
Z Transform (H.1)

Definition

20170130

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Based on
Complex Analysis for Mathematics and Engineering
J. Mathews

z - Transform

$$X(z) = \sum_{k=-\infty}^{+\infty} x[k] z^{-k}$$

$$z = |r| e^{j2\pi F}$$
$$= |r| e^{j\Omega}$$

$$x[n] \longleftrightarrow X(z)$$

OneSided z-transform

$$X(z) = \sum_{k=0}^{+\infty} x[k] z^{-k}$$

Inverse z-Transform

$$X(z) = \mathcal{Z}[\{x_n\}_{n=0}^{\infty}]$$

$$= \sum_{n=0}^{\infty} x_n z^{-n}$$

$$= \sum_{n=0}^{\infty} x[n] z^{-n}$$

$$x[n] \longrightarrow X(z)$$

$$x_n = x[n]$$

$$= \mathcal{Z}^{-1}[X(z)]$$

$$= \frac{1}{2\pi i} \int_{\mathcal{C}} X(z) z^{n+1} dz$$

$$x[n] \longleftarrow X(z)$$

Admissible Form of z -transform

$$X(z) = \sum_{n=0}^{\infty} x[n] z^{-n}$$

$X(z)$: admissible z -transform

if $X(z)$ is a rational function

$$X(z) = \frac{P(z)}{Q(z)} = \frac{b_0 + b_1 z^1 + b_2 z^2 + \dots + b_{p-1} z^{p-1} + b_p z^p}{a_0 + a_1 z^1 + a_2 z^2 + \dots + a_{q-1} z^{q-1} + a_q z^q}$$

$P(z)$: a polynomial of degree p

$Q(z)$: a polynomial of degree q

Residue Theorem

D : Simply connected domain

C : Simple closed contour (CCW) in D

if $f(z)$ is **analytic** inside C and on C
except at the points z_1, z_2, \dots, z_k in C

then

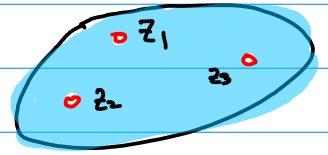
$$\frac{1}{2\pi i} \int_C f(z) dz = \sum_{j=1}^k \text{Res}(f(z), z_j)$$

Singular points of $f(z)$: z_1, z_2, \dots, z_k

Integration of a function of a complex var.

$$\oint_C f(z) dz = 2\pi i \sum_{k=1}^n \text{Res}(f(z), z_k)$$

finite number k of
singular points z_k
residue theorem



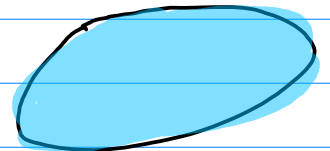
$$\oint_C f(z) dz = 0 \quad \text{if } f(z) = F'(z) \text{ on } C$$

: $F(z)$ is an antiderivative of $f(z)$

fundamental theorem of calculus

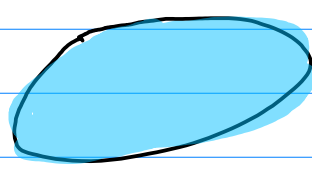
$$\oint_C f(z) dz = 0 \quad \text{if } f(z) \text{ is analytic within and on } C$$

no singularity



$\oint_C f(z) dz = 0$ if $f(z)$ is continuous in D and
 $f(z) = F'(z)$: $F(z)$ is an antiderivative of $f(z)$
fundamental theorem of calculus

$\oint_C f(z) dz = 0$ if $f(z)$ is analytic within and on C
no singularity



can expand $f(z)$ about any point z_m
over powers of $(z - z_m)$

whether or not $f(z)$ is singular at z_m
or at other points between z and z_m

$$f(z) = \sum_{n=n_1}^{\infty} a_n^{(m)} (z - z_m)^n$$

① Laurent Series Expansion of $f(z)$ at z_m
general n_1 - depend on $f(z)$ and z_m

② z -transform of $a_n^{(m)}$
general n_1 - depend on $f(z)$

$$z_m = 0$$

③ Taylor Series Expansion of $f(z)$ at z_m
positive n_1 - depend on $f(z)$ and z_m ($n_1 > 0$)

④ MacLaurin Series Expansion of $f(z)$ at z_m
positive n_1 - depend on $f(z)$ ($n_1 > 0$)

$$z_m = 0$$

* Expansion of $f(z)$ about any point z_m
over powers of $(z - z_m)$

$$f(z) = \sum_{n=n_1}^{\infty} a_n^{(m)} (z - z_m)^n$$

$$a_n^{(m)} = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_m)^{n+1}} dz$$
$$a_n^{(m)} = \sum_k \operatorname{Res} \left(\frac{f(z)}{(z - z_m)^{n+1}}, z_k \right)$$

for general $f(z)$

for general $f(z)$

$$a_n^{(m)} = \frac{1}{n!} f^{(n)}(z_m) \quad n_1 \geq 0$$

for analytic $f(z)$ within C

analytic $f(z) \longrightarrow \frac{f(z)}{(z - z_m)^{n+1}}$ has a pole at z_m
order of $n+1$

$$f(z) = \sum_{n=n_1}^{\infty} a_n^{(m)} (z - z_m)^n$$

z_m : possible poles of $f(z)$
not necessarily poles

$$a_n^{(m)} = \frac{1}{2\pi i} \oint_C \frac{f(z')}{(z' - z_m)^{n+1}} dz'$$
$$= \sum_k \text{Res} \left(\frac{f(z)}{(z - z_m)^{n+1}}, z_k \right)$$

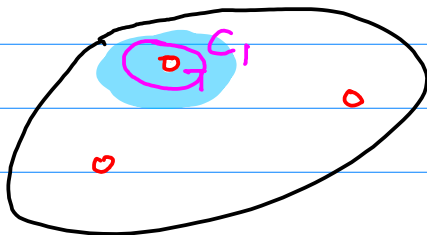
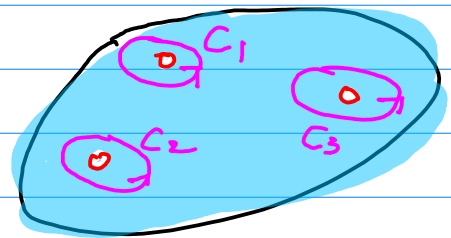
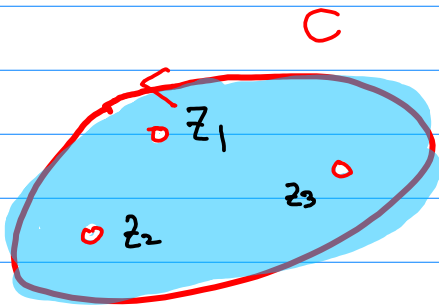
z_k : poles of $\frac{f(z)}{(z - z_m)^{n+1}}$
within C

$$= \frac{1}{n!} f^{(n)}(z_m) \quad n_1 \geq 0$$

Residue Theorem

assumed there are (m) singularities (poles) of $f(z)$ in a region

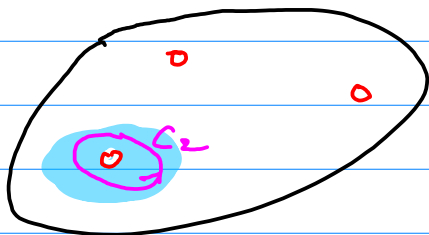
C_m is taken to enclose only one pole z_m



$a_n^{(1)}$ expanded at z_1

C_1 encloses z_1 only

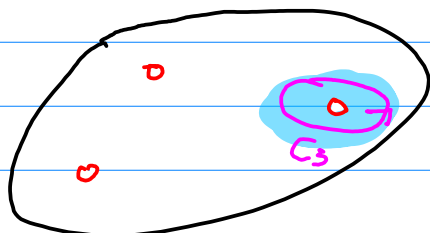
$$\tilde{a}_{-1}^{(1)} = \text{Res}(f(z), z_1)$$



$a_n^{(2)}$ expanded at z_2

C_2 encloses z_2 only

$$\tilde{a}_{-1}^{(2)} = \text{Res}(f(z), z_2)$$



$a_n^{(3)}$ expanded at z_3

C_3 encloses z_3 only

$$\tilde{a}_{-1}^{(3)} = \text{Res}(f(z), z_3)$$

$$\eta = -1$$

$$a_n^{\{m\}} = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_m)^{n+1}} dz$$
$$= \sum_k \operatorname{Res} \left(\frac{f(z)}{(z-z_m)^{n+1}}, z_k \right)$$

$$a_{-1}^{\{m\}} = \frac{1}{2\pi i} \oint_C f(z) dz$$
$$= \sum_k \operatorname{Res} (f(z), z_k)$$

$$a_{-1}^{\{m\}} = \frac{1}{2\pi i} \oint_C f(z) dz = \sum_k \operatorname{Res} (f(z), z_k)$$

if C encloses only one pole

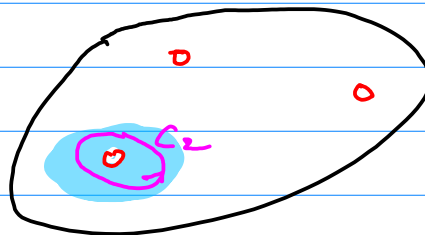
$$a_{-1}^{\{0\}} = \frac{1}{2\pi i} \oint_{C_0} f(z) dz = \operatorname{Res} (f(z), z_0)$$

$$\tilde{a}_{-1}^{\{m\}} = \operatorname{Res} (f(z), z_m)$$

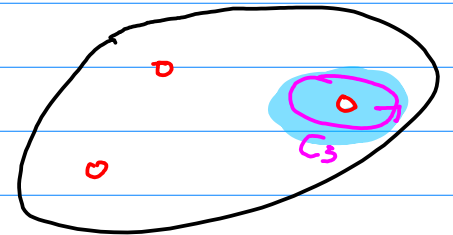
the residue of $f(z)$ at z_m using C_m



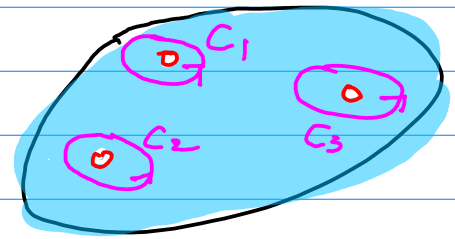
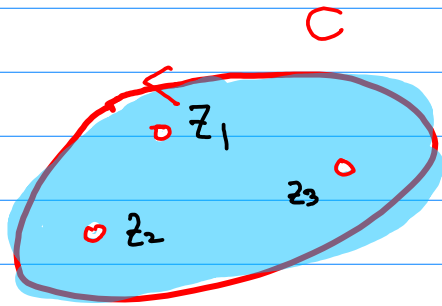
$$\tilde{a}_{-1}^{\{1\}} = \operatorname{Res} (f(z), z_1)$$



