots)$

p^{-1} $\boxed{-\dfrac{p^{-1}}{1-p^{-1}z^{-1}}}$ $-(p^{-1}) \Rightarrow -(p^{-1})^{-n+1} \Rightarrow -p^{n-1}$

$n = 0, -1, -2, \cdots$

$-(p^{-1}z^{0} + p^{-2}z^{-1} + p^{-3}z^{-2} + \cdots)$

Ⓐ $f(z) \longrightarrow a_n$ method 2

p^{-1} $\boxed{-\dfrac{p^{-1}}{1-p^{-1}z}}$ $-(p^{-1})$ $-p^{-(n+1)}$

$z \rightarrow n = 0, 1, 2, \cdots \rightarrow (p)^{-n-1}$

p $\boxed{\dfrac{z^{-1}}{1-pz^{-1}}}$ (p) $p^{-(n+1)}$

$z^{-1}/z^{-1} \rightarrow n = -1, -2, -3, \cdots \rightarrow (p)^{-n-1}$

p $\boxed{\dfrac{z}{1-pz}}$ (p) p^{n-1}

$z/z \rightarrow n = 1, 2, 3, \cdots \rightarrow (p)^{n-1}$

p^{-1} $\boxed{-\dfrac{p^{-1}}{1-p^{-1}z^{-1}}}$ $-(p^{-1})$ $-p^{n-1}$

$z^{-1} \rightarrow n = 0, -1, -2, \cdots \rightarrow (p)^{n-1}$

(B) $f(z) \longrightarrow a_n$

2 formulas

$$\boxed{\frac{p}{z - p}} \qquad \boxed{\frac{p}{z^{-1} - p}}$$

2 representations each

$\boxed{\dfrac{p}{z-p}}$

- causal: $\boxed{-\dfrac{1}{1 - p^{-1} z}} \triangleq f'(z) \qquad -p^{-n} \qquad (n \geq 0)$

$$\|$$

- anti- causal: $\boxed{\dfrac{p z^{-1}}{1 - p z^{-1}}} \triangleq g'(z^{-1}) \qquad p^{-n} \qquad (n < 0)$

$\boxed{\dfrac{p}{z^{-1}-p}}$

- causal: $\boxed{\dfrac{p z}{1 - p z}} \triangleq g'(z) \qquad p^{n} \qquad (n \geq 1)$

$$\|$$

- anti- causal: $\boxed{-\dfrac{1}{1 - p^{-1} z^{-1}}} \triangleq f'(z^{-1}) \qquad -p^{n} \qquad (n < 1)$

$\boxed{\dfrac{2}{2 - z}}$

- $\boxed{\dfrac{1}{1 - 0.5z}}$ causal $\quad 0.5^{n} \ (n \geq 0)$
- $\boxed{\dfrac{-2z^{-1}}{1 - 2z^{-1}}}$ anti $\quad -2^{-n} \ (n < 0)$

$\boxed{\dfrac{z}{z - 0.5}}$

- $\boxed{\dfrac{-2z}{1 - 2z}}$ causal $\quad -2^{n} \ (n \geq 1)$
- $\boxed{\dfrac{1}{1 - 0.5z^{-1}}}$ anti $\quad 0.5^{-n} \ (n < 1)$

(B) $f(z) \longrightarrow a_n$ method 1

p^{-1} $\boxed{-\dfrac{1}{1-p^{-1}z}}$ $-(p^{-1}) \Rightarrow -(p^{-1})^{n} \Rightarrow -p^{-n}$

$n = 0, 1, 2, \cdots$

$-(p^0z^0 + p^{-1}z^1 + p^{-2}z^2 + \cdots)$

p $\boxed{\dfrac{pz^{-1}}{1-pz^{-1}}}$ $(p) \Rightarrow (p)^{-n} \Rightarrow p^{-n}$

$n = -1, -2, -3, \cdots$

$(p^1z^{-1} + p^2z^{-2} + p^3z^{-3} + \cdots)$

p $\boxed{\dfrac{pz}{1-pz}}$ $(p) \Rightarrow (p)^{n} \Rightarrow p^{n}$

$n = 1, 2, 3, \cdots$

$(p^1z^1 + p^2z^2 + p^3z^3 + \cdots)$

p^{-1} $\boxed{-\dfrac{1}{1-p^{-1}z^{-1}}}$ $-(p^{-1}) \Rightarrow -(p^{-1})^{-n} \Rightarrow -p^{n}$

$n = 0, -1, -2, \cdots$

$-(p^0z^0 + p^{-1}z^{-1} + p^{-2}z^{-2} + \cdots)$

Ⓑ $f(z) \longrightarrow a_n$ method 2

p^{-1} $\boxed{-\frac{1}{1-p^{-1}z}}$ $-(p^{-1})$ z $\rightarrow$ $n=0,1,2,\cdots$ $\rightarrow$ $-p^{-n}$ $(p)^{-n}$

p $\boxed{\frac{pz^{-1}}{1-pz^{-1}}}$ (p) z^{-1}/z^{-1} $\rightarrow$ $n=-1,-2,-3,\cdots$ $\rightarrow$ p^{-n} $(p)^{-n}$

p $\boxed{\frac{pz}{1-pz}}$ (p) z/z $\rightarrow$ $n=1,2,3,\cdots$ $\rightarrow$ p^{n} $(p)^{n}$

p^{-1} $\boxed{-\frac{1}{1-p^{-1}z^{-1}}}$ $-(p^{-1})$ z^{-1} $\rightarrow$ $n=0,-1,-2,\cdots$ $\rightarrow$ $-p^{n}$ $(p)^{n}$

2 formulas

$$\boxed{\frac{p}{z-p}} \qquad \boxed{\frac{p}{z^{-1}-p}}$$

2 representations each

causal / anti- causal

$$\frac{p}{z-p} \;\to\; \boxed{-\frac{1}{1-p^{-1}z}} \triangleq f'(z) = X'(z^{-1})$$

$$\| \qquad \|$$

$$\frac{p}{z-p} \;\to\; \boxed{\frac{pz^{-1}}{1-pz^{-1}}} \triangleq Y'(z) = g'(z^{-1})$$

causal / anti- causal

causal / anti- causal

$$\frac{p}{z^{-1}-p} \;\to\; \boxed{\frac{pz}{1-pz}} \triangleq g'(z) = Y'(z^{-1})$$

$$\| \qquad \|$$

$$\frac{p}{z^{-1}-p} \;\to\; \boxed{-\frac{1}{1-p^{-1}z^{-1}}} \triangleq X'(z) = f'(z^{-1})$$

causal / anti- causal

$$
\begin{array}{cccc}
n \in [0,\infty) & |z|<2 & & |z|>0.5 \quad\quad n\in(-\infty,0] \\
\boxed{\begin{array}{c}\text{causal}\\ a_n = 0.5^{n}\end{array}} & \boxed{\dfrac{2}{2-z}} & \xleftrightarrow{z^{-1}} & \boxed{\dfrac{z}{z-0.5}} \quad \boxed{\begin{array}{c}\text{anti-causal}\\ a_{-n} = 0.5^{-n}\end{array}} \\
& \Big\| \;\; \cdot\,\dfrac{p}{p}\,\dfrac{z^{-1}}{z^{-1}} & & \cdot\,\dfrac{p}{p}\,\dfrac{z}{z} \;\; \Big\| \\
\boxed{\begin{array}{c}-a_n = -0.5^{n}\\ \text{anti-causal}\end{array}} & \boxed{\dfrac{2}{2-z}} & \xleftrightarrow{z^{-1}} & \boxed{\dfrac{z}{z-0.5}} \quad \boxed{\begin{array}{c}-a_{-n} = -0.5^{-n}\\ \text{causal}\end{array}} \\
n \in (-\infty,-1] & |z|>2 & & |z|<0.5 \quad\quad n\in[1,\infty)
\end{array}
$$

