

Laurent Series and z-Transform Examples case 5.A

20171212

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relative values

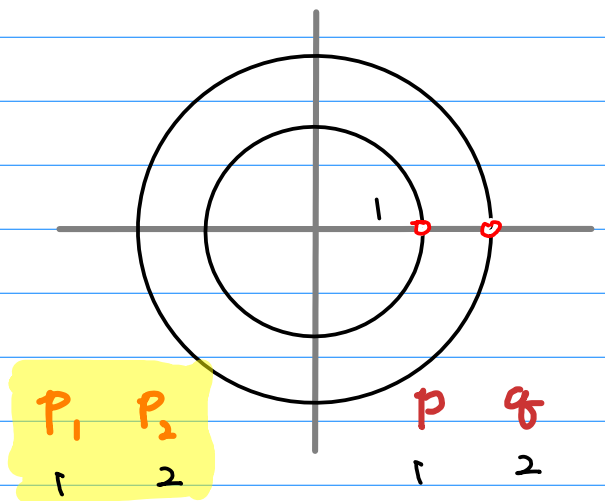
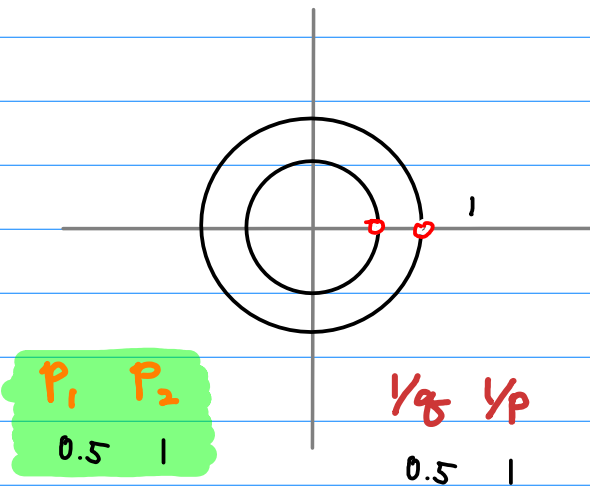
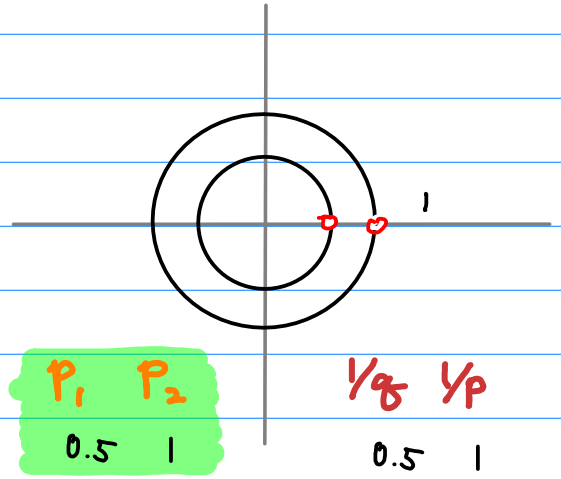
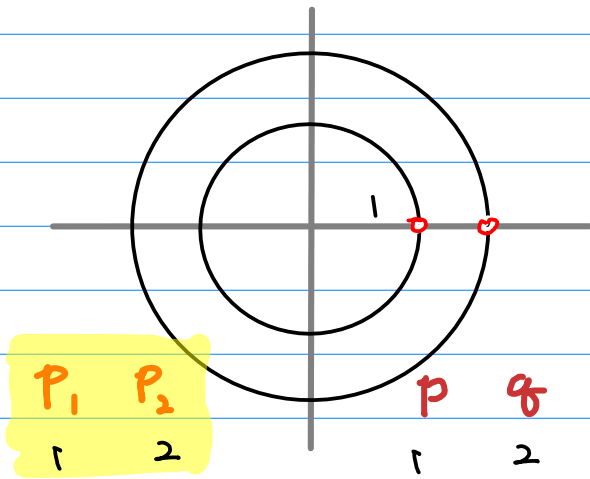
$$p_1 < p_2$$

fixed values

$$p = 1 \quad q = 2$$

$f(z)$

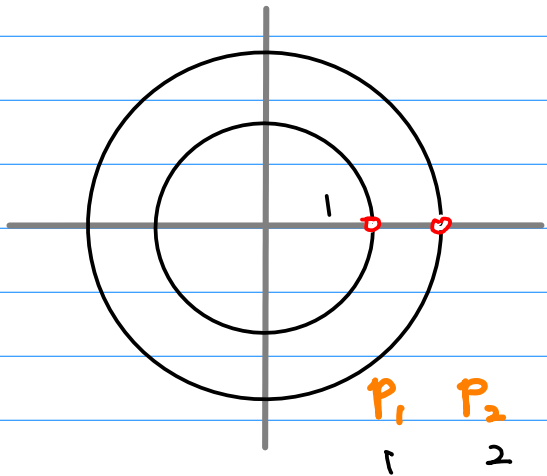
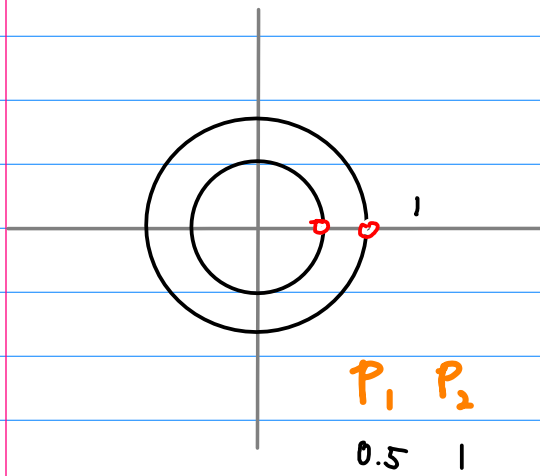
$X(z)$



$f(z)$

$p_1 < p_2$

$p_1 \leq p_2 \leq 1$	2^{n-1}	$1 \leq p_1 \leq p_2$	$(\frac{1}{2})^{n+1}$	
$\begin{cases} (\frac{1}{p_2})^{n-1} - (\frac{1}{p_1})^{n-1} & (n > 0) \\ 0 & (n \leq 0) \end{cases}$		$\begin{cases} (\frac{1}{p_2})^{n+1} - (\frac{1}{p_1})^{n+1} & (n \geq 0) \\ 0 & (n < 0) \end{cases}$		$ z < p_1$
$\begin{cases} 0 & (n > 0) \\ (\frac{1}{p_1})^{n-1} - (\frac{1}{p_2})^{n-1} & (n \leq 0) \end{cases}$		$\begin{cases} 0 & (n \geq 0) \\ (\frac{1}{p_1})^{n+1} - (\frac{1}{p_2})^{n+1} & (n < 0) \end{cases}$		$ z > p_2$
$\begin{cases} (\frac{1}{p_2})^{n-1} & (n > 0) \\ (\frac{1}{p_1})^{n-1} & (n \leq 0) \end{cases}$		$\begin{cases} (\frac{1}{p_2})^{n+1} & (n \geq 0) \\ (\frac{1}{p_1})^{n+1} & (n < 0) \end{cases}$		$p_1 < z < p_2$



2^{n-1}

$\frac{1}{p_1}$	$\frac{1}{p_2}$
2	1

$\frac{1}{p_1}$	$\frac{1}{p_2}$
1	0.5

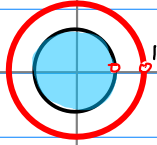
$(\frac{1}{2})^{n+1}$

$$P_1 \leq P_2 \leq 1$$

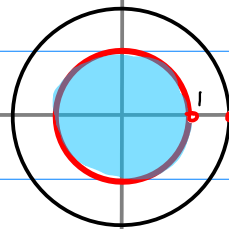
$$2^{n-1}$$

$$1 \leq P_1 \leq P_2$$

$$\left(\frac{1}{2}\right)^{n+1}$$

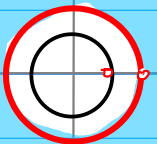


$$n > 0$$

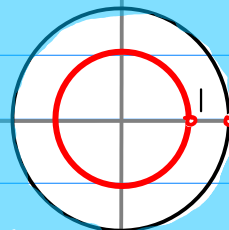


$$n \geq 0$$

$$|z| < P_1$$

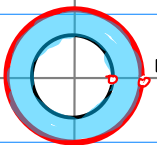


$$n \leq 0$$



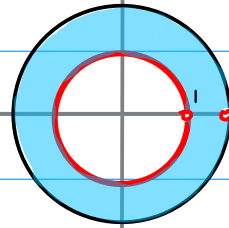
$$n < 0$$

$$|z| > P_2$$



$$n > 0$$

$$n \leq 0$$



$$n \geq 0$$

$$n < 0$$

$$P_1 < |z| < P_2$$

Unit Circle \in ROC \longrightarrow \leq , \geq

$X(z)$ $p_1 < p_2$ $p_1 \leq p_2 \leq 1$ $(\frac{1}{2})^{n+1}$ $1 \leq p_1 \leq p_2$ 2^{n-1}

0	$(n \geq 0)$
$(p_1)^{n+1} - (p_2)^{n+1}$	$(n < 0)$

0	$(n > 0)$
$(p_1)^{n-1} - (p_2)^{n-1}$	$(n \leq 0)$

 $|z| < p_1$

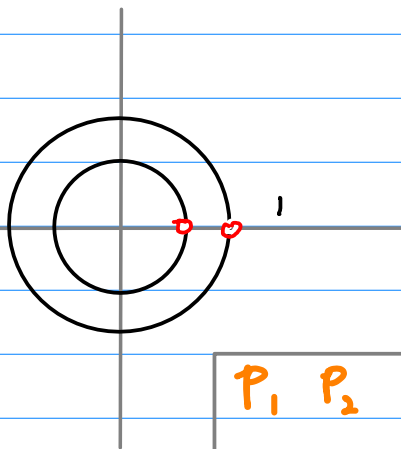
$(p_2)^{n+1} - (p_1)^{n+1}$	$(n \geq 0)$
0	$(n < 0)$