$$\tilde{a}_{-1}^{\{2\}} = \operatorname{Res} (f(z), z_2)$$



$$\tilde{a}_{-1}^{\{3\}} = \operatorname{Res} (f(z), z_3)$$



$$\oint_C f(z) dz = 2\pi j \sum_{k=1}^M \tilde{a}_{-1}^{(k)} = 2\pi j \sum_{k=1}^M \text{Res}(f(z), z_k)$$

residue theorem

$$a_n = \sum_{k=1}^M \text{Res} \left(\frac{f(z)}{(z-z_k)^{n+1}}, z_k \right)$$

Laurent coefficient

C encloses k poles

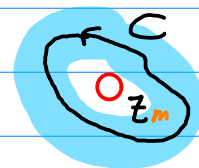
C_k encloses only the k -th pole

$\tilde{a}_{-1}^{(k)}$ the residue of the k -th pole enclosed by C , z_k

$$f(z) = \sum_{n=n_1}^{\infty} a_n (z - z_m)^n$$

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z')}{(z' - z_m)^{n+1}} dz'$$

$$= \sum_k \text{Res} \left(\frac{f(z)}{(z - z_m)^{n+1}}, z_k \right)$$



C is in the same region of analyticity of $f(z)$
typically a circle centered on z_m

z_k within C : singularities of $\frac{f(z)}{(z - z_m)^{n+1}}$

$n_1 = n_{f,m}$ depends on $f(z)$, z_m

a_n depends on $f(z)$, z_m , region of analyticity

Whether $f(z)$ is singular at $z = z_m$ or not
or at other points between z and z_m

We can expand $f(z)$ about any point z_m
over powers of $(z - z_m)$.

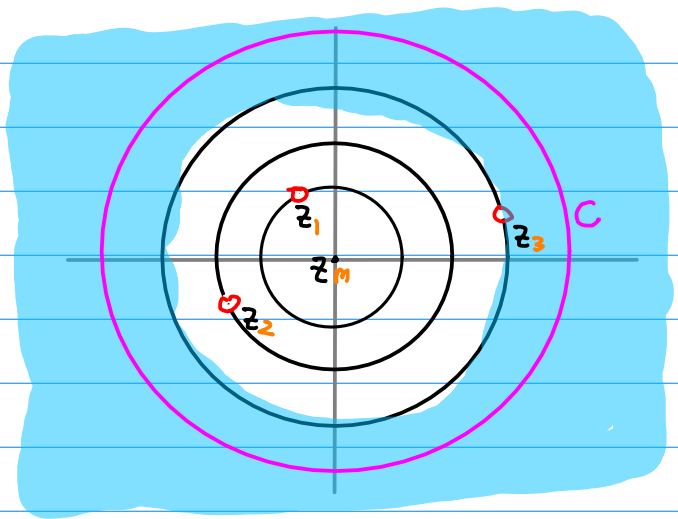
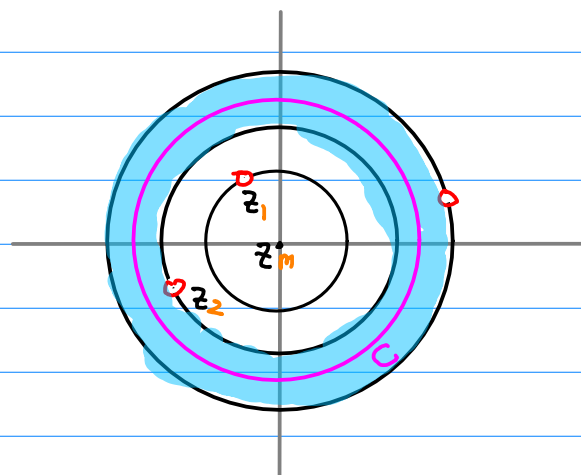
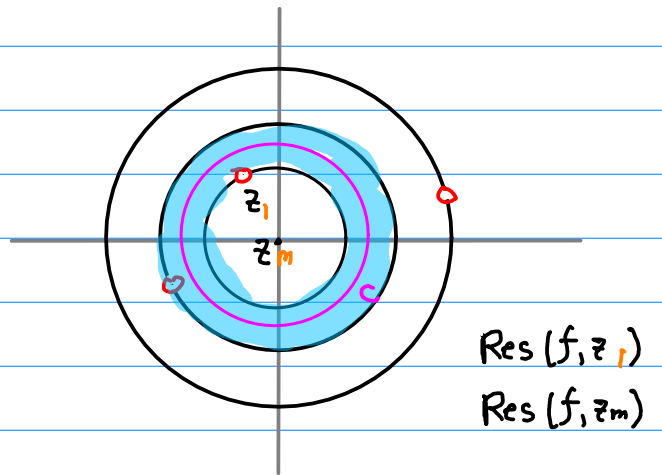
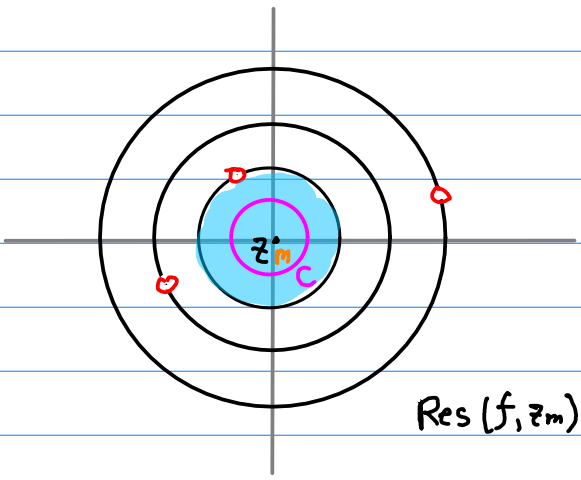
$$f(z) = \sum_{n=n_1}^{\infty} a_n (z - z_m)^n$$

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z')}{(z' - z_m)^{n+1}} dz'$$

$$= \sum_k \operatorname{Res} \left(\frac{f(z)}{(z - z_m)^{n+1}}, z_k \right)$$

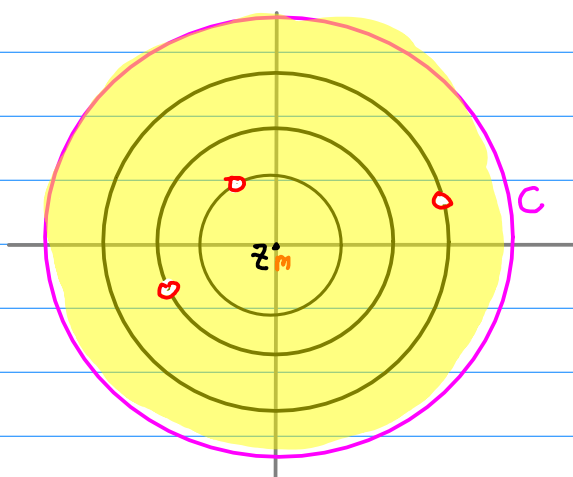
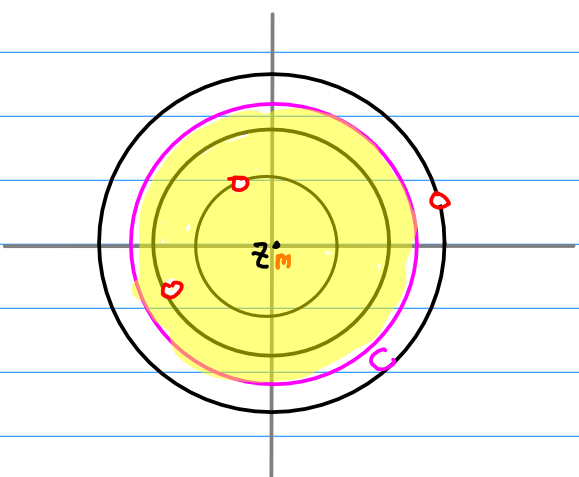
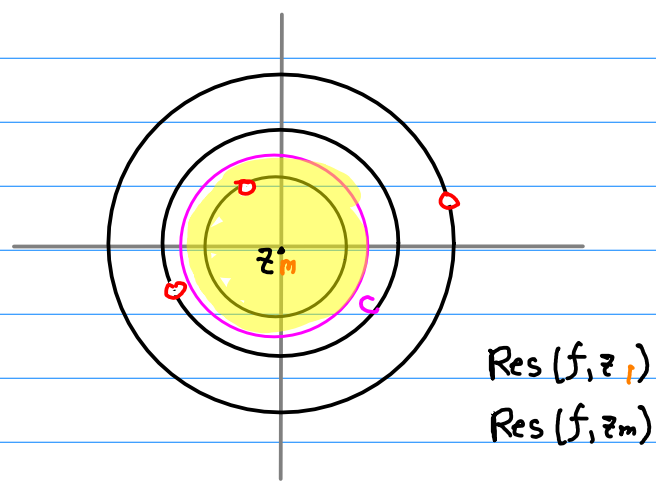
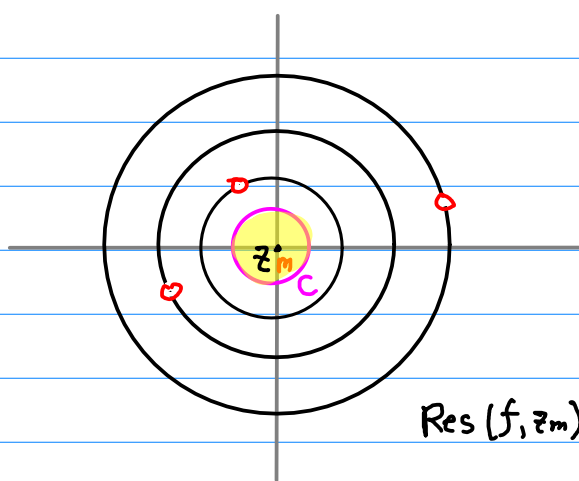
z_k within C : singularities of $\frac{f(z)}{(z - z_m)^{n+1}}$

$$\begin{cases} \text{poles of } f(z) \cup z = z_m & n \geq 0 \\ \text{poles of } f(z) & n < 0 \end{cases}$$