$$
\begin{tikzcd}
\boxed{\dfrac{2}{2-z}} \arrow[r, leftrightarrow, "z^{-1}"] \arrow[d, "-\frac{1}{2}"] & \boxed{\dfrac{z}{z-0.5}} \arrow[d, "-\frac{1}{2}"] \\
\boxed{-\dfrac{1}{2-z}} \arrow[r, leftrightarrow, "z^{-1}"] & \boxed{-\dfrac{0.5z}{z-0.5}}
\end{tikzcd}
$$

$$0.5^{-n+1} = \left(\frac{1}{2}\right)^{-n+1}$$

$$= 2^{+n-1}$$

$$\frac{z}{2-z} \rightarrow \frac{1}{1-0.5z} \;\text{causal: } 0.5^{n}\ (n \geq 0), \qquad \frac{-2z^{-1}}{1-2z^{-1}} \;\text{anti: } -2^{-n}\ (n<0)$$

$$\frac{z}{z-0.5} \rightarrow \frac{-2z}{1-2z} \;\text{causal: } -2^{n}\ (n \geq 1), \qquad \frac{1}{1-0.5z^{-1}} \;\text{anti: } 0.5^{-n}\ (n<1)$$

$$-\frac{1}{2-z} \rightarrow \frac{-0.5}{1-0.5z} \;\text{causal: } -0.5^{n+1}\ (n \geq 0), \qquad \frac{z^{-1}}{1-2z^{-1}} \;\text{anti: } 2^{-n-1}\ (n<1)$$

$$-\frac{0.5z}{z-0.5} \rightarrow \frac{z}{1-2z} \;\text{causal: } 2^{n-1}\ (n \geq 1), \qquad \frac{-0.5}{1-0.5z^{-1}} \;\text{anti: } -0.5^{-n+1}\ (n<1)$$

$\frac{2}{2-z}, \frac{z}{z-0.5}$ vs. $-\frac{0.5}{1-0.5z}, \frac{z}{1-2z}$

$n \in [0,\infty)$	$\lvert z\rvert < 2$		$\lvert z\rvert > 0.5$	$n \in (-\infty, 0]$
causal $a_n = 0.5^n$	$\frac{2}{2-z}$	$\xleftrightarrow{z^{-1}}$	$\frac{z}{z-0.5}$	anti-causal $a_{-n} = 0.5^{-n}$
	$\cdot \frac{p}{p} \frac{z^{-1}}{z^{-1}}$		$\cdot \frac{p}{p} \frac{z}{z}$	
$-a_n = -0.5^n$ anti-causal	$\frac{2}{2-z}$	$\xleftrightarrow{z^{-1}}$	$\frac{z}{z-0.5}$	$-a_{-n} = -0.5^{-n}$ causal
$n \in (-\infty,-1]$	$\lvert z\rvert > 2$		$\lvert z\rvert < 0.5$	$n \in [1,\infty)$

	$n \in [0,\infty)$	$\lvert z\rvert < 2$		$\lvert z\rvert > 0.5$	$n \in (-\infty, 0]$
a_n	causal -0.5^{n+1}	$-\frac{0.5}{1-0.5z}$	$\xleftrightarrow{z^{-1}}$	$-\frac{0.5}{1-0.5z^{-1}}$	anti-causal -0.5^{-n+1} -2^{+n-1}
b.	0.5^{n+1} anti-causal	$\frac{z^{-1}}{1-2z^{-1}}$	$\xleftrightarrow{z^{-1}}$	$\frac{z}{1-2z}$	2^{+n-1} 0.5^{-n+1} causal
	$n \in (-\infty,-1]$	$\lvert z\rvert > 2$		$\lvert z\rvert < 0.5$	$n \in [1,\infty)$

$\frac{2}{2-z}$

$\frac{1}{1-0.5z}$ causal 0.5^n $(n \geq 0)$

$\frac{-2z^{-1}}{1-2z^{-1}}$ anti -2^{-n} $(n<0)$

$\frac{z}{z-0.5}$

$\frac{-2z}{1-2z}$ causal -2^n $(n \geq 1)$

$\frac{1}{1-0.5z^{-1}}$ anti 0.5^{-n} $(n<1)$

$n \in [0,\infty)$ $|z|<2$ $|z|>0.5$ $n \in (-\infty, 0]$

causal $a_n = 0.5^n$ $\quad \frac{2}{2-z} \xleftrightarrow{z^{-1}} \frac{z}{z-0.5} \quad$ anti-causal $a_{-n} = 0.5^{-n}$

$\cdot \frac{p}{p}\frac{z^{-1}}{z^{-1}}$ $\quad \cdot \frac{p}{p}\frac{z}{z}$

$-a_n = -0.5^n$ anti-causal $\quad \frac{2}{2-z} \xleftrightarrow{z^{-1}} \frac{z}{z-0.5} \quad$ $-a_{-n} = -0.5^{-n}$ causal

$n \in (-\infty,-1]$ $|z|>2$ $|z|<0.5$ $n \in [1,\infty)$

$-\dfrac{1}{2-z}$

$\dfrac{-0.5}{1-0.5z}$ causal -0.5^{n+1} $(n\geq 0)$

$\dfrac{z^{-1}}{1-2z^{-1}}$ anti 2^{-n-1} $(n<1)$

$-\dfrac{0.5z}{z-0.5}$

$\dfrac{z}{1-2z}$ causal 2^{n-1} $(n\geq 1)$

$\dfrac{-0.5}{1-0.5z^{-1}}$ anti -0.5^{-n+1} $(n<0)$

	$n\in[0,\infty)$	$\lvert z\rvert<2$		$\lvert z\rvert>0.5$	$n\in(-\infty,0]$
a_n	causal -0.5^{n+1}	$-\dfrac{0.5}{1-0.5z}$	$\xleftrightarrow{z^{-1}}$	$-\dfrac{0.5}{1-0.5z^{-1}}$	anti-causal -0.5^{-n+1}

	$n\in(-\infty,-1]$	$\lvert z\rvert>2$		$\lvert z\rvert<0.5$	$n\in[1,\infty)$
b_n	0.5^{n+1} anti-causal	$\dfrac{z^{-1}}{1-2z^{-1}}$	$\xleftrightarrow{z^{-1}}$	$\dfrac{z}{1-2z}$	0.5^{-n+1} causal

	$n\in[0,\infty)$	$\lvert z\rvert<2$		$\lvert z\rvert>0.5$	$n\in(-\infty,0]$
a_n	causal -0.5^{n+1}	$-\dfrac{0.5}{1-0.5z}$	$\xleftrightarrow{z^{-1}}$	$-\dfrac{0.5}{1-0.5z^{-1}}$	anti-causal -2^{+n-1}