$(p_2)^{n-1} - (p_1)^{n-1}$	$(n > 0)$
0	$(n \leq 0)$

 $|z| > p_2$

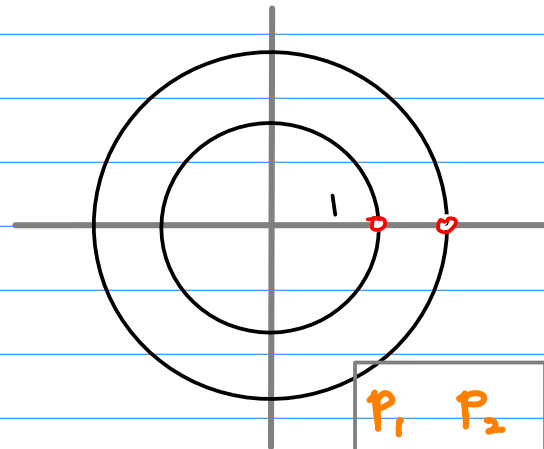
$(p_2)^{n+1}$	$(n \geq 0)$
$(p_1)^{n+1}$	$(n < 0)$

$(p_2)^{n-1}$	$(n > 0)$
$(p_1)^{n-1}$	$(n \leq 0)$

 $p_1 < |z| < p_2$ 

p_1	p_2
0.5	1.0

$\frac{1}{p_1}$	$\frac{1}{p_2}$
2	1



p_1	p_2
1	2

$\frac{1}{p_1}$	$\frac{1}{p_2}$
1	0.5

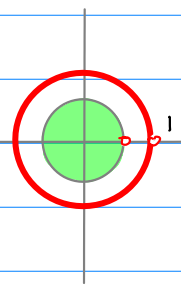
 2^{n-1} $(\frac{1}{2})^{n+1}$

$$P_1 < P_2 < 1$$

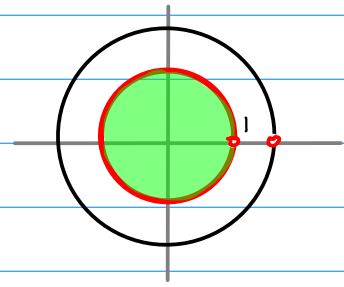
$$2^{n-1}$$

$$1 < P_1 < P_2$$

$$\left(\frac{1}{2}\right)^{n+1}$$

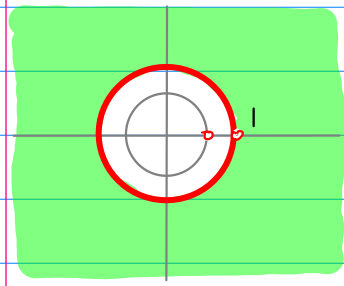


$$n < 0$$

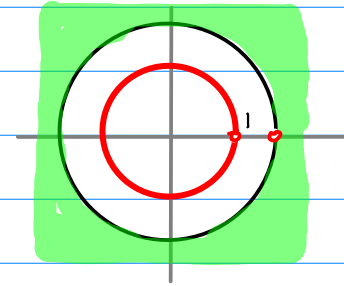


$$n \leq 0$$

$$|z| < P_1$$

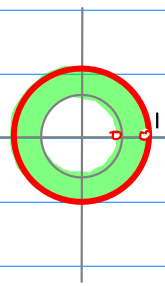


$$n \geq 0$$



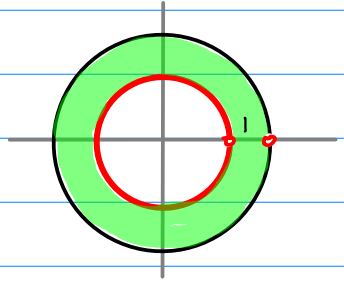
$$n > 0$$

$$|z| > P_2$$



$$n \geq 0$$

$$n < 0$$



$$n > 0$$

$$n < 0$$

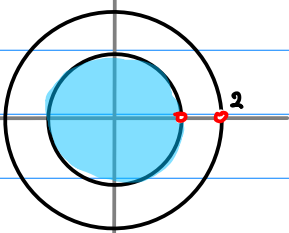
$$P_1 < |z| < P_2$$

Unit Circle \in ROC $\rightarrow \leq, \geq$

(A-1)

$f(z)$, a_n

I

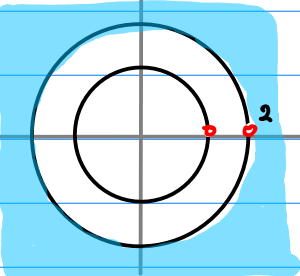


$(\frac{1}{2})^{n+1} - 1$	$(n \geq 0)$
0	$(n < 0)$

$p_1 = 1$
$p_2 = 2$

$(\frac{1}{p_2})^{n+1} - (\frac{1}{p_1})^{n+1}$	$(n \geq 0)$
0	$(n < 0)$

II

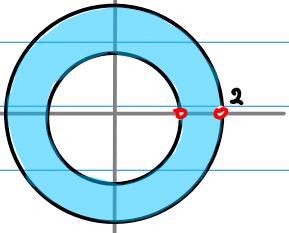


0	$(n \geq 0)$
$1 - (\frac{1}{2})^{n+1}$	$(n < 0)$

$p_1 = 1$
$p_2 = 2$

0	$(n \geq 0)$
$(\frac{1}{p_1})^{n+1} - (\frac{1}{p_2})^{n+1}$	$(n < 0)$

III

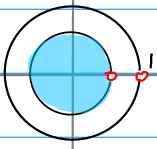


$(\frac{1}{2})^{n+1}$	$(n \geq 0)$
1	$(n < 0)$

$p_1 = 1$
$p_2 = 2$

$(\frac{1}{p_2})^{n+1}$	$(n \geq 0)$
$(\frac{1}{p_1})^{n+1}$	$(n < 0)$

I

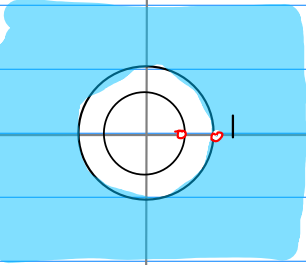


$1 - 2^{n-1}$	$(n > 0)$
0	$(n \leq 0)$

$p_1 = 0.5$
$p_2 = 1$

$(\frac{1}{p_2})^{n-1} - (\frac{1}{p_1})^{n-1}$	$(n > 0)$
0	$(n \leq 0)$

II

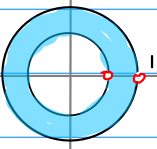


0	$(n > 0)$
$2^{n-1} - 1$	$(n \leq 0)$

$p_1 = 0.5$
$p_2 = 1$

0	$(n > 0)$
$(\frac{1}{p_1})^{n-1} - (\frac{1}{p_2})^{n-1}$	$(n \leq 0)$

III



1	$(n > 0)$
2^{n-1}	$(n \leq 0)$

$p_1 = 0.5$
$p_2 = 1$

$(\frac{1}{p_2})^{n-1}$	$(n > 0)$
$(\frac{1}{p_1})^{n-1}$	$(n \leq 0)$

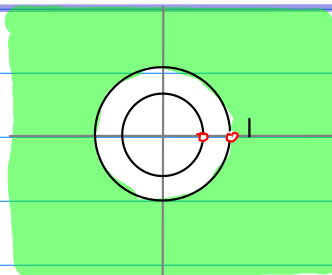
(B-1) $f(z) \xleftrightarrow{z^{-1}} \boxed{X(z)}$