$$\text{Res}(f, z_1) + \text{Res}(f, z_2) + \text{Res}(f, z_m)$$

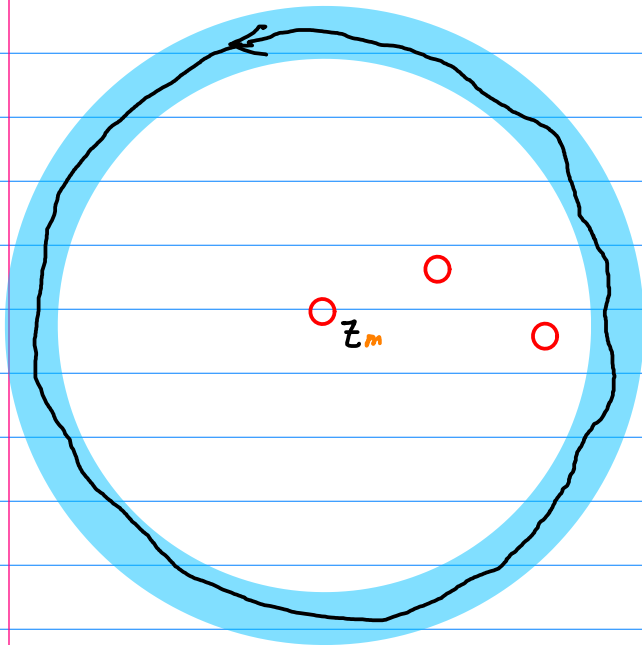
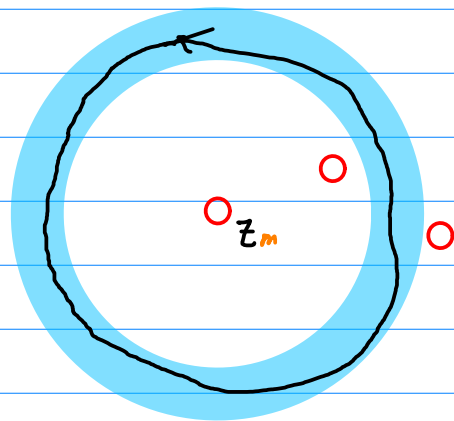
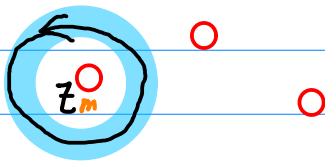
$$\text{Res}(f, z_1) + \text{Res}(f, z_2) + \text{Res}(f, z_3) + \text{Res}(f, z_m)$$



$$f(z) = \sum_{n=-\infty}^{\infty} a_n^{(m)} (z - z_m)^n$$

$$a_n^{(m)} = \frac{1}{2\pi i} \oint_C \frac{f(z')}{(z' - z_m)^{n+1}} dz'$$

$$= \sum_k \operatorname{Res} \left(\frac{f(z)}{(z - z_m)^{n+1}}, z_k \right)$$



Laurent's Theorem

f : analytic within the annular domain D

$$r < |z - z_0| < R$$

then

$$f(z) = \sum_{k=-\infty}^{+\infty} a_k (z - z_0)^k, \text{ valid for } r < |z - z_0| < R$$

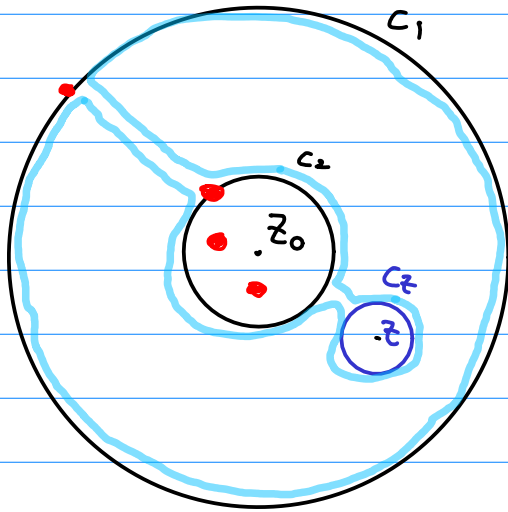
$$a_k = \frac{1}{2\pi i} \oint_C \frac{f(s)}{(s - z_0)^{k+1}} ds, \quad k = 0, \pm 1, \pm 2, \dots$$

C : a simple closed curve
that lies entirely within D
that encloses z_0

$$a_{-1} = \frac{1}{2\pi i} \oint_C f(s) ds \quad \rightarrow \quad \oint_C f(s) ds = 2\pi i \cdot a_{-1}$$

$$a_{-1} = \frac{1}{2\pi i} \oint_C f(s) ds = \text{Res}(f(z), z_0)$$

$$= \begin{cases} \lim_{z \rightarrow z_0} (z - z_0) f(z) & \text{(simple)} \\ \frac{1}{(n-1)!} \lim_{z \rightarrow z_0} \frac{d^{n-1}}{dz^{n-1}} (z - z_0)^n f(z) & \text{(order } n) \end{cases}$$



z_0 : expansion point

z : evaluation point

which poles of $f(z)$ lie between the point of evaluation z and the point z_0 about which the expansion is formed

$\frac{f(z')}{(z' - z_0)}$ is analytic between C_1 & C_2

deformation theorem

$C_1 - C_2$ coincide

common contour \curvearrowright

Cauchy's Residue Theorem

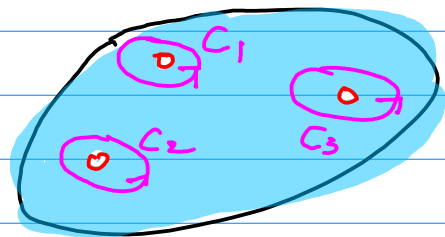
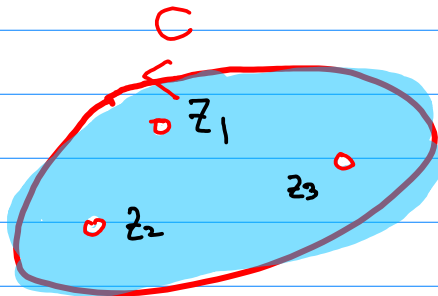
$f(z)$: **analytic** on and within C
except a finite number of **singular points**
 z_1, z_2, \dots, z_n within C

then

$$\int_C f(z) dz = 2\pi i \sum_{k=1}^n \text{Res}(f(z), z_k)$$

D : a simply connected domain

C : a simple closed contour in D



z_1

$$f(z) = \sum_{k=-\infty}^{+\infty} a_k (z - z_1)^k$$

$$a_{-1}^{z_1} = \frac{1}{2\pi i} \oint_{C_1} f(s) ds = \text{Res}(f(z), z_1)$$

z_2

$$f(z) = \sum_{k=-\infty}^{+\infty} a_k (z - z_2)^k$$

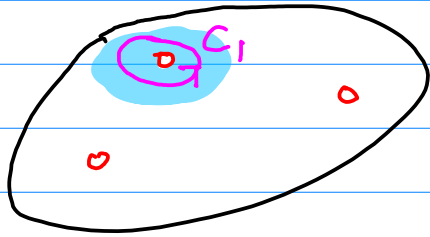
$$a_{-1}^{z_2} = \frac{1}{2\pi i} \oint_{C_2} f(s) ds = \text{Res}(f(z), z_2)$$

z_3

$$f(z) = \sum_{k=-\infty}^{+\infty} a_k (z - z_3)^k$$

$$a_{-1}^{z_3} = \frac{1}{2\pi i} \oint_{C_3} f(s) ds = \text{Res}(f(z), z_3)$$

z_1

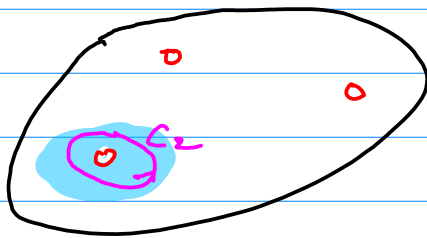


Laurent series expansion at z_1

$$f(z) = \sum_{k=-\infty}^{+\infty} a_k (z - z_1)^k$$

$$a_{-1} = \frac{1}{2\pi i} \oint_{C_1} f(s) ds = \text{Res}(f(z), z_1)$$

z_2

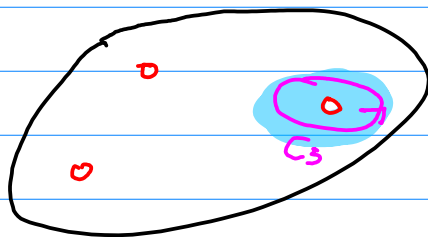


Laurent series expansion at z_2

$$f(z) = \sum_{k=-\infty}^{+\infty} a_k (z - z_2)^k$$

$$a_{-1} = \frac{1}{2\pi i} \oint_{C_2} f(s) ds = \text{Res}(f(z), z_2)$$

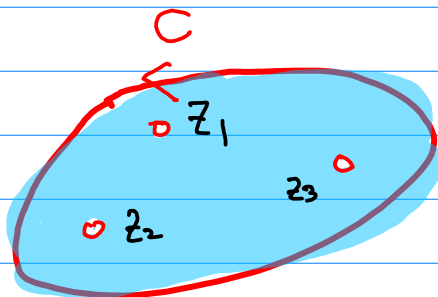
z_3



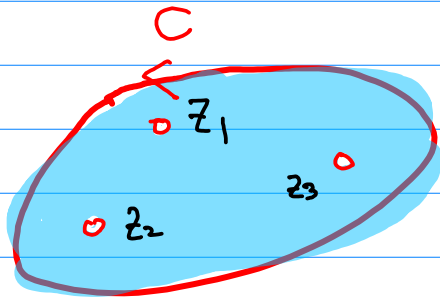
Laurent series expansion at z_3

$$f(z) = \sum_{k=-\infty}^{+\infty} a_k (z - z_3)^k$$

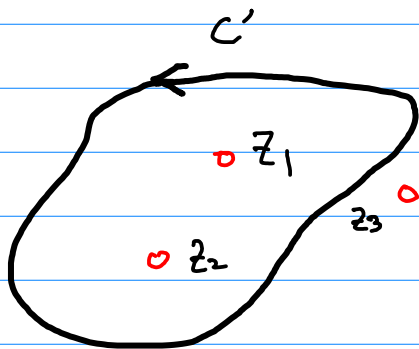
$$a_{-1} = \frac{1}{2\pi i} \oint_{C_3} f(s) ds = \text{Res}(f(z), z_3)$$



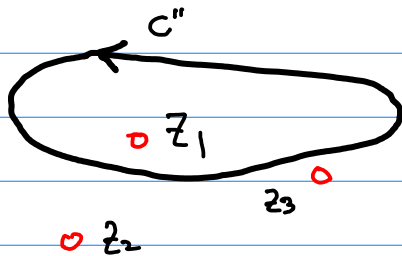
$$\int_C f(z) dz = 2\pi i \sum_{k=1}^n \text{Res}(f(z), z_k)$$