	$n\in(-\infty,-1]$	$\lvert z\rvert>2$		$\lvert z\rvert<0.5$	$n\in[1,\infty)$
b_n	0.5^{n+1} anti-causal	$\dfrac{z^{-1}}{1-2z^{-1}}$	$\xleftrightarrow{z^{-1}}$	$\dfrac{z}{1-2z}$	2^{+n-1} causal

Time Shift

$p = 2$

1 $(n \geq 0)$ $a_n = \left(\frac{1}{2}\right)^n$ $f(z) = \frac{2}{2-z}$ $X(z) = \frac{z}{z-0.5}$

3 $(n < 0)$ $a_n = \left(\frac{1}{2}\right)^n$ $f(z) = -\frac{2}{2-z}$ $X(z) = -\frac{z}{z-0.5}$

5 $(n \geq 1)$ $a_{n-1} = \left(\frac{1}{2}\right)^{n-1}$ $f(z) = \frac{2z}{2-z}$ $X(z) = \frac{1}{z-0.5}$

7 $(n < 1)$ $a_{n-1} = \left(\frac{1}{2}\right)^{n-1}$ $f(z) = -\frac{2z}{2-z}$ $X(z) = -\frac{1}{z-0.5}$

9 $(n \geq -1)$ $a_{n+1} = \left(\frac{1}{2}\right)^{n+1}$ $f(z) = \frac{2}{(2-z)z}$ $X(z) = \frac{z^2}{z-0.5}$

11 $(n < -1)$ $a_{n+1} = \left(\frac{1}{2}\right)^{n+1}$ $f(z) = -\frac{2}{(2-z)z}$ $X(z) = -\frac{z^2}{z-0.5}$

Time Shift

$p = \frac{1}{2}$

2 $(n \geq 0)$ $a_n = (2)^n$ $f(z) = \frac{0.5}{0.5 - z}$ $X(z) = \frac{z}{z-2}$

4 $(n < 0)$ $a_n = (2)^n$ $f(z) = -\frac{0.5}{0.5 - z}$ $X(z) = -\frac{z}{z-2}$

6 $(n \geq 1)$ $a_{n-1} = (2)^{n-1}$ $f(z) = \frac{0.5z}{0.5 - z}$ $X(z) = \frac{1}{z-2}$

8 $(n < 1)$ $a_{n-1} = (2)^{n-1}$ $f(z) = -\frac{0.5z}{0.5 - z}$ $X(z) = -\frac{1}{z-2}$

10 $(n \geq -1)$ $a_{n+1} = (2)^{n+1}$ $f(z) = \frac{0.5}{(0.5 - z)z}$ $X(z) = \frac{z^2}{z-2}$

12 $(n < -1)$ $a_{n+1} = (2)^{n+1}$ $f(z) = -\frac{0.5}{(0.5 - z)z}$ $X(z) = -\frac{z^2}{z-2}$

Time Shift

$2 \longleftrightarrow \frac{1}{2}$

1. $(n \geq 0) \quad a_n = \left(\frac{1}{2}\right)^n \qquad f(z) = \frac{2}{2-z} \qquad X(z) = \frac{z}{z-0.5}$

2. $(n \geq 0) \quad a_n = (2)^n \qquad f(z) = \frac{0.5}{0.5-z} \qquad X(z) = \frac{z}{z-2}$

3. $(n < 0) \quad a_n = \left(\frac{1}{2}\right)^n \qquad f(z) = -\frac{2}{2-z} \qquad X(z) = -\frac{z}{z-0.5}$

4. $(n < 0) \quad a_n = (2)^n \qquad f(z) = -\frac{0.5}{0.5-z} \qquad X(z) = -\frac{z}{z-2}$

5. $(n \geq 1) \quad a_{n-1} = \left(\frac{1}{2}\right)^{n-1} \qquad f(z) = \frac{2z}{2-z} \qquad X(z) = \frac{1}{z-0.5}$

6. $(n \geq 1) \quad a_{n-1} = (2)^{n-1} \qquad f(z) = \frac{0.5z}{0.5-z} \qquad X(z) = \frac{1}{z-2}$

7. $(n < 1) \quad a_{n-1} = \left(\frac{1}{2}\right)^{n-1} \qquad f(z) = -\frac{2z}{2-z} \qquad X(z) = -\frac{1}{z-0.5}$

8. $(n < 1) \quad a_{n-1} = (2)^{n-1} \qquad f(z) = -\frac{0.5z}{0.5-z} \qquad X(z) = -\frac{1}{z-2}$

9. $(n \geq -1) \quad a_{n+1} = \left(\frac{1}{2}\right)^{n+1} \qquad f(z) = \frac{2}{(2-z)z} \qquad X(z) = \frac{z^2}{z-0.5}$

10. $(n \geq -1) \quad a_{n+1} = (2)^{n+1} \qquad f(z) = \frac{0.5}{(0.5-z)z} \qquad X(z) = \frac{z^2}{z-2}$

11. $(n < -1) \quad a_{n+1} = \left(\frac{1}{2}\right)^{n+1} \qquad f(z) = -\frac{2}{(2-z)z} \qquad X(z) = -\frac{z^2}{z-0.5}$

12. $(n < -1) \quad a_{n+1} = (2)^{n+1} \qquad f(z) = -\frac{0.5}{(0.5-z)z} \qquad X(z) = -\frac{z^2}{z-2}$

Shift to the right

shift to the right →		$\times z$	$\times z^{-1}$
delete a_0			

1 $(n \geq 0)$ $a_n = \left(\frac{1}{2}\right)^n$ $f(z) = \frac{2}{2-z}$ $X(z) = \frac{z}{z-0.5}$

5 $(n \geq 1)$ $a_{n-1} = \left(\frac{1}{2}\right)^{n-1}$ $f(z) = \frac{2z}{2-z}$ $X(z) = \frac{1}{z-0.5}$

2 $(n \geq 0)$ $a_n = (2)^n$ $f(z) = \frac{0.5}{0.5-z}$ $X(z) = \frac{z}{z-2}$

6 $(n \geq 1)$ $a_{n-1} = (2)^{n-1}$ $f(z) = \frac{0.5z}{0.5-z}$ $X(z) = \frac{1}{z-2}$

shift to the right →		$\times z$	$\times z^{-1}$
insert a_0			

3 $(n < 0)$ $a_n = \left(\frac{1}{2}\right)^n$ $f(z) = -\frac{2}{2-z}$ $X(z) = -\frac{z}{z-0.5}$

7 $(n < 1)$ $a_{n-1} = \left(\frac{1}{2}\right)^{n-1}$ $f(z) = -\frac{2z}{2-z}$ $X(z) = -\frac{1}{z-0.5}$

4 $(n < 0)$ $a_n = (2)^n$ $f(z) = -\frac{0.5}{0.5-z}$ $X(z) = -\frac{z}{z-2}$

8 $(n < 1)$ $a_{n-1} = (2)^{n-1}$ $f(z) = -\frac{0.5z}{0.5-z}$ $X(z) = -\frac{1}{z-2}$

Shift to the left

Shift to the left $\leftarrow$ $\times z^{-1}$ $\times z$

delete a_0

1 $(n \geq 0)$ $a_n = \left(\frac{1}{2}\right)^n$ $f(z) = \frac{2}{2-z}$ $X(z) = \frac{z}{z-0.5}$

9 $(n \geq -1)$ $a_{n+1} = \left(\frac{1}{2}\right)^{n+1}$ $f(z) = \frac{2}{(2-z)z}$ $X(z) = \frac{z}{z-0.5}$