$a_n = \boxed{x_n}$

I

$p_1 = 0.5$
 $p_2 = 1$

$(p_1)^{n+1} - (p_2)^{n+1}$
0

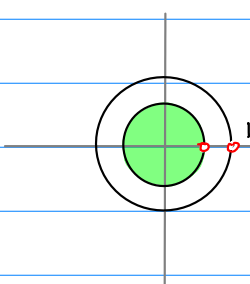


$(\frac{1}{2})^{n+1} - 1 \quad (n \geq 0)$
0 $(n < 0)$

II

$p_1 = 0.5$
 $p_2 = 1$

0
 $(p_2)^{n+1} - (p_1)^{n+1}$

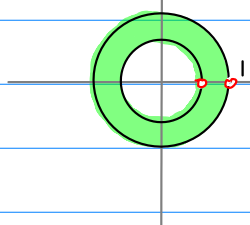


0 $(n \geq 0)$
 $1 - (\frac{1}{2})^{n+1} \quad (n < 0)$

III

$p_1 = 0.5$
 $p_2 = 1$

$(p_1)^{n+1}$
 $(p_2)^{n+1}$

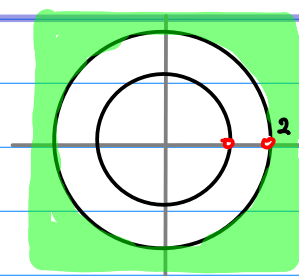


$(\frac{1}{2})^{n+1} \quad (n \geq 0)$
1 $(n < 0)$

I

$p_1 = 1$
 $p_2 = 2$

$(p_1)^{n-1} - (p_2)^{n-1}$
0

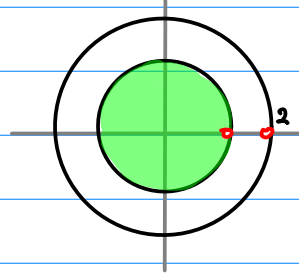


$1 - 2^{n-1} \quad (n > 0)$
0 $(n \leq 0)$

II

$p_1 = 1$
 $p_2 = 2$

0
 $(p_2)^{n-1} - (p_1)^{n-1}$

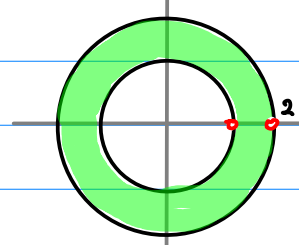


0 $(n > 0)$
 $2^{n-1} - 1 \quad (n \leq 0)$

III

$p_1 = 1$
 $p_2 = 2$

$(p_1)^{n-1}$
 $(p_2)^{n-1}$



1 $(n > 0)$
 $2^{n-1} \quad (n \leq 0)$

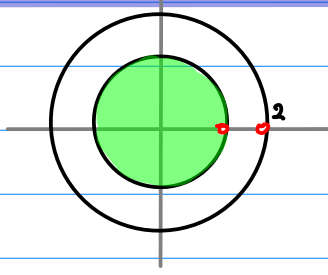
(c-1) $f(z) = X(z)$

$a_n \xleftrightarrow{-n} x_n$

I

$p_1 = 1$
 $p_2 = 2$

$$\begin{matrix} 0 \\ (p_2)^{n-1} - (p_1)^{n-1} \end{matrix}$$

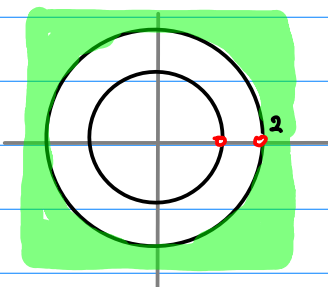


$$\begin{matrix} 0 & (n > 0) \\ 2^{n-1} - 1 & (n \leq 0) \end{matrix}$$

II

$p_1 = 1$
 $p_2 = 2$

$$\begin{matrix} (p_1)^{n-1} - (p_2)^{n-1} \\ 0 \end{matrix}$$

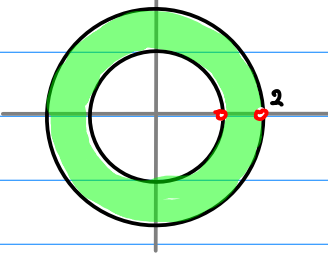


$$\begin{matrix} 1 - 2^{n-1} & (n > 0) \\ 0 & (n \leq 0) \end{matrix}$$

III

$p_1 = 1$
 $p_2 = 2$

$$\begin{matrix} (p_1)^{n-1} \\ (p_2)^{n-1} \end{matrix}$$

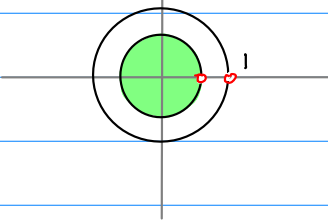


$$\begin{matrix} 1 & (n > 0) \\ 2^{n-1} & (n \leq 0) \end{matrix}$$

I

$p_1 = 0.5$
 $p_2 = 1$

$$\begin{matrix} 0 \\ (p_2)^{n+1} - (p_1)^{n+1} \end{matrix}$$

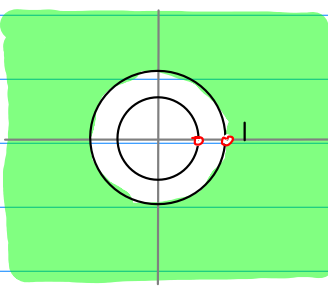


$$\begin{matrix} 0 & (n \geq 0) \\ 1 - (\frac{1}{2})^{n+1} & (n < 0) \end{matrix}$$

II

$p_1 = 0.5$
 $p_2 = 1$

$$\begin{matrix} (p_1)^{n+1} - (p_2)^{n+1} \\ 0 \end{matrix}$$

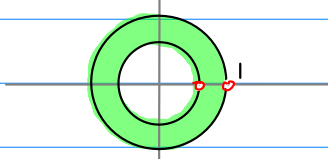


$$\begin{matrix} (\frac{1}{2})^{n+1} - 1 & (n \geq 0) \\ 0 & (n < 0) \end{matrix}$$

III

$p_1 = 0.5$
 $p_2 = 1$

$$\begin{matrix} (p_1)^{n+1} \\ (p_2)^{n+1} \end{matrix}$$



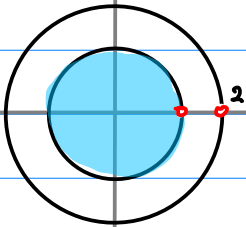
$$\begin{matrix} (\frac{1}{2})^{n+1} & (n \geq 0) \\ 1 & (n < 0) \end{matrix}$$



(A-2) $f(z) \xleftrightarrow{z^{-1}} X(z)$

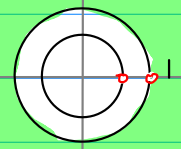
$a_n = x_n$

I

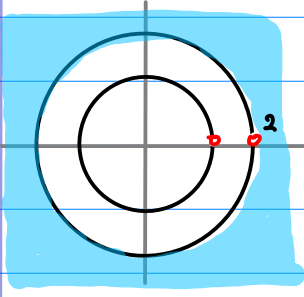


$\left(\frac{1}{p_2}\right)^{n+1} - \left(\frac{1}{p_1}\right)^{n+1}$	$(n \geq 0)$	$(n \geq 0)$	$(p_1)^{n+1} - (p_2)^{n+1}$
0	$(n < 0)$	$(n < 0)$	0

$p_1 = 1$
 $p_2 = 2$
 $p_1 = 0.5$
 $p_2 = 1$

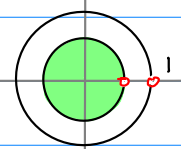


II

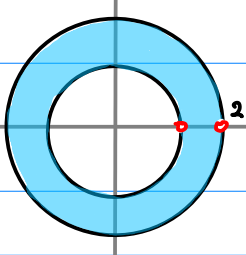


0	$(n \geq 0)$	$(n \geq 0)$	0
$\left(\frac{1}{p_1}\right)^{n+1} - \left(\frac{1}{p_2}\right)^{n+1}$	$(n < 0)$	$(n < 0)$	$(p_2)^{n+1} - (p_1)^{n+1}$

$p_1 = 1$
 $p_2 = 2$
 $p_1 = 0.5$
 $p_2 = 1$

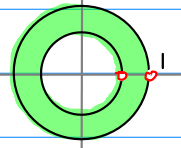


III

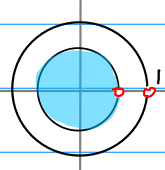


$\left(\frac{1}{p_2}\right)^{n+1}$	$(n \geq 0)$	$(n \geq 0)$	$(p_1)^{n+1}$
$\left(\frac{1}{p_1}\right)^{n+1}$	$(n < 0)$	$(n < 0)$	$(p_2)^{n+1}$

$p_1 = 1$
 $p_2 = 2$
 $p_1 = 0.5$
 $p_2 = 1$

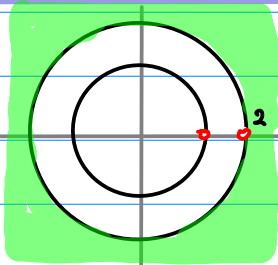


I

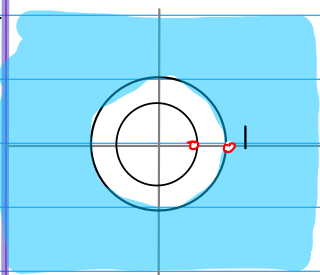


$\left(\frac{1}{p_2}\right)^{n-1} - \left(\frac{1}{p_1}\right)^{n-1}$	$(n > 0)$	$(n > 0)$	$(p_1)^{n-1} - (p_2)^{n-1}$
0	$(n \leq 0)$	$(n \leq 0)$	0

$p_1 = 0.5$
 $p_2 = 1$
 $p_1 = 1$
 $p_2 = 2$

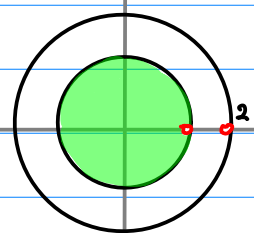


II

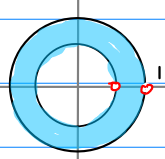


0	$(n > 0)$	$(n > 0)$	0
$\left(\frac{1}{p_1}\right)^{n-1} - \left(\frac{1}{p_2}\right)^{n-1}$	$(n \leq 0)$	$(n \leq 0)$	$(p_2)^{n-1} - (p_1)^{n-1}$

$p_1 = 0.5$
 $p_2 = 1$
 $p_1 = 1$
 $p_2 = 2$

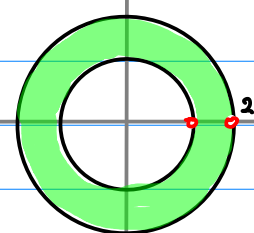


III



$\left(\frac{1}{p_2}\right)^{n-1}$	$(n > 0)$	$(n > 0)$	$(p_1)^{n-1}$
$\left(\frac{1}{p_1}\right)^{n-1}$	$(n \leq 0)$	$(n \leq 0)$	$(p_2)^{n-1}$

$p_1 = 0.5$
 $p_2 = 1$
 $p_1 = 1$
 $p_2 = 2$



(B-2) $f(z) = X(z) \quad a_n \overset{-n}{\longleftrightarrow} x_n$

I

$\left(\frac{1}{p_2}\right)^{n+1} - \left(\frac{1}{p_1}\right)^{n+1}$	$(n \geq 0) \quad (n > 0)$	0
0	$(n < 0) \quad (n \leq 0)$	$(p_2)^{n-1} - (p_1)^{n-1}$

$p_1 = 1$
 $p_2 = 2$

II

0	$(n \geq 0) \quad (n > 0)$	$(p_1)^{n-1} - (p_2)^{n-1}$
$\left(\frac{1}{p_1}\right)^{n+1} - \left(\frac{1}{p_2}\right)^{n+1}$	$(n < 0) \quad (n \leq 0)$	0