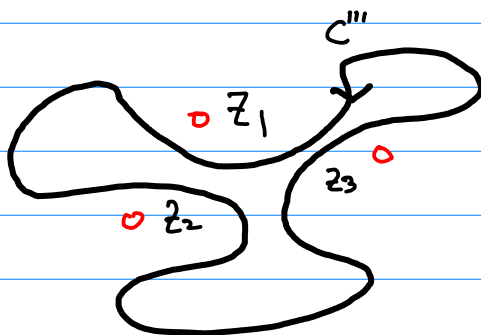
$$\int_C f(z) dz = 2\pi i \operatorname{Res}(f(z), z_1) + 2\pi i \operatorname{Res}(f(z), z_2) + 2\pi i \operatorname{Res}(f(z), z_3)$$



$$\int_{C'} f(z) dz = 2\pi i \operatorname{Res}(f(z), z_1) + 2\pi i \operatorname{Res}(f(z), z_2)$$

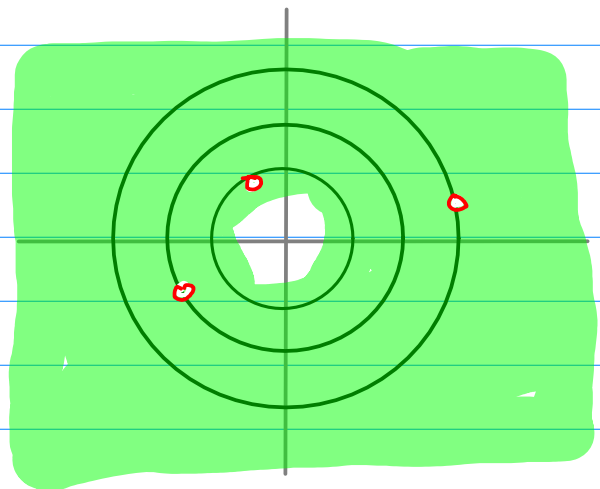
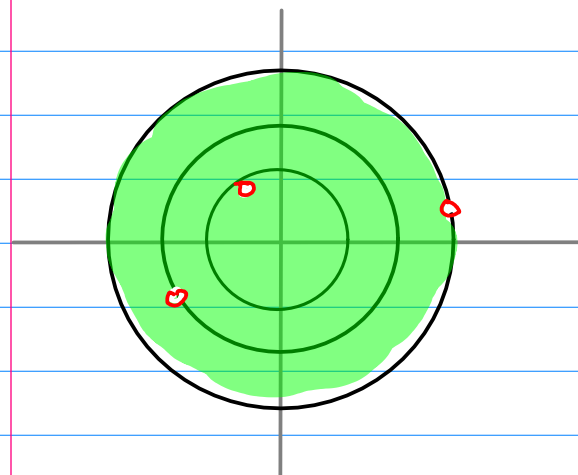
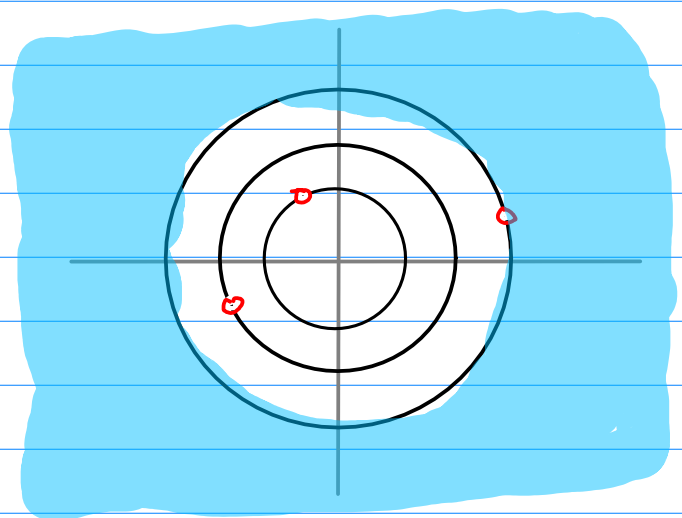
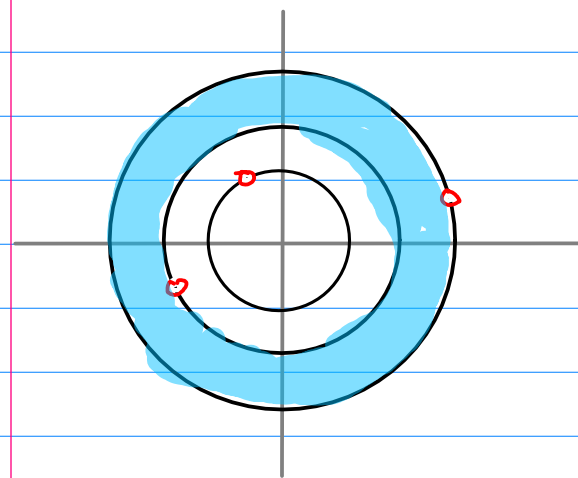
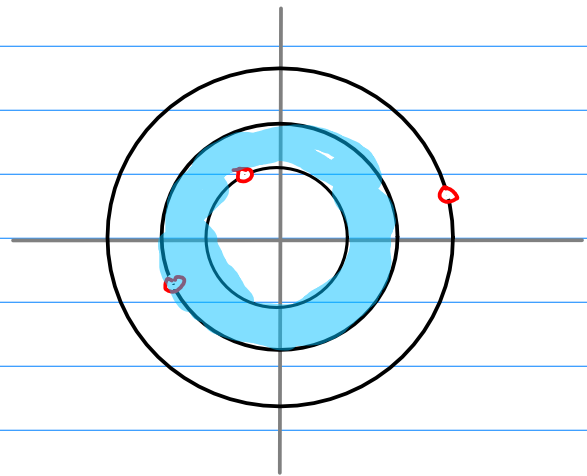
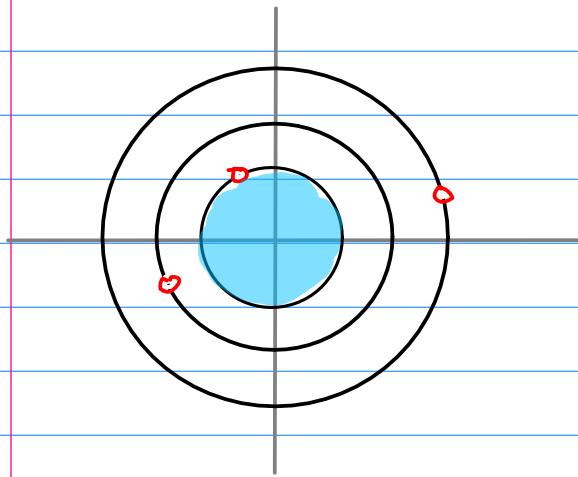


$$\int_{C''} f(z) dz = 2\pi i \operatorname{Res}(f(z), z_1)$$



$$\int_{C'''} f(z) dz = 0$$

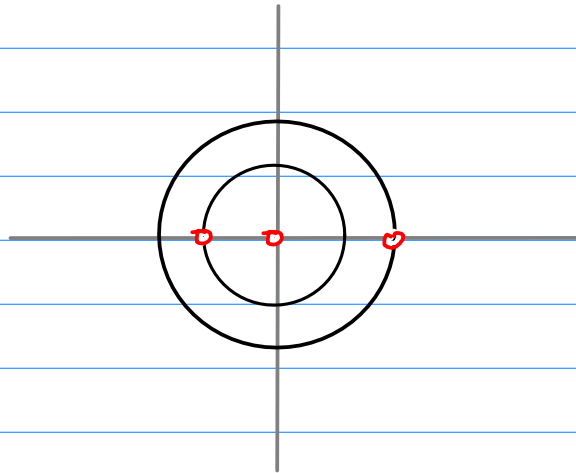
Different D , Different Laurent Series



z -transform

$$f(z) = \frac{12}{z(2-z)(1+z)} = \frac{4}{z} \left(\frac{1}{1+z} + \frac{1}{2-z} \right)$$

pole: $z=0$, $z=2$, $z=-1$



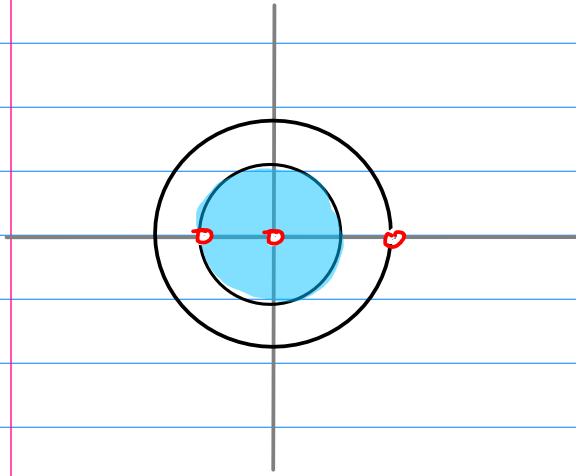
$$0 < |z| < 1$$

$$f(z) = -3 + 9z/2 - 15z^2/4 + 33z^3/8 + \dots + 6/z$$

$$|z| > 2$$

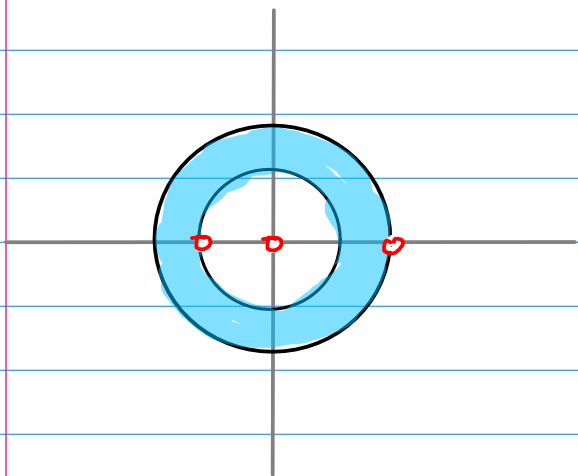
$$\frac{1}{1+z} = \frac{1}{z} \frac{1}{(1+z^{-1})} \quad \frac{1}{2+z} = -\frac{1}{z} \frac{1}{1-2z^{-1}}$$