2 $(n \geq 0)$ $a_n = (2)^n$ $f(z) = \frac{0.5}{0.5-z}$ $X(z) = \frac{z}{z-2}$

10 $(n \geq -1)$ $a_{n+1} = (2)^{n+1}$ $f(z) = \frac{0.5}{(0.5-z)z}$ $X(z) = \frac{z}{z-2}$

Shift to the left $\leftarrow$ $\times z^{-1}$ $\times z$

insert a_0

3 $(n < 0)$ $a_n = \left(\frac{1}{2}\right)^n$ $f(z) = -\frac{2}{2-z}$ $X(z) = -\frac{z}{z-0.5}$

11 $(n < -1)$ $a_{n+1} = \left(\frac{1}{2}\right)^{n+1}$ $f(z) = -\frac{2}{(2-z)z}$ $X(z) = -\frac{z}{z-0.5}$

4 $(n < 0)$ $a_n = (2)^n$ $f(z) = -\frac{0.5}{0.5-z}$ $X(z) = -\frac{z}{z-2}$

12 $(n < -1)$ $a_{n+1} = (2)^{n+1}$ $f(z) = -\frac{0.5}{(0.5-z)z}$ $X(z) = -\frac{z}{z-2}$

$n=-4$	$n=-3$	$n=-2$	$n=-1$	$n=0$	$n=1$	$n=2$	
b^{-3}	b^{-2}	b^{-1}	b^{0}	b^{1}	b^{2}	b^{3}	

b^{n+1} $n=-2,-3,-4,\cdots$ b^{n+1} $n=-1,0,1,\cdots$

$n=-3$	$n=-2$	$n=-1$	$n=0$	$n=1$	$n=2$	$n=3$	
b^{-3}	b^{-2}	b^{-1}	b^{0}	b^{1}	b^{2}	b^{3}	

b^{n} $n=-1,-2,-3,\cdots$ b^{n} $n=0,1,2,\cdots$

$n=-3$	$n=-2$	$n=-1$	$n=0$	$n=1$	$n=2$	$n=3$		
	b^{-3}	b^{-2}	b^{-1}	b^{0}	b^{1}	b^{2}	b^{3}	

b^{n-1} $n=0,-1,-2,\cdots$ b^{n-1} $n=1,2,3,\cdots$

$$1 \longleftrightarrow \frac{1}{1}$$

① $(n \geq 0)$ $a_n = (1)^n$ $f(z) = \dfrac{1}{1-z}$ $X(z) = \dfrac{z}{z-1}$

② $(n \geq 0)$ $a_n = (1^{-1})^n$ $f(z) = \dfrac{1}{1-z}$ $X(z) = \dfrac{z}{z-1}$

③ $(n < 0)$ $a_n = (1)^n$ $f(z) = -\dfrac{1}{1-z}$ $X(z) = -\dfrac{z}{z-1}$

④ $(n < 0)$ $a_n = (1^{-1})^n$ $f(z) = -\dfrac{1}{1-z}$ $X(z) = -\dfrac{z}{z-1}$

⑤ $(n \geq 1)$ $a_{n-1} = (1)^{n-1}$ $f(z) = \dfrac{z}{1-z}$ $X(z) = \dfrac{1}{z-1}$

⑥ $(n \geq 1)$ $a_{n-1} = (1^{-1})^{n-1}$ $f(z) = \dfrac{z}{1-z}$ $X(z) = \dfrac{1}{z-1}$

⑦ $(n < 1)$ $a_{n-1} = (1)^{n-1}$ $f(z) = -\dfrac{z}{1-z}$ $X(z) = -\dfrac{1}{z-1}$

⑧ $(n < 1)$ $a_{n-1} = (1^{-1})^{n-1}$ $f(z) = -\dfrac{z}{1-z}$ $X(z) = -\dfrac{1}{z-1}$

⑨ $(n \geq -1)$ $a_{n+1} = (1)^{n+1}$ $f(z) = \dfrac{1}{(1-z)z}$ $X(z) = \dfrac{z}{z-1}$

⑩ $(n \geq -1)$ $a_{n+1} = (1^{-1})^{n+1}$ $f(z) = \dfrac{1}{(1-z)z}$ $X(z) = \dfrac{z}{z-1}$

⑪ $(n < -1)$ $a_{n+1} = (1)^{n+1}$ $f(z) = -\dfrac{1}{(1-z)z}$ $X(z) = -\dfrac{z}{z-1}$

⑫ $(n < -1)$ $a_{n+1} = (1^{-1})^{n+1}$ $f(z) = -\dfrac{1}{(1-z)z}$ $X(z) = -\dfrac{z}{z-1}$

1 $(n \geq 0)$ $a_n = (1)^n$ $f(z) = \frac{1}{1-z}$ $X(z) = \frac{z}{z-1}$

2 $(n \geq 0)$ $a_n = (1^{-1})^n$ $f(z) = \frac{1}{1-z}$ $X(z) = \frac{z}{z-1}$

shift to the right $\rightarrow$ $\times z$ $\times z^{-1}$

delete a_0

5 $(n \geq 1)$ $a_{n-1} = (1)^{n-1}$ $f(z) = \frac{z}{1-z}$ $X(z) = \frac{1}{z-1}$

6 $(n \geq 1)$ $a_{n-1} = (1^{-1})^{n-1}$ $f(z) = \frac{z}{1-z}$ $X(z) = \frac{1}{z-1}$

3 $(n < 0)$ $a_n = (1)^n$ $f(z) = -\frac{1}{1-z}$ $X(z) = -\frac{z}{z-1}$

4 $(n < 0)$ $a_n = (1^{-1})^n$ $f(z) = -\frac{1}{1-z}$ $X(z) = -\frac{z}{z-1}$

shift to the right $\rightarrow$ $\times z$ $\times z^{-1}$

insert a_0

7 $(n < 1)$ $a_{n-1} = (1)^{n-1}$ $f(z) = -\frac{z}{1-z}$ $X(z) = -\frac{1}{z-1}$

8 $(n < 1)$ $a_{n-1} = (1^{-1})^{n-1}$ $f(z) = -\frac{z}{1-z}$ $X(z) = -\frac{1}{z-1}$

① $(n \geq 0)$ $a_n = (1)^n$ $f(z) = \frac{1}{1-z}$ $X(z) = \frac{z}{z-1}$

② $(n \geq 0)$ $a_n = (1^{-1})^n$ $f(z) = \frac{1}{1-z}$ $X(z) = \frac{z}{z-1}$

shift to the left ← $\times z^{-1}$ $\times z$

delete a_0

⑨ $(n \geq -1)$ $a_{n+1} = (1)^{n+1}$ $f(z) = \frac{1}{(1-z)z}$ $X(z) = \frac{z}{z-1}$

⑩ $(n \geq -1)$ $a_{n+1} = (1^{-1})^{n+1}$ $f(z) = \frac{1}{(1-z)z}$ $X(z) = \frac{z}{z-1}$

③ $(n < 0)$ $a_n = (1)^n$ $f(z) = -\frac{1}{1-z}$ $X(z) = -\frac{z}{z-1}$

④ $(n < 0)$ $a_n = (1^{-1})^n$ $f(z) = -\frac{1}{1-z}$ $X(z) = -\frac{z}{z-1}$

shift to the left ← $\times z^{-1}$ $\times z$

insert a_0

⑪ $(n < -1)$ $a_{n+1} = (1)^{n+1}$ $f(z) = -\frac{1}{(1-z)z}$ $X(z) = -\frac{z}{z-1}$