$p_1 = 1$
 $p_2 = 2$

III

$\left(\frac{1}{p_2}\right)^{n+1}$	$(n \geq 0) \quad (n > 0)$	$(p_1)^{n-1}$
$\left(\frac{1}{p_1}\right)^{n+1}$	$(n < 0) \quad (n \leq 0)$	$(p_2)^{n-1}$

$p_1 = 1$
 $p_2 = 2$

I

$\left(\frac{1}{p_2}\right)^{n-1} - \left(\frac{1}{p_1}\right)^{n-1}$	$(n > 0) \quad (n \geq 0)$	0
0	$(n \leq 0) \quad (n < 0)$	$(p_2)^{n+1} - (p_1)^{n+1}$

$p_1 = 0.5$
 $p_2 = 1$

II

0	$(n > 0) \quad (n \geq 0)$	$(p_1)^{n+1} - (p_2)^{n+1}$
$\left(\frac{1}{p_1}\right)^{n-1} - \left(\frac{1}{p_2}\right)^{n-1}$	$(n \leq 0) \quad (n < 0)$	0

$p_1 = 0.5$
 $p_2 = 1$

III

$\left(\frac{1}{p_2}\right)^{n-1}$	$(n > 0) \quad (n \geq 0)$	$(p_1)^{n+1}$
$\left(\frac{1}{p_1}\right)^{n-1}$	$(n \leq 0) \quad (n < 0)$	$(p_2)^{n+1}$

$p_1 = 0.5$
 $p_2 = 1$

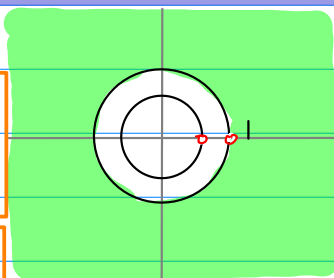
(C-2) $a_n = x_n$

$a_n \overset{-n}{\longleftrightarrow} x_n$

①

$(n \geq 0) (p_1)^{n+1} - (p_2)^{n+1}$
 $(n < 0) 0$

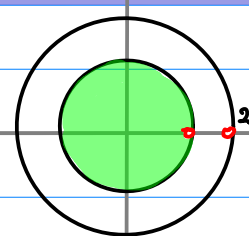
$p_1 = 0.5$
 $p_2 = 1$



⑦

$(n > 0) 0$
 $(n \leq 0) (p_2)^{n-1} - (p_1)^{n-1}$

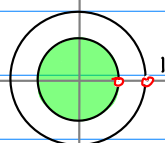
$p_1 = 1$
 $p_2 = 2$



②

$(n \geq 0) 0$
 $(n < 0) (p_2)^{n+1} - (p_1)^{n+1}$

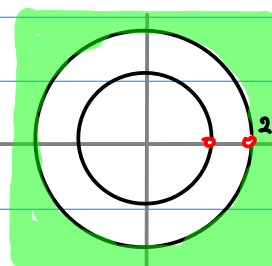
$p_1 = 0.5$
 $p_2 = 1$



⑧

$(n > 0) (p_1)^{n-1} - (p_2)^{n-1}$
 $(n \leq 0) 0$

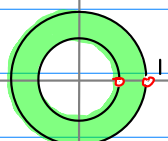
$p_1 = 1$
 $p_2 = 2$



③

$(n \geq 0) (p_1)^{n+1}$
 $(n < 0) (p_2)^{n+1}$

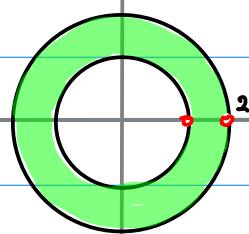
$p_1 = 0.5$
 $p_2 = 1$



⑨

$(n > 0) (p_1)^{n-1}$
 $(n \leq 0) (p_2)^{n-1}$

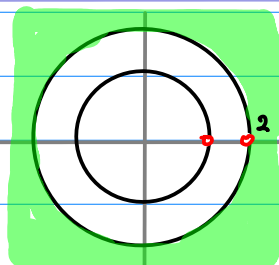
$p_1 = 1$
 $p_2 = 2$



④

$(n > 0) (p_1)^{n-1} - (p_2)^{n-1}$
 $(n \leq 0) 0$

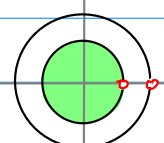
$p_1 = 1$
 $p_2 = 2$



⑩

$(n \geq 0) 0$
 $(n < 0) (p_2)^{n+1} - (p_1)^{n+1}$

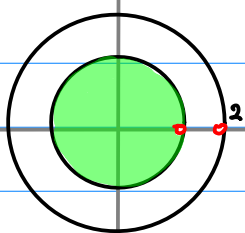
$p_1 = 0.5$
 $p_2 = 1$



⑤

$(n > 0) 0$
 $(n \leq 0) (p_2)^{n-1} - (p_1)^{n-1}$

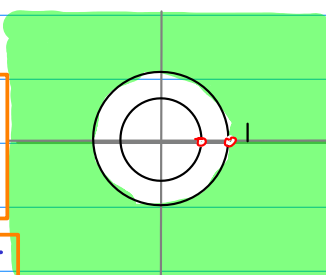
$p_1 = 1$
 $p_2 = 2$



⑪

$(n \geq 0) (p_1)^{n+1} - (p_2)^{n+1}$
 $(n < 0) 0$

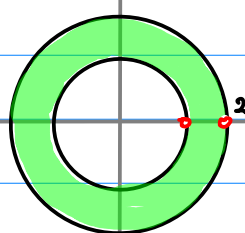
$p_1 = 0.5$
 $p_2 = 1$



⑥

$(n > 0) (p_1)^{n-1}$
 $(n \leq 0) (p_2)^{n-1}$

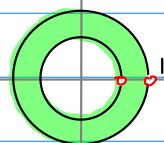
$p_1 = 1$
 $p_2 = 2$



⑫

$(n \geq 0) (p_1)^{n+1}$
 $(n < 0) (p_2)^{n+1}$

$p_1 = 0.5$
 $p_2 = 1$

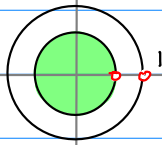


(C-3) $a_n = x_n$

$a_n \overset{-n}{\longleftrightarrow} x_n$

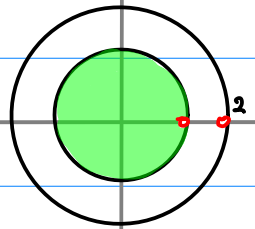
②

$(n \geq 0) \quad 0$
 $(n < 0) \quad (p_2)^{n+1} - (p_1)^{n+1}$
 $p_1 = 0.5$
 $p_2 = 1$



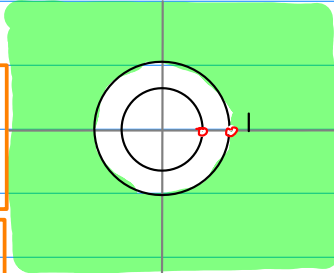
⑦

$(n > 0) \quad 0$
 $(n \leq 0) \quad (p_2)^{n-1} - (p_1)^{n-1}$
 $p_1 = 1$
 $p_2 = 2$



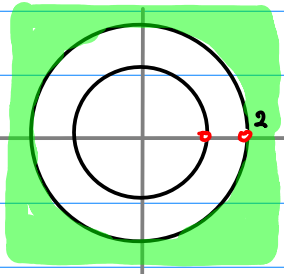
①

$(n \geq 0) \quad (p_1)^{n+1} - (p_2)^{n+1}$
 $(n < 0) \quad 0$
 $p_1 = 0.5$
 $p_2 = 1$



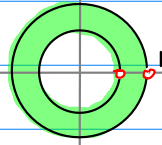
⑧

$(n > 0) \quad (p_1)^{n-1} - (p_2)^{n-1}$
 $(n \leq 0) \quad 0$
 $p_1 = 1$
 $p_2 = 2$



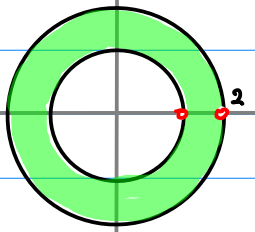
③

$(n \geq 0) \quad (p_1)^{n+1}$
 $(n < 0) \quad (p_2)^{n+1}$
 $p_1 = 0.5$
 $p_2 = 1$



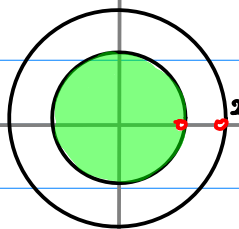
⑨

$(n > 0) \quad (p_1)^{n-1}$
 $(n \leq 0) \quad (p_2)^{n-1}$
 $p_1 = 1$
 $p_2 = 2$



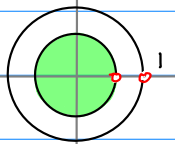
⑤

$(n > 0) \quad 0$
 $(n \leq 0) \quad (p_2)^{n-1} - (p_1)^{n-1}$
 $p_1 = 1$
 $p_2 = 2$



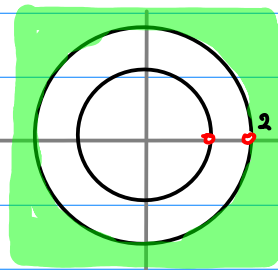
⑩

$(n \geq 0) \quad 0$
 $(n < 0) \quad (p_2)^{n+1} - (p_1)^{n+1}$
 $p_1 = 0.5$
 $p_2 = 1$