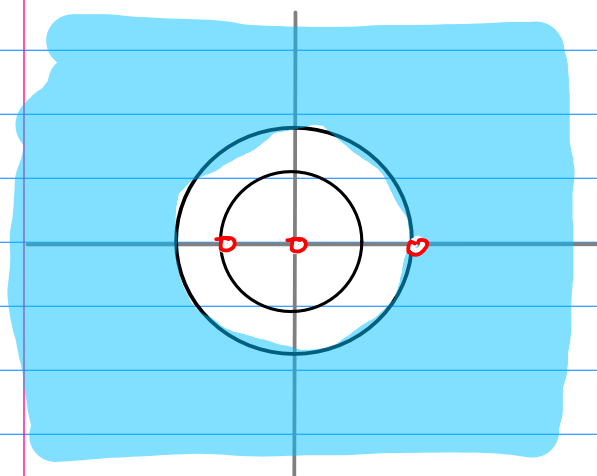
$$f(z) = -(12/z^3) (1 + 1/z + 3/z^2 + 5/z^3 + 11/z^4 + \dots)$$



$$0 < |z| < 1$$

$$f(z) = -3 + 9z/2 - 15z^2/4 + 33z^3/8 + \dots + 6/z$$





$$|z| > 2$$

$$\frac{1}{1+z} = \frac{1}{z} \frac{1}{(1+z^{-1})}$$

$$\frac{1}{2+z} = -\frac{1}{z} \frac{1}{1-2z^{-1}}$$

$$f(z) = \frac{-1}{(z-1)(z-2)}$$

Complex Variables and Ap
Brown & Churchill

$$f(z) = \frac{-1}{(z-1)(z-2)} = \frac{1}{z-1} - \frac{1}{z-2}$$

$$D_1: |z| < 1$$

$$D_2: 1 < |z| < 2$$

$$D_3: 2 < |z|$$

① D_1 $|z| < 1, \quad \left|\frac{z}{2}\right| < 1$

$$\begin{aligned} f(z) &= \frac{1}{z-1} - \frac{1}{z-2} = \frac{-1}{1-z} + \frac{1}{1-\left(\frac{z}{2}\right)} \\ &= -\sum_{n=0}^{\infty} z^n + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} = \sum_{n=0}^{\infty} (2^{-n-1} - 1)z^n \quad |z| < 1 \end{aligned}$$

② D_2 $1 < |z| < 2 \Rightarrow \left|\frac{1}{z}\right| < 1, \quad \left|\frac{z}{2}\right| < 1$

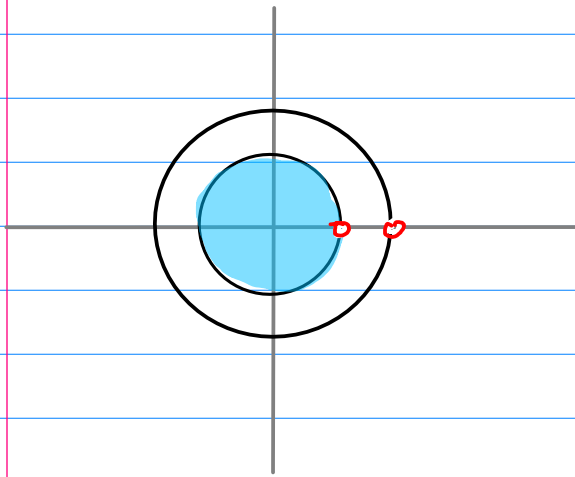
$$\begin{aligned} f(z) &= \frac{1}{z-1} - \frac{1}{z-2} = \frac{1}{z} \cdot \frac{1}{1-\left(\frac{1}{z}\right)} + \frac{1}{2} \frac{1}{1-\left(\frac{z}{2}\right)} \\ &= \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} \\ &= \sum_{n=1}^{\infty} \frac{1}{z^n} + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} \end{aligned}$$

③ D_3 $2 < |z| \quad \left|\frac{z}{2}\right| < 1 \quad \left|\frac{1}{z}\right| < 1$

$$\begin{aligned} f(z) &= \frac{1}{z-1} - \frac{1}{z-2} = \frac{1}{z} \frac{1}{1-\left(\frac{1}{z}\right)} - \frac{1}{z} \frac{1}{1-\left(\frac{z}{2}\right)} \\ &= \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} - \sum_{n=0}^{\infty} \frac{z^n}{z^{n+1}} = \sum_{n=0}^{\infty} \frac{1-2^n}{z^{n+1}} \\ &= \sum_{n=1}^{\infty} \frac{1-2^{n-1}}{z^n} \end{aligned}$$

$$f(z) = \frac{-1}{(z-1)(z-2)}$$

① $D_1 \quad |z| < 1, \quad \left|\frac{z}{2}\right| < 1$



$$\frac{f(z)}{z^{n+1}} = \frac{1}{(z-1)(z-2)z^{n+1}}$$

$$f(z) = \frac{1}{z-1} - \frac{1}{z-2} = \frac{-1}{1-z} + \frac{1}{1-\left(\frac{z}{2}\right)}$$

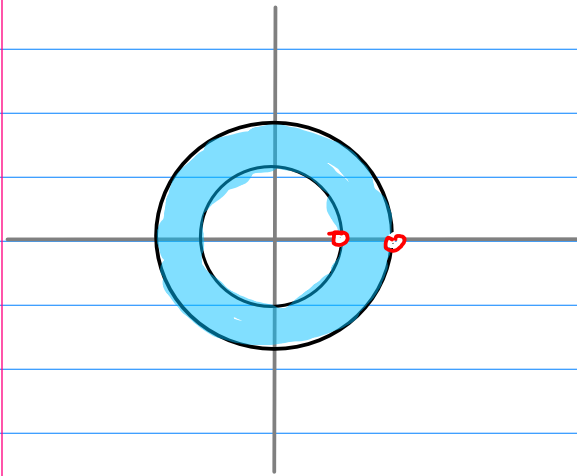
$$= -\sum_{n=0}^{\infty} z^n + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} = \sum_{n=0}^{\infty} (2^{-n-1} - 1)z^n \quad |z| < 1$$

$$a_n = \frac{f(z)}{z^{n+1}} = \frac{1}{(z-1)(z-2)z^{n+1}}$$

$$a_n = \sum_{k=1}^M \text{Res} \left(\frac{f(z)}{(z-z_k)^{n+1}}, z_k \right)$$

$$f(z) = \frac{-1}{(z-1)(z-2)}$$

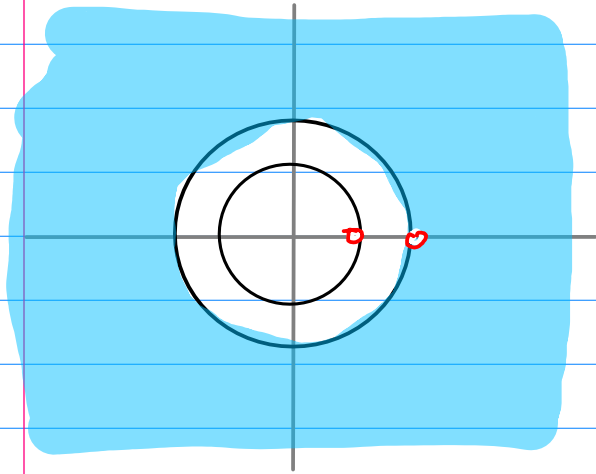
$$\textcircled{2} D_2 \quad 1 < |z| < 2 \Rightarrow \left| \frac{1}{z} \right| < 1, \quad \left| \frac{z}{2} \right| < 1$$



$$\begin{aligned} f(z) &= \frac{1}{z-1} - \frac{1}{z-2} = \frac{1}{z} \cdot \frac{1}{1-\left(\frac{1}{z}\right)} + \frac{1}{2} \frac{1}{1-\left(\frac{z}{2}\right)} \\ &= \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} \\ &= \sum_{n=1}^{\infty} \frac{1}{z^n} + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} \end{aligned}$$

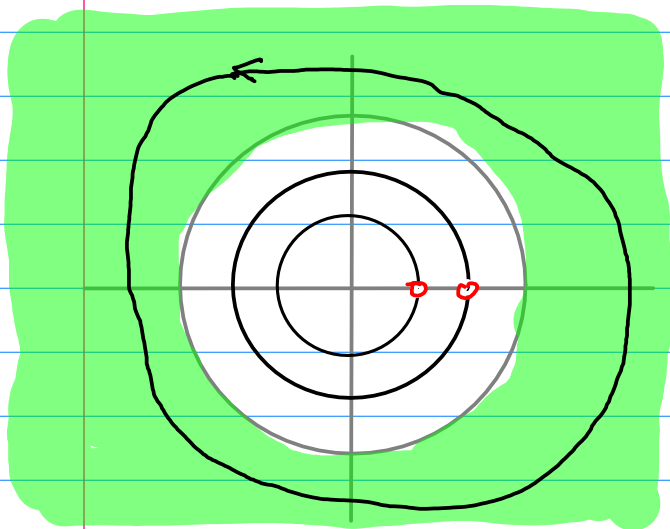
$$f(z) = \frac{-1}{(z-1)(z-2)}$$

③ D_3 $2 < |z|$ $|\frac{z}{2}| < 1$ $|\frac{1}{z}| < 1$



$$\begin{aligned} f(z) &= \frac{1}{z-1} - \frac{1}{z-2} = \frac{1}{z} \frac{1}{1-(\frac{1}{z})} - \frac{1}{z} \frac{1}{1-(\frac{1}{2z})} \\ &= \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} - \sum_{n=0}^{\infty} \frac{2^n}{z^{n+1}} = \sum_{n=0}^{\infty} \frac{1-2^n}{z^{n+1}} \\ &= \sum_{n=1}^{\infty} \frac{1-2^{n-1}}{z^n} \end{aligned}$$

$$f(z) = \frac{-1}{(z-1)(z-2)}$$



$$x[n]$$

$$= \frac{1}{2\pi i} \int_C \boxed{X(z) z^{n-1}} dz$$

$$= \sum_{j=1}^k \text{Res}(\boxed{X(z) z^{n-1}}, z_j)$$

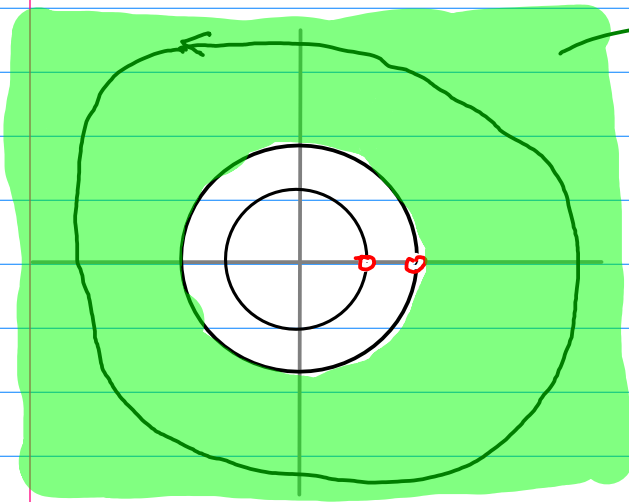
$$X(z) = \frac{-1}{(z-1)(z-2)}$$

$$X(z) z^{n-1} = \frac{-1}{(z-1)(z-2)} z^{n-1}$$

$$\text{Res}(\boxed{X(z) z^{n-1}}, 1) = (z-2) \frac{-1}{(z-1)(z-2)} z^{n-1} \Big|_{z=1} = 1$$

$$\text{Res}(\boxed{X(z) z^{n-1}}, 2) = (z-1) \frac{-1}{(z-1)(z-2)} z^{n-1} \Big|_{z=2} = -2^{n-1}$$

$$x[n] = 1 - 2^{n-1}$$



ROC (Region of Convergence)

$$|z| > 2 \Rightarrow \frac{2}{|z|} < 1$$

$$\left(\frac{2}{z}\right)^0 + \left(\frac{2}{z}\right)^1 + \left(\frac{2}{z}\right)^2 + \dots \longrightarrow \frac{1}{1 - \frac{2}{z}}$$