⑫ $(n < -1)$ $a_{n+1} = (1^{-1})^{n+1}$ $f(z) = -\frac{1}{(1-z)z}$ $X(z) = -\frac{z}{z-1}$

Causality

$f(z)\ (|z|<p) \leftrightarrow a_n\ (n \geq 0)$ $\quad -(p^{-1}, p^{-2}, p^{-3}, \cdots)$

$X(z^{-1})\ (|z|<p) \leftrightarrow x_{-n}\ (n < 1)$ $\quad -(p^{-1}, p^{-2}, p^{-3}, \cdots)$

$f(z^{-1})\ (|z|>p^{-1}) \leftrightarrow a_{-n}\ (n < 1)$ $\quad -(p^{-1}, p^{-2}, p^{-3}, \cdots)$

$X(z)\ (|z|>p^{-1}) \leftrightarrow x_n\ (n \geq 0)$ $\quad -(p^{-1}, p^{-2}, p^{-3}, \cdots)$

$f(z)\ (|z|>p) \leftrightarrow -a_n\ (n < 0)$ $\quad (p^0, p^1, p^2, \cdots)$

$X(z^{-1})\ (|z|>p) \leftrightarrow -x_{-n}\ (n \geq 1)$ $\quad (p^0, p^1, p^2, \cdots)$

$f(z^{-1})\ (|z|<p^{-1}) \leftrightarrow -a_{-n}\ (n \geq 1)$ $\quad (p^0, p^1, p^2, \cdots)$

$X(z)\ (|z|<p^{-1}) \leftrightarrow -x_n\ (n < 0)$ $\quad (p^0, p^1, p^2, \cdots)$

$$f(z) \quad f(z^{-1})$$
$$g(z^{-1}) \quad g(z)$$

$$X(z^{-1}) \quad X(z)$$
$$Y(z) \quad Y(z^{-1})$$

$$a_n \quad a_{-n}$$
$$b_{-n} \quad b_n$$

$$x_{-n} \quad x_n$$
$$y_n \quad y_{-n}$$

$$f(z) \quad f(z^{-1})$$
$$f(z) \quad f(z^{-1})$$

$$X(z^{-1}) \quad X(z)$$
$$X(z^{-1}) \quad X(z)$$

$$-(p^{-1}, p^{-2}, p^{-3}, \dots) \quad -(p^{-1}, p^{-2}, p^{-3}, \dots)$$
$$(p^{0}, p^{1}, p^{2}, \dots) \quad (p^{0}, p^{1}, p^{2}, \dots)$$

$$-(p^{-1}, p^{-2}, p^{-3}, \dots) \quad -(p^{-1}, p^{-2}, p^{-3}, \dots)$$
$$(p^{0}, p^{1}, p^{2}, \dots) \quad (p^{0}, p^{1}, p^{2}, \dots)$$

$$-\frac{p^{-1}}{1-p^{-1}z} \quad -\frac{p^{-1}}{1-p^{-1}z^{-1}}$$
$$\frac{z^{-1}}{1-pz^{-1}} \quad \frac{z}{1-pz}$$

$$-\frac{p^{-1}}{1-p^{-1}z} \quad -\frac{p^{-1}}{1-p^{-1}z^{-1}}$$
$$\frac{z^{-1}}{1-pz^{-1}} \quad \frac{z}{1-pz}$$

$$\boxed{\begin{array}{cc} f(z) & g(z) \\ f(z) & g(z) \end{array}} \quad \boxed{\begin{array}{cc} Y(z) & X(z) \\ Y(z) & X(z) \end{array}} \quad \boxed{\begin{array}{cc} a_n & a_{-n} \\ -a_n & -a_{-n} \end{array}} \quad \boxed{\begin{array}{cc} x_{-n} & x_n \\ -x_{-n} & -x_n \end{array}}$$

$$\boxed{\begin{array}{cc} |z| < p & |z| > p^{-1} \\ |z| > p & |z| < p^{-1} \end{array}} \quad \boxed{\begin{array}{cc} |z| < p & |z| > p^{-1} \\ |z| > p & |z| < p^{-1} \end{array}}$$

$$\boxed{\begin{array}{cc} [0,\infty) & (-\infty,0] \\ (-\infty,-1] & [1,\infty) \end{array}} \quad \boxed{\begin{array}{cc} (-\infty,0] & [0,\infty) \\ [1,\infty) & (-\infty,-1] \end{array}}$$

$$\boxed{\begin{array}{c} -(p^{-1}, p^{-2}, p^{-3}, \cdots) \\ (p^{0}, p^{1}, p^{2}, \cdots) \end{array}} \quad \boxed{\begin{array}{c} -(p^{-1}, p^{-2}, p^{-3}, \cdots) \\ (p^{0}, p^{1}, p^{2}, \cdots) \end{array}}$$

$$\boxed{\begin{array}{cc} a_n & a_{-n} \\ -a_n & -a_{-n} \end{array}} \qquad \boxed{\begin{array}{cc} 2^{-n} & 2^{n} \\ -2^{-n} & -2^{n} \end{array}} \qquad a_n = -2^{-n}$$

$$\boxed{\begin{array}{cc} x_{-n} & x_n \\ -x_{-n} & -x_n \end{array}} \qquad \boxed{\begin{array}{cc} 2^{-n} & 2^{n} \\ -2^{-n} & -2^{n} \end{array}} \qquad x_n = -2^{n}$$

$$\boxed{\begin{array}{cc} -(p^{-1}, p^{-2}, p^{-3}, \cdots) & -(p^{-1}, p^{-2}, p^{-3}, \cdots) \\ (p^{0}, p^{1}, p^{2}, \cdots) & (p^{0}, p^{1}, p^{2}, \cdots) \end{array}} \qquad \boxed{\begin{array}{cc} -(-2^{-1}, -2^{-2}, -2^{-3}, \cdots) & -(-2^{-1}, -2^{-2}, -2^{-3}, \cdots) \\ (-2^{0}, -2^{1}, -2^{2}, \cdots) & (-2^{0}, -2^{1}, -2^{2}, \cdots) \end{array}}$$

$$\boxed{\begin{array}{cc} -\dfrac{p^{-1}}{1-p^{-1}z} & -\dfrac{p^{-1}}{1-p^{-1}z^{-1}} \\ \dfrac{z^{-1}}{1-pz^{-1}} & \dfrac{z}{1-pz} \end{array}} \qquad \boxed{\begin{array}{cc} \dfrac{2^{-1}}{1-2^{-1}z} & \dfrac{2^{-1}}{1-2^{-1}z^{-1}} \\ -\dfrac{z^{-1}}{1-2z^{-1}} & -\dfrac{z}{1-2z} \end{array}} \qquad \begin{array}{cc} \dfrac{\left(\frac{1}{2}\right)}{1-\left(\frac{z}{2}\right)} & \dfrac{\left(\frac{1}{2}\right)}{1-\left(\frac{1}{2z}\right)} \\ -\dfrac{\left(\frac{1}{z}\right)}{1-\left(\frac{2}{z}\right)} & -\dfrac{z}{1-2z} \end{array}$$

$$\boxed{\begin{array}{cc} |z| < p & |z| > p^{-1} \\ |z| > p & |z| < p^{-1} \end{array}} \qquad \boxed{\begin{array}{cc} |z| < 2 & |z| > 2^{-1} \\ |z| > 2 & |z| < 2^{-1} \end{array}}$$

$$\boxed{\begin{array}{cc} [0, \infty) & (-\infty, 0] \\ (-\infty, -1] & [1, \infty) \end{array}} \qquad \boxed{\begin{array}{cc} [0, \infty) & (-\infty, 0] \\ (-\infty, -1] & [1, \infty) \end{array}}$$

