

Characteristics of Multiple Random Variables

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June 24, 2019

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Based on
Probability, Random Variables and Random Signal Principles,
P.Z. Peebles,Jr. and B. Shi

Outline

1 Joint Gaussian Random Variables

Bivariate Gaussian Density

two random variables

Definition

The two random variables X and Y are said to be jointly Gaussian, if their joint density function is

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \cdot$$

$$\exp \left\{ \frac{-1}{2(1-\rho^2)} \cdot \left[\frac{(x-\bar{X})^2}{\sigma_X^2} - \frac{2\rho(x-\bar{X})(y-\bar{Y})}{\sigma_X\sigma_Y} + \frac{(y-\bar{Y})^2}{\sigma_Y^2} \right] \right\}$$

$$\bar{X} = E[X], \quad \bar{Y} = E[Y], \quad \sigma_X^2 = E[(X-\bar{X})^2], \quad \sigma_Y^2 = E[(Y-\bar{Y})^2],$$

$$\rho = E[(X-\bar{X})(Y-\bar{Y})]/\sigma_X\sigma_Y$$

Bivariate Gaussian Density - Maximum value

two random variables

$$f_{X,Y}(x,y) \leq f_{X,Y}(\bar{X}, \bar{Y}) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}}.$$

Bivariate Gaussian Density - Uncorrelated

two random variables

$f_{X,Y}(x,y) = f_X(x)f_Y(x)$ is sufficient to guarantee that X and Y are statistically independent. Any uncorrelated Gaussian random variables are also statistically independent a coordinate rotation (linear transformation of X and Y) through the angle

$$\theta = \frac{1}{2} \tan^{-1} \left[\frac{2\rho\sigma_X\sigma_Y}{\sigma_X^2 - \sigma_Y^2} \right]$$

is sufficient to convert correlated random variables X and Y having σ_X^2 and σ_Y^2 , respectively, correlation coefficient ρ , and the joint density of $f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \cdot \exp[\dots]$ into two statistically independent Gaussian random variables

Multi-variate Gaussian Density

N random variables

N random variables X_1, X_2, \dots, X_N are called jointly Gaussian if their joint density function can be written as

$$f_{X_1, \dots, X_N}(x_1, \dots, x_N) = \frac{|[C_X]^{-1}|^{1/2}}{(2\pi)^{N/2}} \exp \left\{ -\frac{[x - \bar{X}]^t [C_X] [x - \bar{X}]}{2} \right\}$$

where

$$[x - \bar{X}] = \begin{bmatrix} x_1 - \bar{X}_1 \\ x_2 - \bar{X}_2 \\ \vdots \\ x_N - \bar{X}_N \end{bmatrix}, \quad [C_X] = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1N} \\ C_{21} & C_{22} & \cdots & C_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ C_{N1} & C_{N2} & \cdots & C_{NN} \end{bmatrix}$$

Multi-variate Gaussian Density - notations

 N random variables

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where $[\bullet]^t$ denotes a matrix transposition,

$[\bullet]^{-1}$ denotes a matrix inversion

$|\bullet|$ denotes a matrix determinant

Covariance Matrix

N random variables

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$$f_{X_1, \dots, X_N}(x_1, \dots, x_N) = \frac{|[C_X]^{-1}|^{1/2}}{(2\pi)^{N/2}} \exp \left\{ -\frac{[x - \bar{X}]^t [C_X] [x - \bar{X}]}{2} \right\}$$

where $[C_X]$ is called the covariance matrix of N random variables

$$C_{ij} = E[(X_i - \bar{X}_i)(X_j - \bar{X}_j)] = \begin{cases} \sigma_{X_i}^2 & i = j \\ C_{X_i X_j} & i \neq j \end{cases}$$

Covariance Matrix ($N = 2$) N random variables

$$f_{X_1 X_2}(x_1, x_2) = \frac{|[C_X]^{-1}|^{1/2}}{(2\pi)^{2/2}} \exp \left\{ -\frac{[x - \bar{X}]^t [C_X] [x - \bar{X}]}{2} \right\}$$

$$[C_X] = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} \sigma_{X_1}^2 & \rho \sigma_{X_1} \sigma_{X_2} \\ \rho \sigma_{X_1} \sigma_{X_2} & \sigma_{X_2}^2 \end{bmatrix}$$

$$[C_X]^{-1} = \frac{1}{1 - \rho^2} \begin{bmatrix} \sigma_{X_1}^2 & -\rho / \sigma_{X_1} \sigma_{X_2} \\ -\rho / \sigma_{X_1} \sigma_{X_2} & \sigma_{X_2}^2 \end{bmatrix}$$

$$|[C_X]^{-1}| = 1 / \sigma_{X_1}^2 \sigma_{X_2}^2 (1 - \rho^2)$$