Converge

$$|z| > 2 \Rightarrow \frac{1}{|z|} < 1$$

$$\left(\frac{1}{z}\right)^0 + \left(\frac{1}{z}\right)^1 + \left(\frac{1}{z}\right)^2 + \dots \longrightarrow \frac{1}{1 - \frac{1}{z}}$$

Converge

$$\begin{aligned} f(z) &= \frac{1}{z-1} - \frac{1}{z-2} = \frac{1}{z} \frac{1}{1 - (\frac{1}{z})} - \frac{1}{z} \frac{1}{1 - (\frac{2}{z})} \\ &= \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} - \sum_{n=0}^{\infty} \frac{2^n}{z^{n+1}} = \sum_{n=0}^{\infty} \frac{1 - 2^n}{z^{n+1}} \\ &= \sum_{n=1}^{\infty} \frac{1 - 2^{n-1}}{z^n} \end{aligned}$$

$$\left(\frac{1}{z}\right) + \left(\frac{1}{z}\right)^2 + \left(\frac{1}{z}\right)^3 + \dots + \frac{1}{2} \left\{ \left(\frac{2}{z}\right) + \left(\frac{2}{z}\right)^2 + \left(\frac{2}{z}\right)^3 + \dots \right\} \longrightarrow \frac{1}{z-1} - \frac{1}{z-2} = \frac{-1}{(z-1)(z-2)}$$

Converge

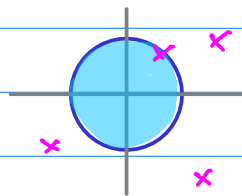
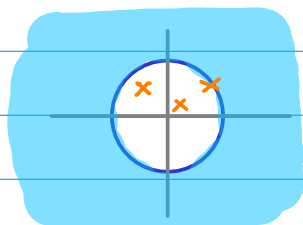
$$(1-2^0)z^{-1} + (1-2^1)z^{-2} + (1-2^2)z^{-3} + \dots \longrightarrow \frac{-1}{(z-1)(z-2)} \quad (|z| > 2)$$

Converge

$$x[n] = 1 - 2^n \quad \longleftrightarrow \quad X(z) = \frac{-1}{(z-1)(z-2)} \quad (|z| > 2)$$

causal $x[n]=0$ ($n < 0$)

anti-causal $x[n]=0$ ($n > 0$)

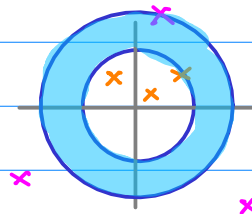


ROC: outside a circle

ROC: inside a circle

bi-causal $x[n]$

Overlapped ROC







$$f(z) = \sum_{n=-\infty}^{\infty} a_n^{(m)} (z - z_m)^n$$

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n \quad z_m = 0 \quad a_n^{(0)} \rightarrow a_n$$

Laurent Series at $z=0$

$$f(z) = \dots + a_{-2} z^{-2} + a_{-1} z^{-1} + a_0 z^0 + a_1 z^1 + a_2 z^2 + a_3 z^3 + \dots$$

z-transform

Bi-causal

$$X(z) = \dots + x[-2] z^2 + x[-1] z^1 + x[0] z^0 + x[1] z^{-1} + x[2] z^{-2} + x[3] z^{-3} + \dots$$

Causal

$$X(z) = x[0] z^0 + x[1] z^{-1} + x[2] z^{-2} + x[3] z^{-3} + \dots$$

Anti-causal

$$X(z) = \dots + x[-2] z^2 + x[-1] z^1 + x[0] z^0$$

$$a_n \leftrightarrow x[-n]$$

$$a_{-n} \leftrightarrow x[n]$$

$$f(z) = \sum_{n=n_1}^{\infty} a_n^{\{m\}} (z - z_m)^n$$

$$a_n^{\{m\}} = \frac{1}{2\pi i} \oint_C \frac{f(z')}{(z' - z_m)^{n+1}} dz'$$

$$= \sum_k \operatorname{Res} \left(\frac{f(z)}{(z - z_m)^{n+1}}, z_k \right)$$

Analytic at z_m

$$n_1 \geq 0$$

Taylor Series

$$\text{general } n_1, \quad z_m = 0$$

MacLaurin Series

Singular at z_m

$$\text{general } n_1$$

Laurent Series

$$\text{general } n_1, \quad z_m = 0$$

z-Transform

$$f(z) = \sum_{n=-\infty}^{\infty} a_n^{(m)} (z - z_m)^n$$

$$\begin{aligned} a_n^{(m)} &= \frac{1}{2\pi i} \oint_c \frac{f(z')}{(z' - z_m)^{n+1}} dz' \\ &= \sum_k \operatorname{Res} \left(\frac{f(z)}{(z - z_m)^{n+1}}, z_k \right) \end{aligned}$$

$$z_m = 0$$

$$a_{-n}^{(0)} = h(n)$$

$$n \rightarrow -n$$

$$H(z) = \sum_{n=-\infty}^{\infty} h(-n) z^n$$

$$\begin{aligned} h(n) &= \frac{1}{2\pi i} \oint_c \frac{H(z')}{z'^{n+1}} dz' \\ &= \sum_k \operatorname{Res} \left(\frac{H(z)}{z^{n+1}}, z_k \right) \end{aligned}$$

$$H(z) = \sum_{n=-\infty}^{\infty} h(n) z^{-n}$$

$$\begin{aligned} h(n) &= \frac{1}{2\pi i} \oint_c H(z') z'^{n-1} dz' \\ &= \sum_k \operatorname{Res} (H(z) z^{n-1}, z_k) \end{aligned}$$

C is in the same region of analyticity of $f(z)$
typically a circle centered on z_m

z_k within C : singularities of $\frac{f(z)}{(z-z_k)^{n+1}}$

C is in the same region of analyticity of $H(z)$
typically a circle centered on z_m

generally a circle centered on the origin
may enclose any or all singularities of $H(z)$
often the unit circle

z_k within C : singularities of $H(z)z^{n-1}$

$$H(z) = \sum_{n=-\infty}^{\infty} h(n) z^{-n} \quad z \in \text{R.O.C}$$

$$h(n) = \frac{1}{2\pi i} \oint_C H(z') z'^{n-1} dz' \quad C \text{ in R.O.C.}$$

$$= \sum_k \text{Res}(H(z) z^{n-1}, z_k)$$

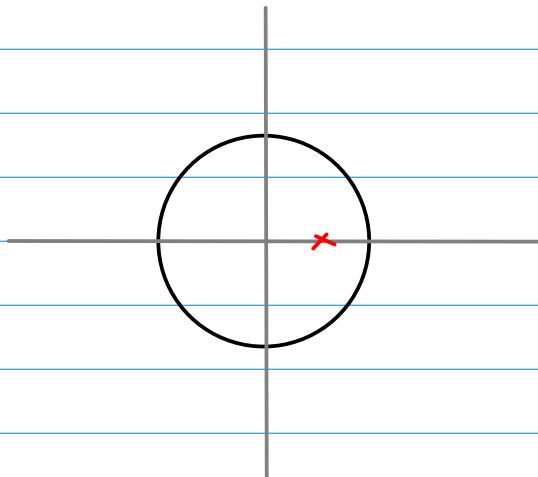
- ① a power series representation of a function $f(z)$ of a complex variable z
- ② a transform $H(z)$ of a sequence of 1

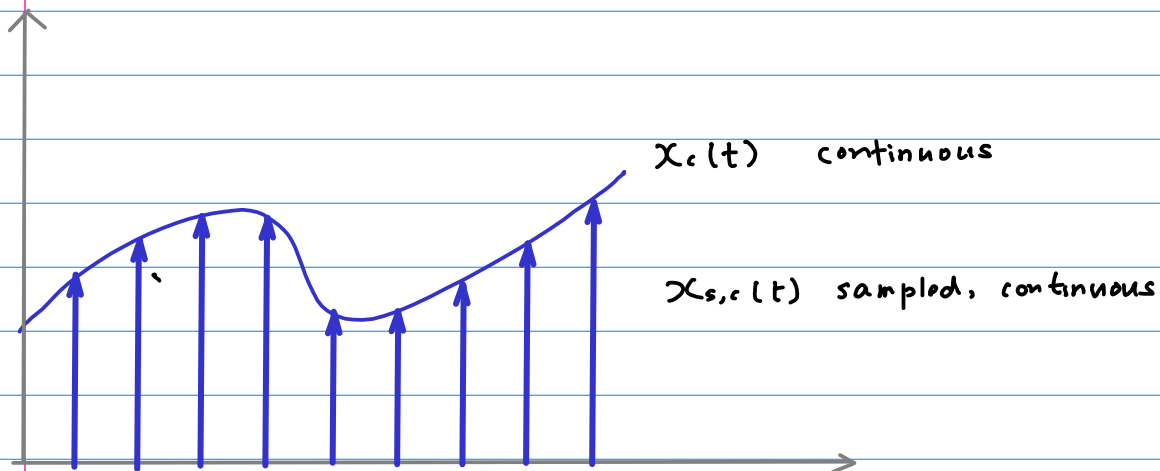
$$X(z) = \frac{z}{z - \frac{1}{2}} \quad \text{pole } z_0 = \frac{1}{2}$$