Causal $a_n = \left(\frac{1}{2}\right)^n, \ (2)^n \quad (n \geq 0)$

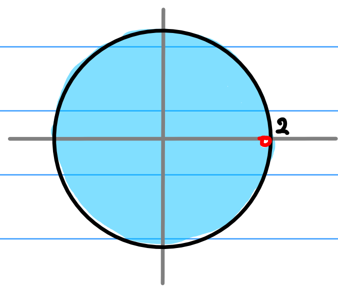
	$f(z)$		$X(z)$

$\left(\frac{1}{2}\right)^n$

$a_n = \left(\frac{1}{2}\right)^n \quad (n \geq 0)$

$$f(z) = \frac{1}{1-\frac{z}{2}} \quad |z| < 2$$

$$= \frac{z^{-1}}{z^{-1}-0.5} = \frac{2}{2-z}$$

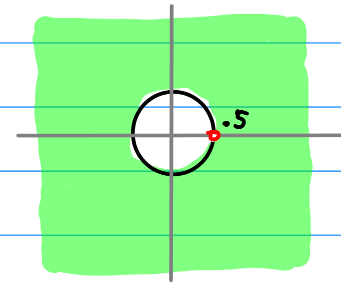


$\xleftrightarrow{z^{-1}}$

$a_n = \left(\frac{1}{2}\right)^n \quad (n \geq 0)$

$$X(z) = \frac{1}{1-\frac{1}{2z}} \quad |z| > 0.5$$

$$= \frac{z}{z-0.5}$$



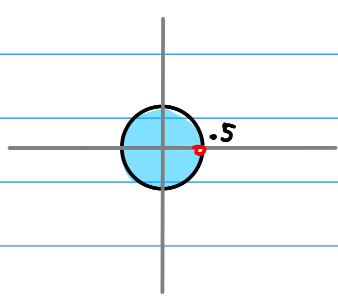
$\updownarrow p^{-1}$ $\qquad\qquad$ $\updownarrow p^{-1}$

$(2)^n$

$a_n = (2)^n \quad (n \geq 0)$

$$f(z) = \frac{1}{1-2z} \quad |z| < 0.5$$

$$= \frac{z^{-1}}{z^{-1}-2} = \frac{0.5}{0.5-z}$$

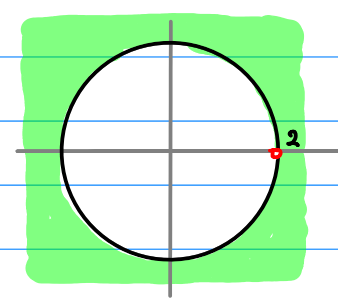


$\xleftrightarrow{z^{-1}}$

$a_n = (2)^n \quad (n \geq 0)$

$$X(z) = \frac{1}{1-\frac{2}{z}} \quad |z| > 2$$

$$= \frac{z}{z-2}$$



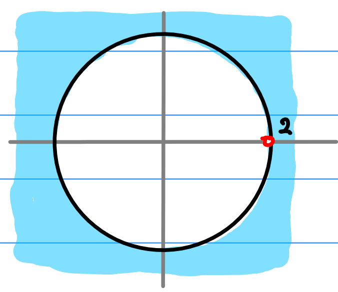
Anti-causal $a_n = \left(\frac{1}{2}\right)^n,\ (2)^n \quad (n < 0)$

	$f(z)$		$X(z)$
$\left(\frac{1}{2}\right)^n$	$a_n = \left(\frac{1}{2}\right)^n \quad (n<0)$		$a_n = \left(\frac{1}{2}\right)^n \quad (n<0)$

$a_n = \left(\frac{1}{2}\right)^n \quad (n < 0)$

$$f(z) = \frac{\frac{2}{z}}{1-\frac{2}{z}} \qquad |z| > 2$$

$$= -\frac{2}{2-z}$$

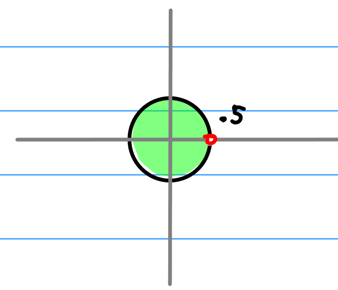


$\overset{z^{-1}}{\longleftrightarrow}$

$a_n = \left(\frac{1}{2}\right)^n \quad (n < 0)$

$$X(z) = \frac{2z}{1-2z} \qquad |z| < 0.5$$

$$= -\frac{z}{z-0.5}$$



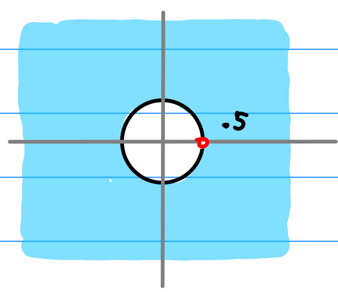
$\updownarrow p^{-1}$ $\qquad$ $\updownarrow p^{-1}$

$(2)^n$

$a_n = (2)^n \quad (n < 0)$

$$f(z) = \frac{\frac{1}{2z}}{1-\frac{1}{2z}} \qquad |z| > 0.5$$

$$= -\frac{0.5}{0.5-z}$$

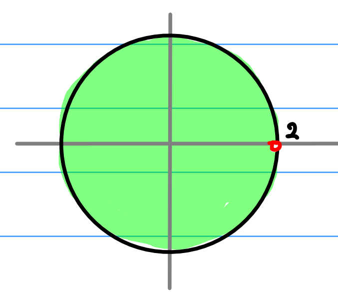


$\overset{z^{-1}}{\longleftrightarrow}$

$a_n = (2)^n \quad (n < 0)$

$$X(z) = \frac{\frac{2}{z}}{1-\frac{2}{z}} \qquad |z| < 2$$

$$= -\frac{z}{z-2}$$



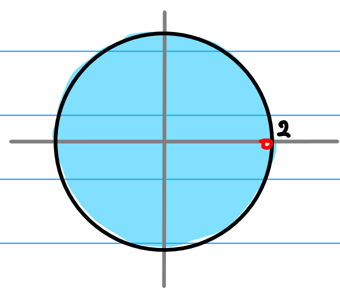
Causal $f(z)$ $\xleftrightarrow{-1}$ Anti-causal $f(z)$