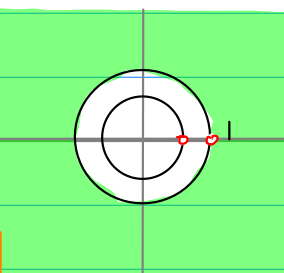
④

$(n > 0) \quad (p_1)^{n-1} - (p_2)^{n-1}$
 $(n \leq 0) \quad 0$
 $p_1 = 1$
 $p_2 = 2$



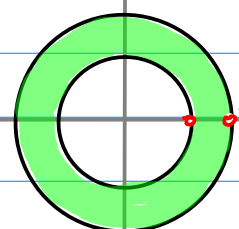
⑪

$(n \geq 0) \quad (p_1)^{n+1} - (p_2)^{n+1}$
 $(n < 0) \quad 0$
 $p_1 = 0.5$
 $p_2 = 1$



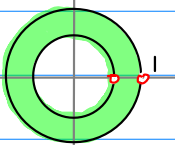
⑥

$(n > 0) \quad (p_1)^{n-1}$
 $(n \leq 0) \quad (p_2)^{n-1}$
 $p_1 = 1$
 $p_2 = 2$



⑫

$(n \geq 0) \quad (p_1)^{n+1}$
 $(n < 0) \quad (p_2)^{n+1}$
 $p_1 = 0.5$
 $p_2 = 1$



(C-4)

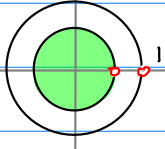
$n \geq 0$
$n < 0$

$n > 0$
$n \leq 0$

2

$(n \geq 0)$ 0
$(n < 0)$ $(p_2)^{n+1} - (p_1)^{n+1}$

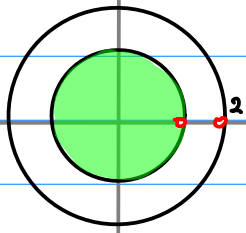
$p_1 = 0.5$
$p_2 = 1$



7

$(n > 0)$ 0
$(n \leq 0)$ $(p_2)^{n-1} - (p_1)^{n-1}$

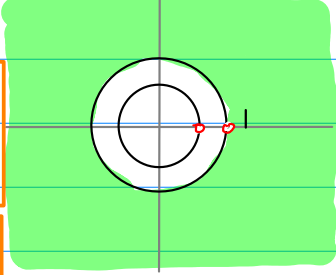
$p_1 = 1$
$p_2 = 2$



1

$(n \geq 0)$ $(p_1)^{n+1} - (p_2)^{n+1}$
$(n < 0)$ 0

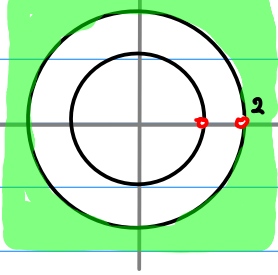
$p_1 = 0.5$
$p_2 = 1$



8

$(n > 0)$ $(p_1)^{n-1} - (p_2)^{n-1}$
$(n \leq 0)$ 0

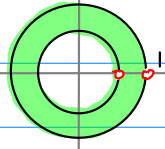
$p_1 = 1$
$p_2 = 2$



3

$(n \geq 0)$ $(p_1)^{n+1}$
$(n < 0)$ $(p_2)^{n+1}$

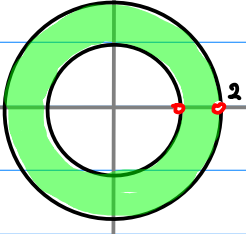
$p_1 = 0.5$
$p_2 = 1$



9

$(n > 0)$ $(p_1)^{n-1}$
$(n \leq 0)$ $(p_2)^{n-1}$

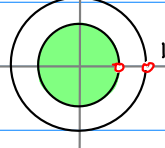
$p_1 = 1$
$p_2 = 2$



10

$(n \geq 0)$ 0
$(n < 0)$ $(p_2)^{n+1} - (p_1)^{n+1}$

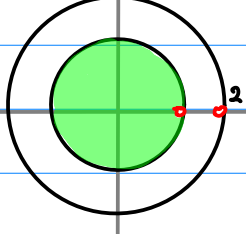
$p_1 = 0.5$
$p_2 = 1$



5

$(n > 0)$ 0
$(n \leq 0)$ $(p_2)^{n-1} - (p_1)^{n-1}$

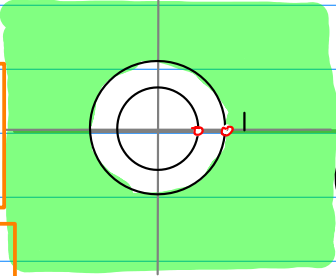
$p_1 = 1$
$p_2 = 2$



11

$(n \geq 0)$ $(p_1)^{n+1} - (p_2)^{n+1}$
$(n < 0)$ 0

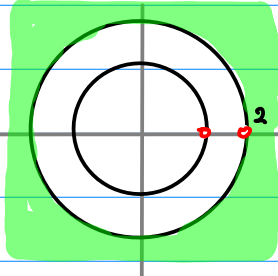
$p_1 = 0.5$
$p_2 = 1$



4

$(n > 0)$ $(p_1)^{n-1} - (p_2)^{n-1}$
$(n \leq 0)$ 0

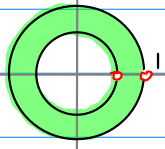
$p_1 = 1$
$p_2 = 2$



12

$(n \geq 0)$ $(p_1)^{n+1}$
$(n < 0)$ $(p_2)^{n+1}$

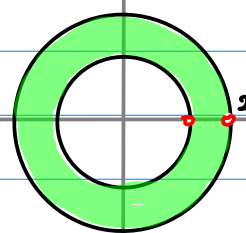
$p_1 = 0.5$
$p_2 = 1$



6

$(n > 0)$ $(p_1)^{n-1}$
$(n \leq 0)$ $(p_2)^{n-1}$

$p_1 = 1$
$p_2 = 2$



$$\begin{array}{|l} n \geq 0 \\ n < 0 \end{array}$$

$$\begin{array}{|l} n > 0 \\ n \leq 0 \end{array}$$

$$\begin{array}{|l} 0 \\ (p_2)^{n+1} - (p_1)^{n+1} \end{array}$$

$$\begin{array}{|l} 0 \\ (p_2)^{n-1} - (p_1)^{n-1} \end{array}$$

$$\begin{array}{|l} (p_1)^{n+1} - (p_2)^{n+1} \\ 0 \end{array}$$

$$\begin{array}{|l} (p_1)^{n-1} - (p_2)^{n-1} \\ 0 \end{array}$$

$$\begin{array}{|l} (p_1)^{n+1} \\ (p_2)^{n+1} \end{array}$$

$$\begin{array}{|l} (p_1)^{n-1} \\ (p_2)^{n-1} \end{array}$$

$$p_1 = 0.5 \quad p_2 = 1$$

$$p_1 = 1 \quad p_2 = 2$$

$$(\quad)^{n+1}$$

$$(\quad)^{n-1}$$

$$\begin{array}{|l} n \geq 0 \\ n < 0 \end{array}$$

$$\begin{array}{|l} n > 0 \\ n \leq 0 \end{array}$$

$$p_1 = 0.5, p_2 = 1$$

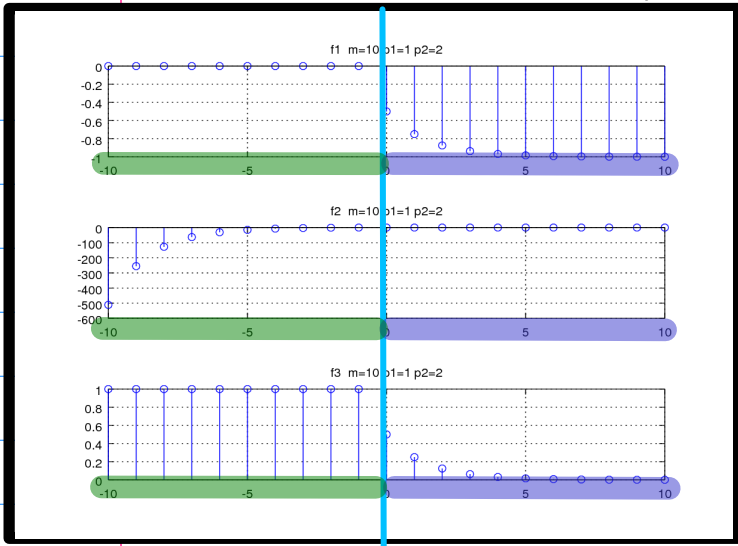
$$p_1 = 1, p_2 = 2$$

$$(p_1)^{n+1}, (p_2)^{n+1}$$

$$(p_1)^{n-1}, (p_2)^{n-1}$$

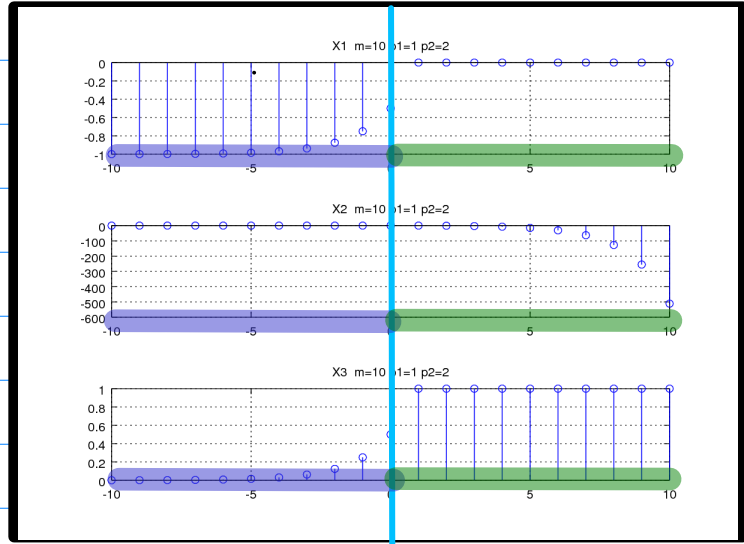
$f(z)$

(1, 2)