$$\begin{aligned} x[n] &= \text{Res} \left(X(z) z^{n-1}, z_0 \right) = \text{Res} \left(\frac{z}{z - \frac{1}{2}} z^{n-1}, \frac{1}{2} \right) \\ &= \text{Res} \left(\frac{z^n}{z - \frac{1}{2}}, \frac{1}{2} \right) = \lim_{z \rightarrow \frac{1}{2}} (z - \frac{1}{2}) \frac{z^n}{z - \frac{1}{2}} = \left(\frac{1}{2} \right)^n \end{aligned}$$

$$x[n] = \frac{1}{2^n} \quad n \geq 0$$

$$\begin{aligned} \left(\frac{1}{2} \right)^0 z^0 + \left(\frac{1}{2} \right)^1 z^{-1} + \left(\frac{1}{2} \right)^2 z^{-2} + \left(\frac{1}{2} \right)^3 z^{-3} + \dots &= \frac{1}{1 - \left(\frac{1}{2} z^{-1} \right)} \\ &= \frac{z}{z - \frac{1}{2}} \end{aligned}$$





$$x_{s,c}(t) = \sum_{n=-\infty}^{+\infty} x(n) \delta_c(t - n\Delta t)$$

$$\begin{aligned} X_{s,c}(s) &= \mathcal{L}\{x_{s,c}(t)\} = \int_{-\infty}^{\infty} \left[\sum_{n=-\infty}^{+\infty} x(n) \delta_c(t - n\Delta t) \right] e^{-st} dt \\ &= \sum_{n=-\infty}^{+\infty} x(n) \int_{-\infty}^{\infty} \delta_c(t - n\Delta t) e^{-st} dt \\ &= \sum_{n=-\infty}^{+\infty} x(n) e^{-sn\Delta t} \quad e^{s\Delta t} \triangleq z \end{aligned}$$

$$X_{s,c}(s) = \sum_{n=-\infty}^{+\infty} x(n) z^{-n} \Big|_{z=e^{s\Delta t}}$$

$$X_{s,c}(s) = X(z) \Big|_{z=e^{s\Delta t}}$$

$$X_{s,c}(s) = \mathcal{L}\{x_{s,c}(t)\} = X(z) \Big|_{z=e^{s\Delta t}}$$

$x_{s,c}(t)$ an impulse train

whose coefficients are given by $x[n] = x_c(n\Delta t)$

z-transform : a special Laurent series

$$z_m = 0$$

$$a_{-n}^{\{0\}} = h(n)$$

$$n \rightarrow -n$$

$$f(z) = \sum_{n=-\infty}^{\infty} a_n^{\{m\}} (z - z_m)^n$$

$$a_n^{\{m\}} = \frac{1}{2\pi i} \oint_c \frac{f(z')}{(z' - z_m)^{n+1}} dz'$$

$$= \sum_k \operatorname{Res} \left(\frac{f(z)}{(z - z_m)^{n+1}}, z_k \right)$$

Time Reversal \leftarrow Laplace Transform

the transform functions

$$X(s) = \int \text{over negative powers } e^{-st} \quad \text{for } t > 0$$

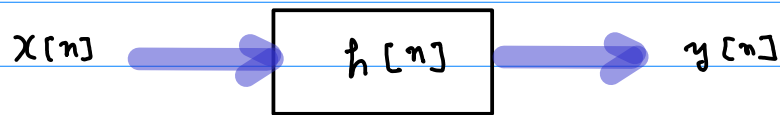
$$X(z) = \int \text{over negative powers } z^{-n} \quad \text{for } n > 0$$

the time expansion functions

$$x(t) = \int \text{over negative powers } e^{-st} \quad \text{for } t > 0$$

$$x[n] = \int \text{over negative powers } z^{-n} \quad \text{for } n > 0$$

Time Reversal \leftarrow z^{-1} : unit delay, char eq (modes in z^k)



Stable system : $h[n]$ must be absolutely summable

$$|e^{j\omega n}| = 1$$

$$|z^n| \quad z = 1$$

$$\infty > M_h > \sum_{n=-\infty}^{\infty} |h[n]| \quad \text{absolutely summable}$$

$$= \sum_{n=-\infty}^{+\infty} |h[n] e^{-j\omega n}|$$

$$\geq \left| \sum_{n=-\infty}^{+\infty} h[n] e^{-j\omega n} \right|$$

$$= \left| H(z) \Big|_{z=e^{j\omega}} \right|$$

$$\infty > \left| H(z) \Big|_{z=e^{j\omega}} \right|$$

a stable system,

$H(z)$ must converge on the unit circle $|z|=1$

ROC (Region of Convergence) must include the unit circle

regardless of causality of $h[n]$

$$H(z) \Big|_{|z|=1} = H(e^{j\hat{\omega}}) \quad \text{DTFT of } h[n]$$

discrete all stable sequence must have convergent DTFTs

continuous all stable signal must have convergent CTFTs

$$C \leftarrow \text{unit circle} \quad z = e^{j\hat{\omega}}$$

ZT⁻¹ DTFT⁻¹ identical formulas

$h[n]$ causal

$$H(z) = \sum_{n=-\infty}^{+\infty} h[n] z^{-n} = \sum_{n=0}^{+\infty} h[n] z^{-n} \quad n \in [0, \infty)$$