$\left(\frac{1}{2}\right)^n$

$$a_n = \left(\frac{1}{2}\right)^n \quad (n \geq 0)$$

$$f(z) = \frac{1}{1-\frac{z}{2}} \qquad |z| < 2$$

$$= \frac{z^{-1}}{z^{-1}-0.5} = \frac{2}{2-z}$$

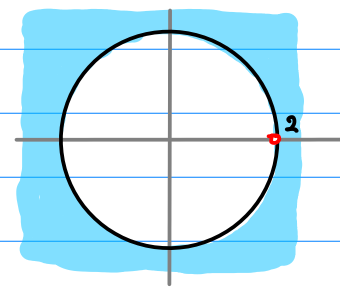


$\xleftrightarrow{-1}$

$$a_n = \left(\frac{1}{2}\right)^n \quad (n < 0)$$

$$f(z) = \frac{\frac{2}{z}}{1-\frac{2}{z}} \qquad |z| > 2$$

$$= -\frac{2}{2-z}$$



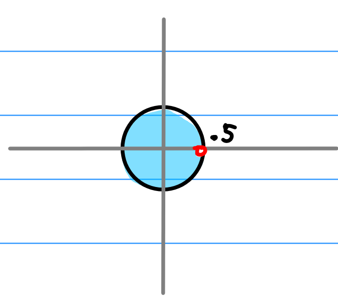
$\updownarrow p^{-1}$ $\qquad$ $\updownarrow p^{-1}$

$(2)^n$

$$a_n = (2)^n \quad (n \geq 0)$$

$$f(z) = \frac{1}{1-2z} \qquad |z| < 0.5$$

$$= \frac{z^{-1}}{z^{-1}-2} = \frac{0.5}{0.5-z}$$

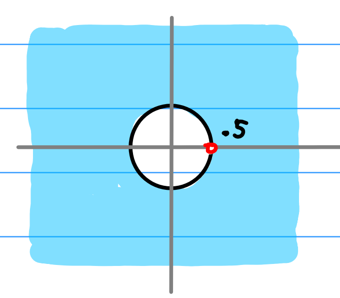


$\xleftrightarrow{-1}$

$$a_n = (2)^n \quad (n < 0)$$

$$f(z) = \frac{\frac{1}{2z}}{1-\frac{1}{2z}} \qquad |z| > 0.5$$

$$= -\frac{0.5}{0.5-z}$$



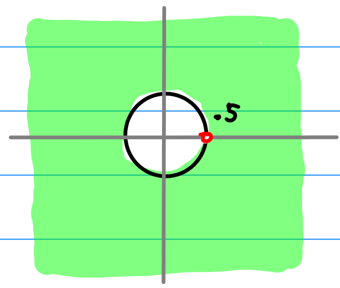
Causal $X(z)$ $\xleftrightarrow{-1}$ Anti-causal $X(z)$

$\left(\frac{1}{2}\right)^n$

$a_n = \left(\frac{1}{2}\right)^n \quad (n \geq 0)$

$$X(z) = \frac{1}{1 - \frac{1}{2z}} \qquad |z| > 0.5$$

$$= \frac{z}{z - 0.5}$$

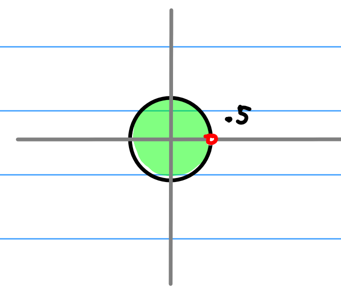


$\xleftrightarrow{-1}$

$a_n = \left(\frac{1}{2}\right)^n \quad (n < 0)$

$$X(z) = \frac{2z}{1 - 2z} \qquad |z| < 0.5$$

$$= -\frac{z}{z - 0.5}$$



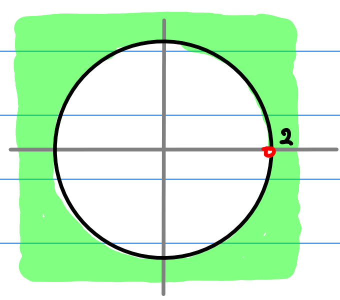
$\updownarrow p^{-1}$ $\qquad$ $\updownarrow p^{-1}$

$(2)^n$

$a_n = (2)^n \quad (n \geq 0)$

$$X(z) = \frac{1}{1 - \frac{2}{z}} \qquad |z| > 2$$

$$= \frac{z}{z - 2}$$

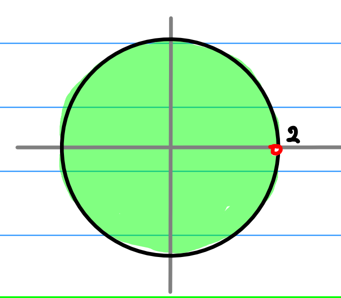


$\xleftrightarrow{-1}$

$a_n = (2)^n \quad (n < 0)$

$$X(z) = \frac{\frac{2}{z}}{1 - \frac{2}{z}} \qquad |z| < 2$$

$$= -\frac{z}{z - 2}$$



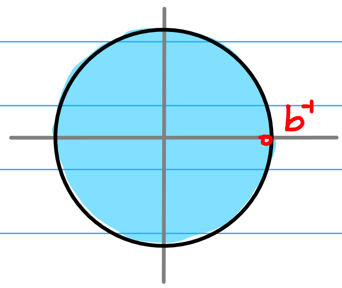
Causal b^n | Anti-causal b^n

$f(z)$

$$a_n = (b)^n \quad (n \geq 0)$$

$$f(z) = \frac{1}{1 - bz} \qquad |z| < b^{-1}$$

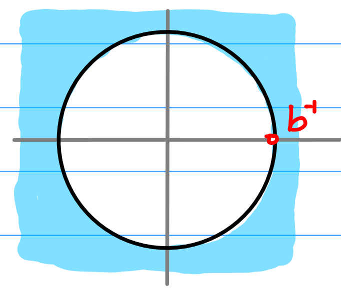
$$= \frac{b^{-1}}{b^{-1} - z}$$



$$a_n = (b)^n \quad (n < 0)$$

$$f(z) = \frac{b^{-1} z^{-1}}{1 - b^{-1} z^{-1}} \qquad |z| > b^{-1}$$

$$= - \frac{b^{-1}}{b^{-1} - z}$$

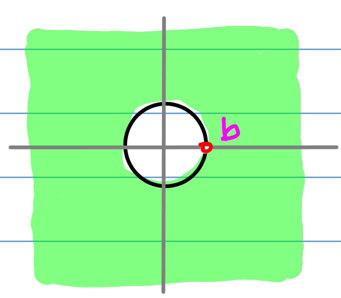


$X(z)$

$$a_n = (b)^n \quad (n \geq 0)$$

$$X(z) = \frac{1}{1 - bz^{-1}} \qquad |z| > b$$

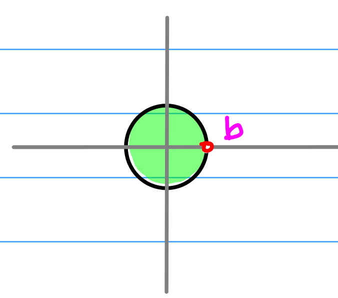
$$= \frac{z}{z - b}$$



$$a_n = (b)^n \quad (n < 0)$$

$$X(z) = \frac{b^{-1} z}{1 - b^{-1} z} \qquad |z| < b$$

$$= - \frac{z}{z - b}$$



Causal $(n \geq 0)$ $(1)^n$, $(1^{-1})^n$

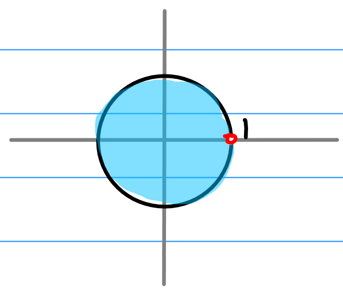
	$f(z)$		$X(z)$
$(1)^n$	$a_n = (1)^n \quad (n \geq 0)$		$a_n = (1)^n \quad (n \geq 0)$