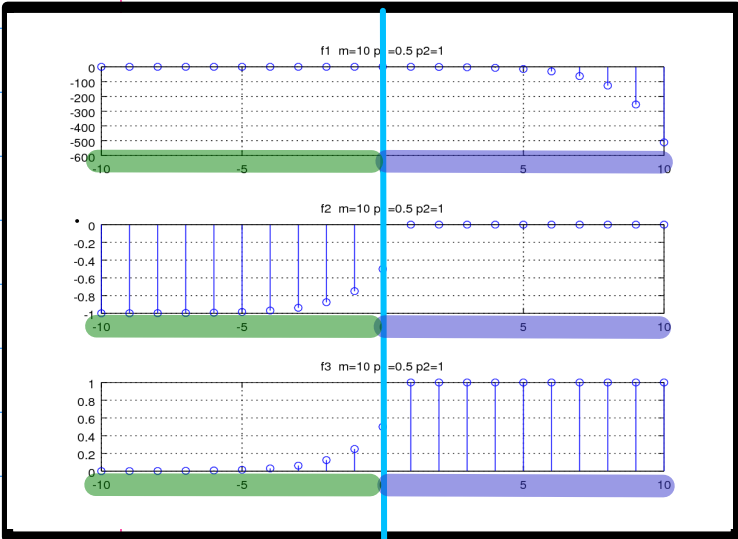


$X(z)$

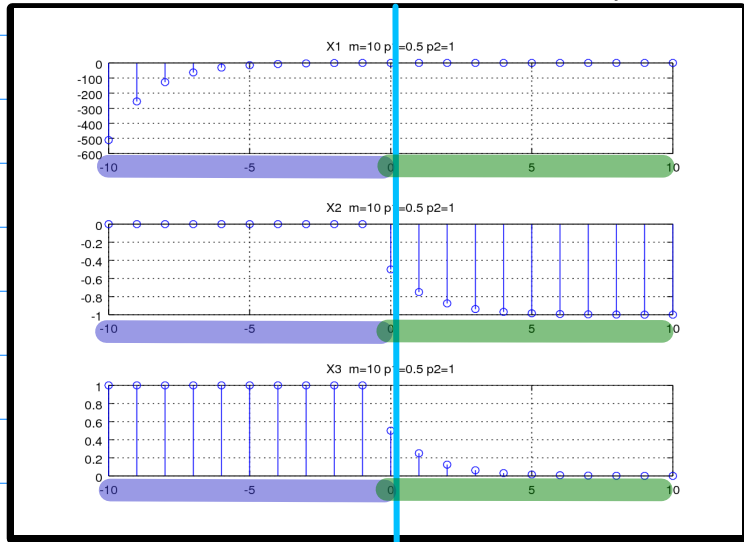
(1, 2)



(0.5, 1)



(0.5, 1)



$$\begin{array}{ll} \left(\frac{1}{2}\right)^{n+1} - 1 & (n \geq 0) \\ 0 & (n < 0) \end{array}$$

$$\begin{array}{ll} 0 & (n \geq 0) \\ 1 - \left(\frac{1}{2}\right)^{n+1} & (n < 0) \end{array}$$

$$\begin{array}{ll} \left(\frac{1}{2}\right)^{n+1} & (n \geq 0) \\ 1 & (n < 0) \end{array}$$

$$\begin{array}{ll} 0 & (n > 0) \\ 2^{n-1} - 1 & (n \leq 0) \end{array}$$

$$\begin{array}{ll} 1 - 2^{n-1} & (n > 0) \\ 0 & (n \leq 0) \end{array}$$

$$\begin{array}{ll} 1 & (n > 0) \\ 2^{n-1} & (n \leq 0) \end{array}$$

$$\begin{array}{ll} 0 & \left(\frac{1}{2}\right)^{n+1} - 1 \\ 1 - \left(\frac{1}{2}\right)^{n+1} & 0 \\ 1 & \left(\frac{1}{2}\right)^{n+1} \end{array}$$

$$\begin{array}{ll} 2^{n-1} - 1 & 0 \\ 0 & 1 - 2^{n-1} \\ 2^{n-1} & 1 \end{array}$$

$$\begin{array}{ll} 1 - 2^{n-1} & (n > 0) \\ 0 & (n \leq 0) \end{array}$$

$$\begin{array}{ll} 0 & (n > 0) \\ 2^{n-1} - 1 & (n \leq 0) \end{array}$$

$$\begin{array}{ll} 1 & (n > 0) \\ 2^{n-1} & (n \leq 0) \end{array}$$

$$\begin{array}{ll} 0 & (n \geq 0) \\ 1 - \left(\frac{1}{2}\right)^{n+1} & (n < 0) \end{array}$$

$$\begin{array}{ll} \left(\frac{1}{2}\right)^{n+1} - 1 & (n \geq 0) \\ 0 & (n < 0) \end{array}$$

$$\begin{array}{ll} \left(\frac{1}{2}\right)^{n+1} & (n \geq 0) \\ 1 & (n < 0) \end{array}$$

$$\begin{array}{ll} 0 & 1 - 2^{n-1} \\ 2^{n-1} - 1 & 0 \\ 2^{n-1} & 1 \end{array}$$

$$\begin{array}{ll} 1 - \left(\frac{1}{2}\right)^{n+1} & 0 \\ 0 & \left(\frac{1}{2}\right)^{n+1} - 1 \\ 1 & \left(\frac{1}{2}\right)^{n+1} \end{array}$$

plotseq1(m, p1, p2) : Laurent

[-m, +m]

```
% Laurent Series and sequences
function plotseq1(m=1, p1=2, p2=2.1)

if (p1 >= 1 && p2 >= 1 && p1 < p2)
    t1n = -m: -1;
    t1p = 0: m;
    t1 = [t1n, t1p];
    f1 = [zeros(1,m), ((1/p2).^(t1p+1) - (1/p1).^(t1p+1))];
    f2 = [((1/p1).^(t1n+1) - (1/p2).^(t1n+1)), zeros(1,m+1)];
    f3 = [(1/p1).^(t1n+1), (1/p2).^(t1p+1)];
endif

if (p1 <= 1 && p2 <= 1 && p1 < p2)
    t1n = -m: 0;
    t1p = +1: m;
    t1 = [t1n, t1p];
    f1 = [zeros(1,m+1), ((1/p2).^(t1p-1) - (1/p1).^(t1p-1))];
    f2 = [((1/p1).^(t1n-1) - (1/p2).^(t1n-1)), zeros(1,m)];
    f3 = [(1/p1).^(t1n-1), (1/p2).^(t1p-1)];
endif

subplot(3, 1, 1);
stem(t1, f1);
grid on
%axis([0, m])
title(sprintf("f1 m=%d p1=%g p2=%g", m, p1, p2))

subplot(3, 1, 2);
stem(t1, f2);
grid on
%axis([0, m])
title(sprintf("f2 m=%d p1=%g p2=%g", m, p1, p2))

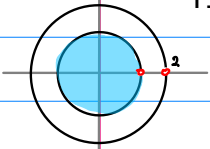
subplot(3, 1, 3);
stem(t1, f3);
grid on
%axis([0, m])
title(sprintf("f3 m=%d p1=%g p2=%g", m, p1, p2))

endfunction
```

f(z) Laurent

```
if (p1 >= 1 && p2 >= 1 && p1 < p2)
  t1n = -m: -1;
  t1p = 0: m;
  t1 = [t1n, t1p];
```

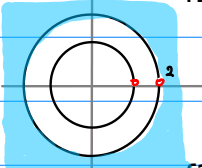
```
f1 = [zeros(1,m), ((1/p2).^ (t1p+1) - (1/p1).^ (t1p+1))];
```



$$\begin{cases} \left(\frac{1}{p_2}\right)^{n+1} - \left(\frac{1}{p_1}\right)^{n+1} \\ 0 \end{cases}$$

$$\begin{cases} \left(\frac{1}{p_2}\right)^{n+1} - \left(\frac{1}{p_1}\right)^{n+1} & (n \geq 0) \\ 0 & (n < 0) \end{cases}$$

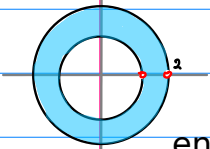
```
f2 = [((1/p1).^ (t1n+1) - (1/p2).^ (t1n+1)), zeros(1,m+1)];
```



$$\begin{cases} 0 \\ \left(\frac{1}{p_1}\right)^{n+1} - \left(\frac{1}{p_2}\right)^{n+1} \end{cases}$$

$$\begin{cases} 0 & (n \geq 0) \\ \left(\frac{1}{p_1}\right)^{n+1} - \left(\frac{1}{p_2}\right)^{n+1} & (n < 0) \end{cases}$$

```
f3 = [(1/p1).^ (t1n+1), (1/p2).^ (t1p+1)];
```



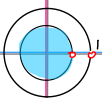
$$\begin{cases} \left(\frac{1}{p_2}\right)^{n+1} \\ \left(\frac{1}{p_1}\right)^{n+1} \end{cases}$$

$$\begin{cases} \left(\frac{1}{p_2}\right)^{n+1} & (n \geq 0) \\ \left(\frac{1}{p_1}\right)^{n+1} & (n < 0) \end{cases}$$

```
endif
```

```
if (p1 <= 1 && p2 <= 1 && p1 < p2)
  t1n = -m: 0;
  t1p = +1: m;
  t1 = [t1n, t1p];
```