for finite values of n ,

each term must be finite as long as $z \neq 0$

For the sum to converge,

$h[n] z^{-n}$ must vanish as $n \rightarrow \infty$

$$|z| > r_h \quad z_h = r_h e^{j\theta}$$

z_h^n is the largest magnitude

geometrically increasing component

$n^m z^n$: the most general term

for impulse responses

$n \rightarrow \infty$ z^n dominant over n^m for finite m

Geometric components — as poles

$$\sum \{ z_0^n u[n] \} = \frac{1}{1 - (\frac{z_0}{z})} = \frac{z}{z - z_0}$$

ROC of a causal sequence $h[n]$

outside the radius of the largest magnitude pole of $H(z)$

ROC of a causal signal $h(t)$

to the right of the rightmost pole of $H_c(s)$

if $h[n]$ is a stable, causal sequence,

the unit circle must be included in the ROC

• Causal $h[n]$

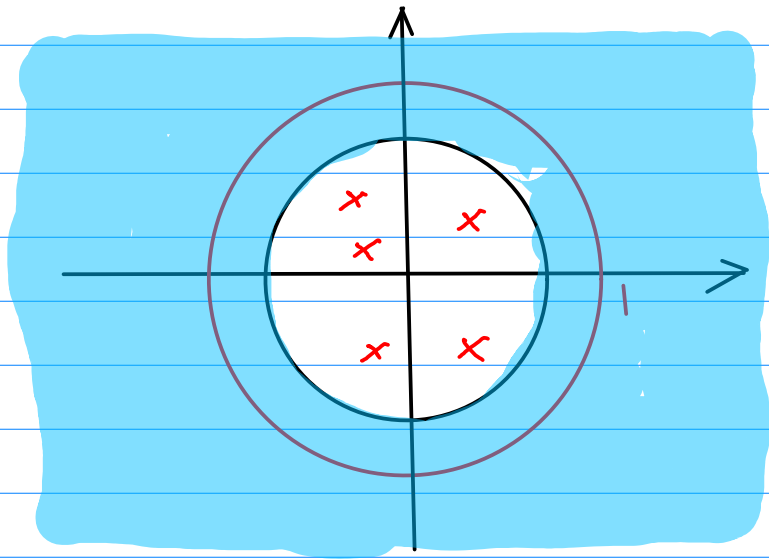
ROC: outside of
a circle

• Stable $h[n]$

all poles inside
the unit circle

ROC circle must be

smaller than the unit circle



⇒ all the geometric components of $h[n]$: modes
must decay with increasing n

all the poles of $H(z)$ must be within the unit circle

all the poles of $H_c(s)$ must be in the left half plane

• anti-causal $h[n]$

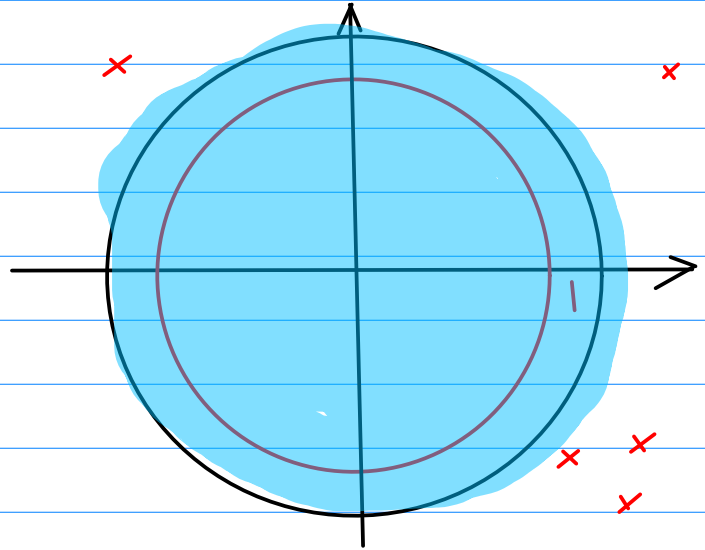
ROC: inside of
a circle

• Stable $h[n]$

all poles outside
the unit circle

ROC circle must be

larger than the unit circle



⇒ all the geometric components of $h[n]$: modes
must decay with decreasing n

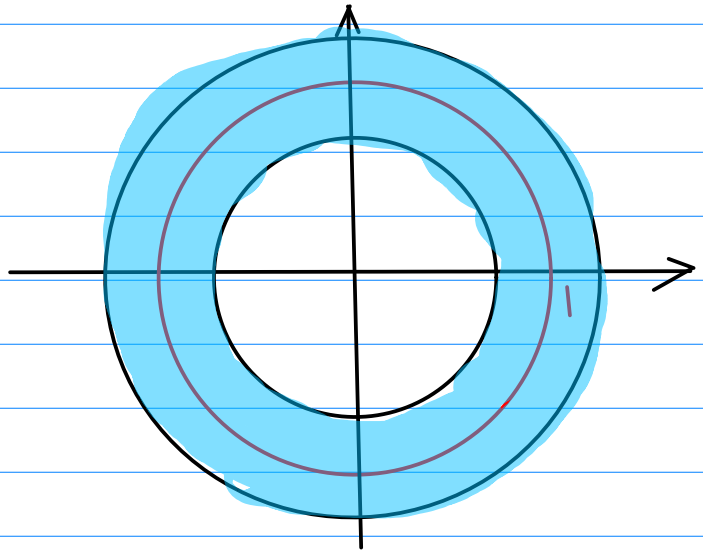
• bi-causal $h[n]$

$h_c[n] + h_{ac}[n]$

outside inside

max mag < min mag

Overlapped ROC



• Stable $h[n]$

all poles outside

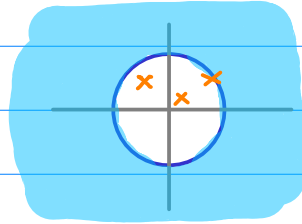
the unit circle

ROC circle must include the unit circle

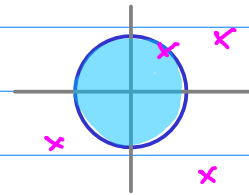
• bi-causal $h[n]$

$$h[n] = h_c[n] + h_{ac}[n]$$

causal comp. anti-causal comp



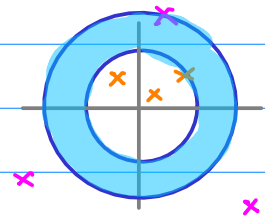
outside a circle



inside a circle

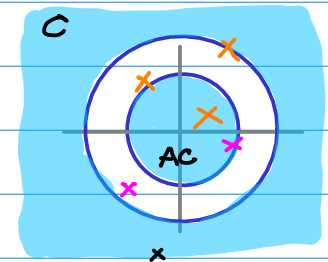
$$\text{max mag} < \text{min mag}$$

Overlapped ROC



$$\text{max mag} > \text{min mag}$$

non-overlapping ROC



• Stable $h[n]$

all poles outside the large circle
inside the small circle

ROC circle must include the unit circle

only one annulus include the unit circle

only one stable sequence

Existence of the z-Transform

$$X(z) = \sum_{n=0}^{\infty} x[n]z^{-n} = \sum_{n=0}^{\infty} \frac{x[n]}{z^n}$$

the existence of the z-transform is guaranteed if

$$|X(z)| \leq \sum_{n=0}^{\infty} \frac{|x[n]|}{|z|^n} < \infty \quad \text{for some } |z|$$

any signal $x[n]$ that grows no faster than an exponential signal r_0^n , for some r_0 satisfies the above condition

if $|x[n]| \leq r_0^n$ for some r_0

$$\text{then } |X(z)| \leq \sum_{n=0}^{\infty} \left(\frac{r_0}{|z|}\right)^n = \frac{1}{1 - \frac{r_0}{|z|}} \quad |z| > r_0$$

therefore $X(z)$ exists for $|z| > r_0$

Almost all practical signals satisfy this condition

$$|x[n]| \leq r_0^n \quad \text{for some } r_0$$

and z-transformable

Some signal models (e.g. r^{n^2}) grow faster than the exponential signal r_0^n (for any r_0) and do not satisfy this condition and are not z-transformable

Such signals are of little practical or theoretical interest. Even such signals over a finite interval are z-transformable.

Region of Convergence

Laplace Transform	$Ae^{\alpha t}u(t)$	$\alpha > 0$
z - Transform	$A\alpha^n u[n]$	$ \alpha > 0$
DTFT (X)		

$$X(z) = A \sum_{n=-\infty}^{\infty} \alpha^n u[n] z^{-n} = A \sum_{n=0}^{\infty} \alpha^n z^{-n} = A \sum_{n=0}^{\infty} \left(\frac{\alpha}{z}\right)^n$$

converge $\left|\frac{\alpha}{z}\right| < 1$ $|z| > |\alpha|$
open exterior of
a circle of radius $|\alpha|$

the sum of a geometric series

$$X(z) = A \frac{1}{1 - \frac{\alpha}{z}} = \frac{A}{1 - \alpha z^{-1}} = A \frac{z}{z - \alpha} \quad |z| > |\alpha|$$

DTFT

$$X(j\hat{\omega}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\hat{\omega}n}$$

DTFT

DTFT of the unit sequence $u[n]$

$$X(e^{-j\hat{\omega}n}) = \sum_{n=-\infty}^{+\infty} u[n] e^{-j\hat{\omega}n} = \sum_{n=0}^{\infty} e^{-j\hat{\omega}n}$$

not converge

$$\hat{\omega} = 0 \quad \sum_{n=0}^{\infty} 1^n \quad \text{diverge}$$

$$\hat{\omega} = \pi \quad \sum_{n=0}^{\infty} (-1)^n \quad \text{oscillates}$$

$$\hat{\omega} = \frac{\pi}{2} \quad \sum_{n=0}^{\infty} (j)^n$$

The DTFTs of some commonly used functions do not exist in the strict sense.

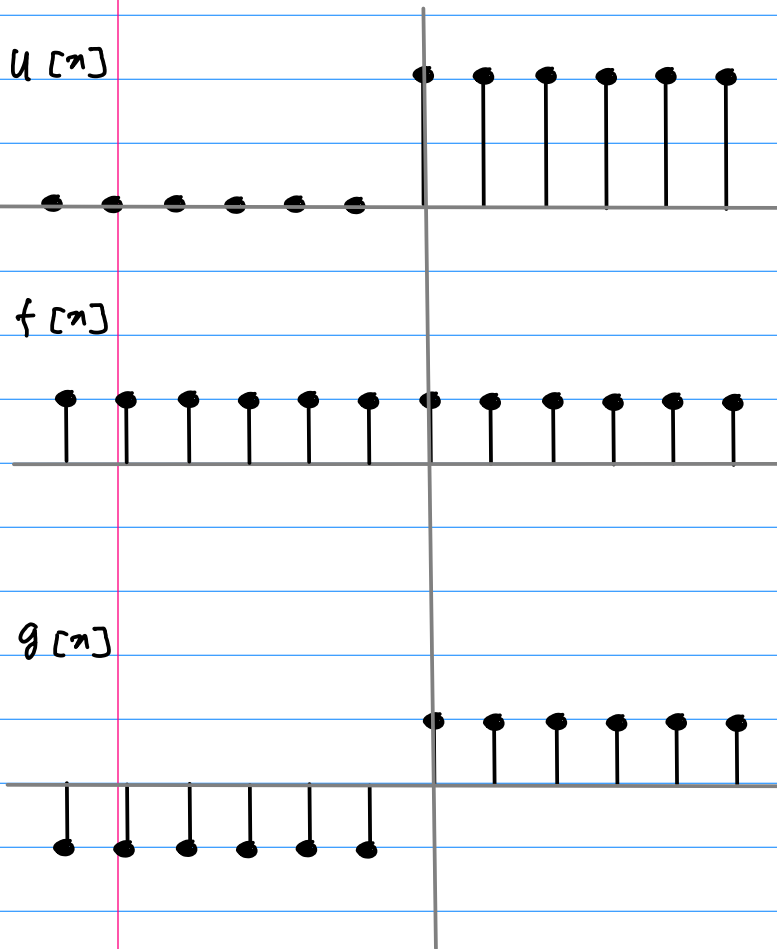
But even though the DTFT does not exist, the z -transform does exist.

$$X(z) = \sum_{n=-\infty}^{+\infty} u[n] z^{-n} = \sum_{n=0}^{\infty} z^{-n}$$

$$|z| > 1 \quad X(z) = \frac{z}{z-1} = \frac{1}{1-z^{-1}}$$

$$X(z) = \frac{z}{z-1} \quad \text{pole } z=1, \quad \text{zero } z=0$$

$$X(z) = \frac{1}{1-z^{-1}} \quad \text{useful when a system is synthesized from a } z\text{-domain transfer function}$$



$$f[n] = \frac{1}{2} \quad -\infty < n < \infty$$

$$g[n] = \begin{cases} \frac{1}{2} & n \geq 0 \\ -\frac{1}{2} & n < 0 \end{cases}$$

$$u[n] = f[n] + g[n]$$

$$\delta[n] = g[n] - g[n-1]$$

$$1 = G(e^{j\omega}) - e^{-j\omega} G(e^{j\omega})$$

$$G(e^{j\omega}) = \frac{1}{1 - e^{-j\omega}}$$

$$F(e^{j\omega}) = \pi \sum_{k=-\infty}^{+\infty} \delta(\omega - 2\pi k) \quad (\text{impulse train})$$

$$U(e^{j\omega}) = \frac{1}{1 - e^{-j\omega}} + \pi \sum_{k=-\infty}^{+\infty} \delta(\omega - 2\pi k)$$

Discrete Time Exponential r^n

continuous time exponential $e^{\lambda t}$

$$\begin{aligned}e^{\lambda t} &= r^t & (e^{\lambda})^t &= r^t \\e^{\lambda} &= r \\ \lambda &= \ln r\end{aligned}$$

$$\begin{aligned}e^{-0.03t} &= (0.9408)^t \\ 4^t &= e^{1.386t}\end{aligned}$$

continuous time analysis $e^{\lambda t}$

discrete time analysis r^n

$$\begin{aligned}e^{\lambda n} &= r^n & (e^{\lambda})^n &= r^n \\e^{\lambda} &= r \\ \lambda &= \ln r\end{aligned}$$

$e^{\lambda n}$

Exponentially grows if $\text{Re } \lambda > 0$ (λ in RHP)

exponentially decays if $\text{Re } \lambda < 0$ (λ in LHP)

Oscillates or constant if $\text{Re } \lambda = 0$ (λ in imag axis)

the location of λ in the complex plain indicates whether

① $e^{\lambda t}$ will grow exponentially

② $e^{\lambda t}$ will decay exponentially

③ $e^{\lambda t}$ will oscillates with constant amplitude

constant signal : oscillation with zero frequency

$e^{j\Omega n}$ $\lambda = j\Omega$ imaginary axis

(constant amplitude oscillating signal)

$$e^{j\Omega n} = (e^{j\Omega})^n = \gamma^n \quad \gamma = e^{j\Omega} \quad |\gamma| = 1$$

$\lambda = j\Omega$ imaginary axis $\rightarrow |\gamma| = 1$ unit circle

if γ lies on the unit circle,

γ^n oscillates with constant amplitude

the imaginary axis in the λ plane

the unit circle in the γ plane

$e^{\lambda n}$ $\lambda = a + jb$ in the LHP ($a < 0$)
exponentially decaying

$$r = e^{\lambda} = e^{a+jb} = e^a e^{jb}$$

$$|r| = |e^{\lambda}| = |e^a \cdot e^{jb}| = |e^a| = e^a$$

$|r| = e^a < 1$ inside the Unit circle

r^n : exponentially decaying

$|r| = e^a > 1$ outside the Unit circle

r^n : exponentially growing

λ -plane

the imaginary axis

the LHP

the RHP



z -plane

the unit circle

inside of the unit circle

outside of the unit circle