$f(z)$:

$$a_n = (1)^n \quad (n \geq 0)$$

$$f(z) = \frac{1}{1-z} \qquad |z| < 1$$

$$= \frac{z^{-1}}{z^{-1}-1} = \frac{1}{1-z}$$



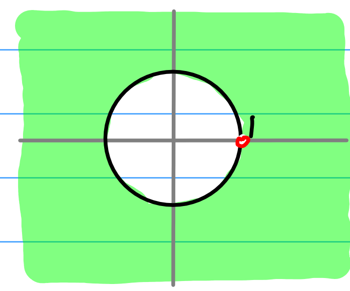
$\xleftrightarrow{z^{-1}}$

$X(z)$:

$$a_n = (1)^n \quad (n \geq 0)$$

$$X(z) = \frac{1}{1-\frac{1}{z}} \qquad |z| > 1$$

$$= \frac{z}{z-1}$$



$\updownarrow p^{-1}$ $\updownarrow p^{-1}$

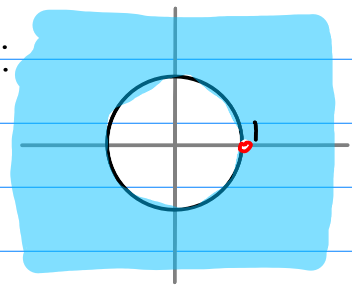
$(1^{-1})^n$

Anti-causal $(n<0)$ $(1)^n$, $(1^{-1})^n$

	$f(z)$		$X(z)$
$(1)^n$	$a_n = (1)^n \quad (n<0)$		$a_n = (1)^n \quad (n<0)$

$f(z) = \dfrac{\frac{1}{z}}{1-\frac{1}{z}} \quad |z| > 1$

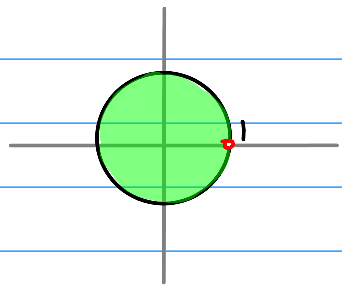
$= -\dfrac{1}{1-z}$



$\xleftrightarrow{z^{-1}}$

$X(z) = \dfrac{z}{1-z} \quad |z| < 1$

$= -\dfrac{z}{z-1}$



$\updownarrow p^{-1}$ $\updownarrow p^{-1}$

$(1^{-1})^n$

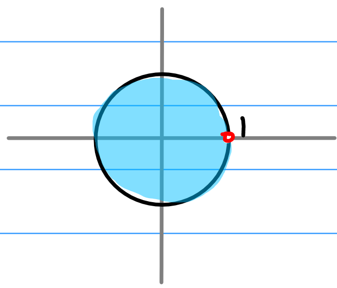
$f(z)$ Causal $(n \geq 0)$ Anti-causal $(n < 0)$

$(1)^n$

$a_n = (1)^n \quad (n \geq 0)$

$$f(z) = \frac{1}{1-z} \qquad |z| < 1$$

$$= \frac{z^{-1}}{z^{-1}-1} = \frac{1}{1-z}$$

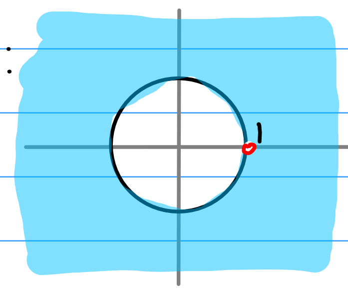


$\xleftrightarrow{-1}$

$a_n = (1)^n \quad (n < 0)$

$$f(z) = \frac{\frac{1}{z}}{1-\frac{1}{z}} \qquad |z| > 1$$

$$= -\frac{1}{1-z}$$



$\updownarrow p^{-1}$

$\updownarrow p^{-1}$

$(1^{-1})^n$

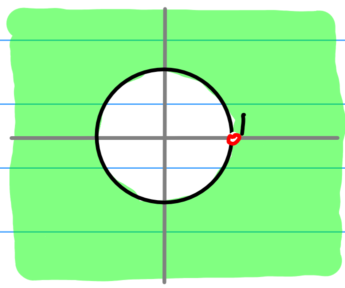
$X(z)$ Causal $(n \geq 0)$ Anti-causal $(n < 0)$

$(1)^n$

$a_n = (1)^n \quad (n \geq 0)$

$X(z) = \dfrac{1}{1 - \frac{1}{z}} \qquad |z| > 1$

$= \dfrac{z}{z - 1}$

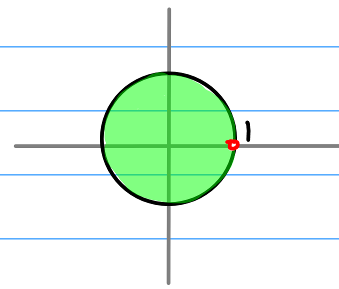


$\xleftrightarrow{-1}$

$a_n = (1)^n \quad (n < 0)$

$X(z) = \dfrac{z}{1 - z} \qquad |z| < 1$

$= - \dfrac{z}{z - 1}$



$\updownarrow p^{-1}$ $\qquad$ $\updownarrow p^{-1}$

$(1^{-1})^n$

Causal b^n | Anti-causal b^n

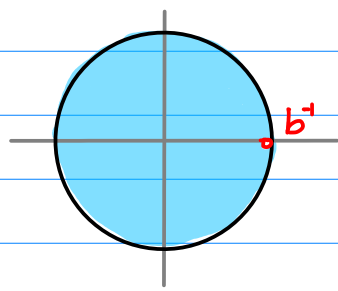
$f(z)$

Causal

$$a_n = (b)^n \quad (n \geq 0)$$

$$f(z) = \frac{1}{1-bz} \qquad |z| < b^{-1}$$

$$= \frac{b^{-1}}{b^{-1}-z}$$

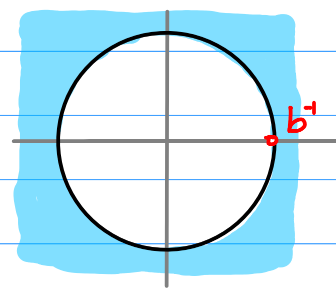


Anti-causal

$$a_n = (b)^n \quad (n < 0)$$

$$f(z) = \frac{b^{-1}z^{-1}}{1-b^{-1}z^{-1}} \qquad |z| > b^{-1}$$

$$= -\frac{b^{-1}}{b^{-1}-z}$$



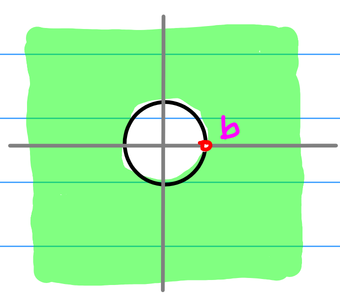
$X(z)$

Causal

$$a_n = (b)^n \quad (n \geq 0)$$

$$X(z) = \frac{1}{1-bz^{-1}} \qquad |z| > b$$

$$= \frac{z}{z-b}$$



Anti-causal

$$a_n = (b)^n \quad (n < 0)$$

$$X(z) = \frac{b^{-1}z}{1-b^{-1}z} \qquad |z| < b$$

$$= -\frac{z}{z-b}$$