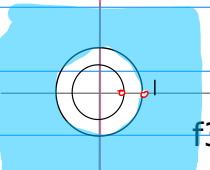
```
f1 = [zeros(1,m+1), ((1/p2).^ (t1p-1) - (1/p1).^ (t1p-1))];
```



$$\begin{cases} \left(\frac{1}{p_2}\right)^{n-1} - \left(\frac{1}{p_1}\right)^{n-1} \\ 0 \end{cases}$$

$$\begin{cases} \left(\frac{1}{p_2}\right)^{n-1} - \left(\frac{1}{p_1}\right)^{n-1} & (n > 0) \\ 0 & (n \leq 0) \end{cases}$$

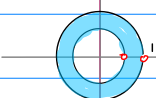
```
f2 = [((1/p1).^ (t1n-1) - (1/p2).^ (t1n-1)), zeros(1,m)];
```



$$\begin{cases} 0 \\ \left(\frac{1}{p_1}\right)^{n-1} - \left(\frac{1}{p_2}\right)^{n-1} \end{cases}$$

$$\begin{cases} 0 & (n > 0) \\ \left(\frac{1}{p_1}\right)^{n-1} - \left(\frac{1}{p_2}\right)^{n-1} & (n \leq 0) \end{cases}$$

```
f3 = [(1/p1).^ (t1n-1), (1/p2).^ (t1p-1)];
```



$$\begin{cases} \left(\frac{1}{p_2}\right)^{n-1} \\ \left(\frac{1}{p_1}\right)^{n-1} \end{cases}$$

$$\begin{cases} \left(\frac{1}{p_2}\right)^{n-1} & (n > 0) \\ \left(\frac{1}{p_1}\right)^{n-1} & (n \leq 0) \end{cases}$$

```
endif
```

plotseq2(m, p1, p2) : Z-trans

[-m, +m]

```
% z-Transform and sequences
function plotseq2(m=1, p1=2, p2=2.1)

cla;

if (p1 <= 1 && p2 <= 1 && p1 < p2)
    t1n = -m: -1;
    t1p = 0: m;
    t1 = [t1n, t1p];
    X1 = [((p2).^(t1n+1) -(p1).^(t1n+1)), zeros(1,m+1)];
    X2 = [zeros(1,m), ((p1).^(t1p+1) - (p2).^(t1p+1))];
    X3 = [(p2).^(t1n+1), (p1).^(t1p+1)];
endif

if (p1 >= 1 && p2 >= 1 && p1 < p2)
    t1n = -m: 0;
    t1p = +1: m;
    t1 = [t1n, t1p];
    X1 = [((p2).^(t1n-1) -(p1).^(t1n-1)), zeros(1,m)];
    X2 = [zeros(1,m+1), ((p1).^(t1p-1) - (p2).^(t1p-1))];
    X3 = [(p2).^(t1n-1), (p1).^(t1p-1)];
endif

subplot(3, 1, 1);
stem(t1, X1);
grid on
%axis([0, m])
title(sprintf("X1 m=%d p1=%g p2=%g", m, p1, p2))

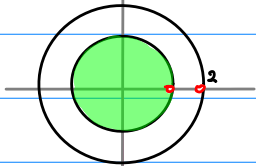
subplot(3, 1, 2);
stem(t1, X2);
grid on
%axis([0, m])
title(sprintf("X2 m=%d p1=%g p2=%g", m, p1, p2))

subplot(3, 1, 3);
stem(t1, X3);
grid on
%axis([0, m])
title(sprintf("X3 m=%d p1=%g p2=%g", m, p1, p2))

endfunction
```

X(z) z-Trans

```
if (p1 <= 1 && p2 <= 1 && p1 < p2)
  t1n = -m: -1;
  t1p = 0: m;
  t1 = [t1n, t1p];
```

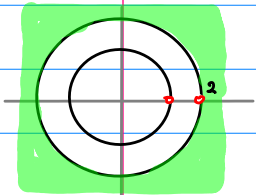


```
X1 = [zeros(1,m), ((p1).^t1p+1 - (p2).^t1p+1)];
```

$$\begin{cases} 0 \\ (p_2)^{n+1} - (p_1)^{n+1} \end{cases}$$

$$\begin{cases} (n > 0) & 0 \\ (n \leq 0) & (p_2)^{n-1} - (p_1)^{n-1} \end{cases}$$

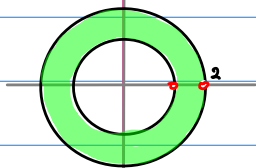
```
X2 = [((p2).^t1n+1 - (p1).^t1n+1), zeros(1,m+1)];
```



$$\begin{cases} (p_1)^{n+1} - (p_2)^{n+1} \\ 0 \end{cases}$$

$$\begin{cases} (n > 0) & (p_1)^{n-1} - (p_2)^{n-1} \\ (n \leq 0) & 0 \end{cases}$$

```
X3 = [(p2).^t1n+1, (p1).^t1p+1];
```

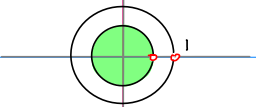


$$\begin{cases} (p_1)^{n+1} \\ (p_2)^{n+1} \end{cases}$$

$$\begin{cases} (n > 0) & (p_1)^{n-1} \\ (n \leq 0) & (p_2)^{n-1} \end{cases}$$

endif

```
if (p1 >= 1 && p2 >= 1 && p1 < p2)
  t1n = -m: 0;
  t1p = +1: m;
  t1 = [t1n, t1p];
```

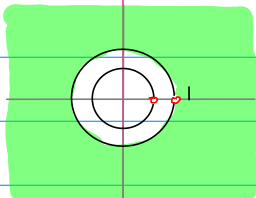


```
X1 = [((p2).^t1n-1 - (p1).^t1n-1), zeros(1,m)];
```

$$\begin{cases} 0 \\ (p_2)^{n-1} - (p_1)^{n-1} \end{cases}$$

$$\begin{cases} (n \geq 0) & 0 \\ (n < 0) & (p_2)^{n+1} - (p_1)^{n+1} \end{cases}$$

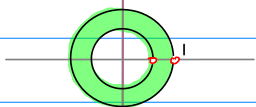
```
X2 = [zeros(1,m+1), ((p1).^t1p-1 - (p2).^t1p-1)];
```



$$\begin{cases} (p_1)^{n+1} - (p_2)^{n+1} \\ 0 \end{cases}$$

$$\begin{cases} (n \geq 0) & (p_1)^{n+1} - (p_2)^{n+1} \\ (n < 0) & 0 \end{cases}$$

```
X3 = [(p2).^t1n-1, (p1).^t1p-1];
```



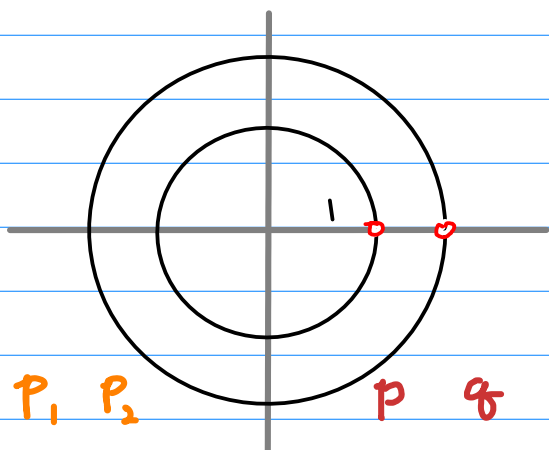
$$\begin{cases} (p_1)^{n+1} \\ (p_2)^{n-1} \end{cases}$$

$$\begin{cases} (n \geq 0) & (p_1)^{n+1} \\ (n < 0) & (p_2)^{n+1} \end{cases}$$

endif

$f(z)$

$X(z)$

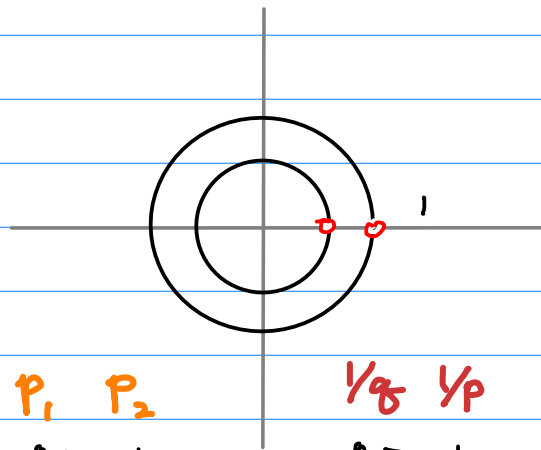


P_1, P_2
1 2

p, q
1 2

$\frac{1}{P_1}$	$\frac{1}{P_2}$
1	0.5

$\frac{1}{p}$	$\frac{1}{q}$
1	0.5



P_1, P_2
0.5 1

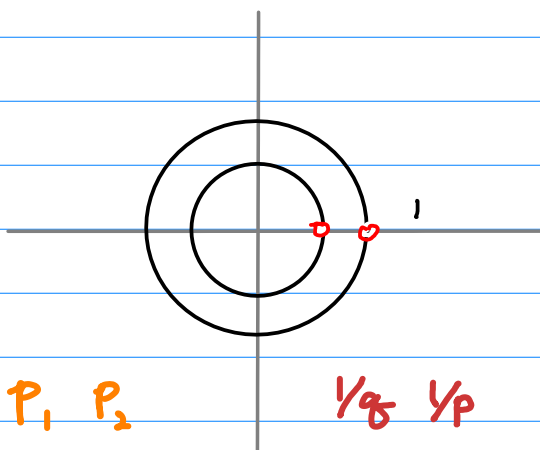
$\frac{1}{q}, \frac{1}{p}$
0.5 1

P_1	P_2
0.5	1

$\frac{1}{q}$	$\frac{1}{p}$
0.5	1

$(\frac{1}{2})^{n+1}$

$(\frac{1}{2})^{n+1}$

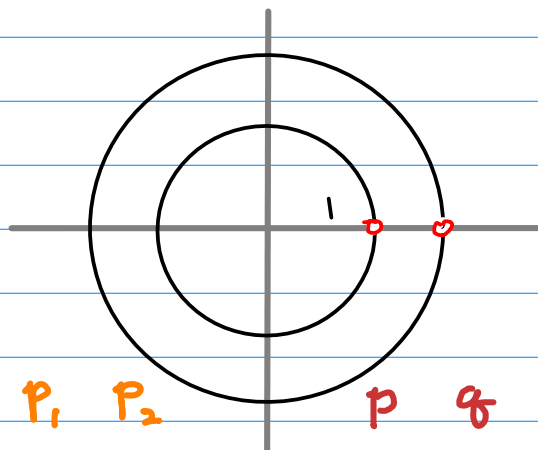


P_1, P_2
0.5 1

$\frac{1}{q}, \frac{1}{p}$
0.5 1

$\frac{1}{P_1}$	$\frac{1}{P_2}$
2	1

q	p
2	1



P_1, P_2
1 2

p, q
1 2

P_1	P_2
1	2

p	q
1	2

2^{n-1}

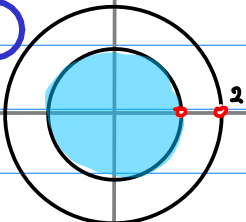
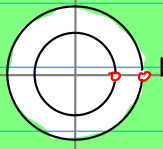
2^{n-1}

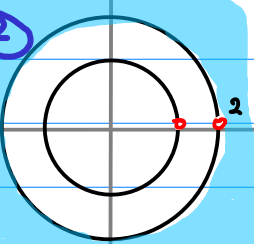
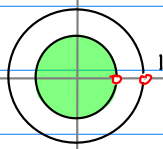
relative values

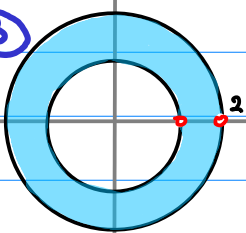
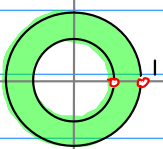
$$p_1 < p_2$$

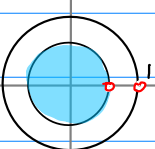
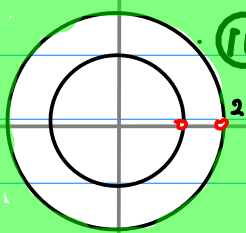
fixed values

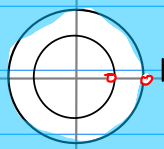
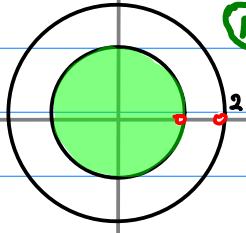
$$p = 1 \quad q = 2$$

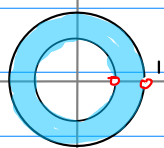
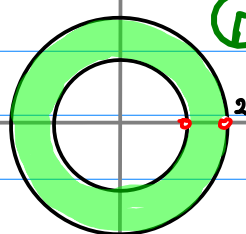
I **1**  $\left(\frac{1}{q}\right)^{n+1} - \left(\frac{1}{p}\right)^{n+1}$ ($n \geq 0$) ($n > 0$) 0
 0 ($n < 0$) ($n \leq 0$) $\left(\frac{1}{q}\right)^{n+1} - \left(\frac{1}{p}\right)^{n+1}$
 $p_1 = 1 = p$ $p_2 = 2 = q$ $p_1 = 0.5 = \frac{1}{q}$ $p_2 = 1 = \frac{1}{p}$ 

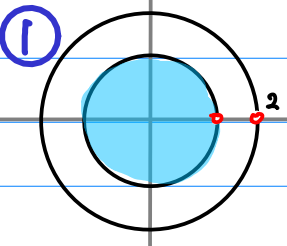
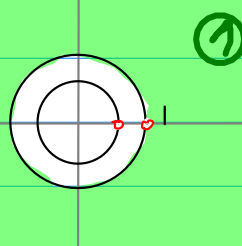
II **2**  0 ($n \geq 0$) ($n > 0$) $\left(\frac{1}{p}\right)^{n+1} - \left(\frac{1}{q}\right)^{n+1}$
 $\left(\frac{1}{p}\right)^{n+1} - \left(\frac{1}{q}\right)^{n+1}$ ($n < 0$) ($n \leq 0$) 0
 $p_1 = 1 = p$ $p_2 = 2 = q$ $p_1 = 0.5 = \frac{1}{q}$ $p_2 = 1 = \frac{1}{p}$ 

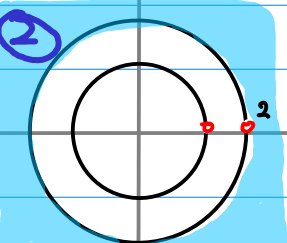
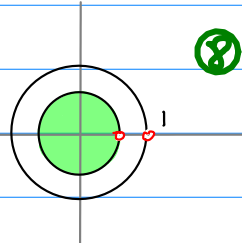
III **3**  $\left(\frac{1}{q}\right)^{n+1}$ ($n \geq 0$) ($n > 0$) $\left(\frac{1}{q}\right)^{n+1}$
 $\left(\frac{1}{p}\right)^{n+1}$ ($n < 0$) ($n \leq 0$) $\left(\frac{1}{p}\right)^{n+1}$
 $p_1 = 1 = p$ $p_2 = 2 = q$ $p_1 = 0.5 = \frac{1}{q}$ $p_2 = 1 = \frac{1}{p}$ 

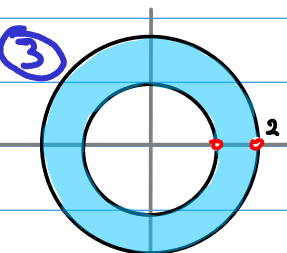
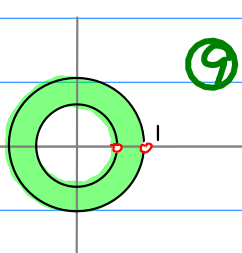
I **4**  $(p)^{n-1} - (q)^{n-1}$ ($n > 0$) ($n \geq 0$) 0
 0 ($n \leq 0$) ($n < 0$) $(p)^{n-1} - (q)^{n-1}$
 $p_1 = 0.5 = \frac{1}{q}$ $p_2 = 1 = \frac{1}{p}$ $p_1 = 1 = p$ $p_2 = 2 = q$ 

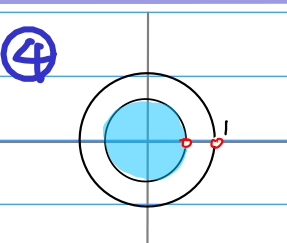
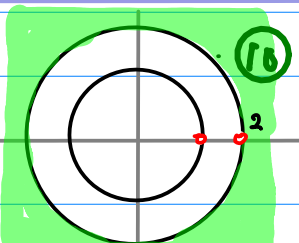
II **5**  0 ($n > 0$) ($n \geq 0$) $(q)^{n-1} - (p)^{n-1}$
 $(q)^{n-1} - (p)^{n-1}$ ($n \leq 0$) ($n < 0$) 0
 $p_1 = 0.5 = \frac{1}{q}$ $p_2 = 1 = \frac{1}{p}$ $p_1 = 1 = p$ $p_2 = 2 = q$ 

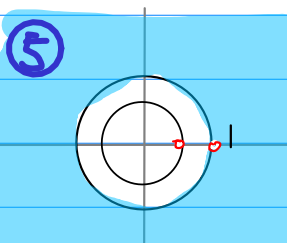
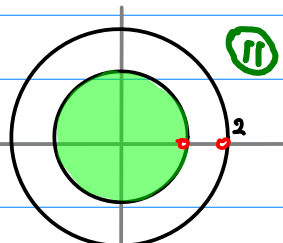
III **6**  $(p)^{n-1}$ ($n > 0$) ($n \geq 0$) $(p)^{n-1}$
 $(q)^{n-1}$ ($n \leq 0$) ($n < 0$) $(q)^{n-1}$
 $p_1 = 0.5 = \frac{1}{q}$ $p_2 = 1 = \frac{1}{p}$ $p_1 = 1 = p$ $p_2 = 2 = q$ 

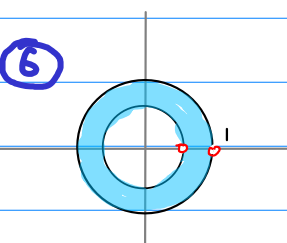
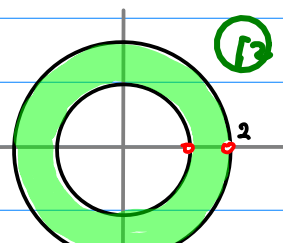
I **1**  $\left(\frac{1}{q}\right)^{n+1} - \left(\frac{1}{p}\right)^{n+1}$ ($n \geq 0$) ($n > 0$) 0
 0 ($n < 0$) ($n \leq 0$) $\left(\frac{1}{q}\right)^{n+1} - \left(\frac{1}{p}\right)^{n+1}$
 $p_1 = 1 = p$ $p_2 = 2 = q$ $p_1 = 0.5 = \frac{1}{q}$ $p_2 = 1 = \frac{1}{p}$ 

II **2**  0 ($n \geq 0$) ($n > 0$) $\left(\frac{1}{p}\right)^{n+1} - \left(\frac{1}{q}\right)^{n+1}$
 $\left(\frac{1}{p}\right)^{n+1} - \left(\frac{1}{q}\right)^{n+1}$ ($n < 0$) ($n \leq 0$) 0
 $p_1 = 1 = p$ $p_2 = 2 = q$ $p_1 = 0.5 = \frac{1}{q}$ $p_2 = 1 = \frac{1}{p}$ 

III **3**  $\left(\frac{1}{q}\right)^{n+1}$ ($n \geq 0$) ($n > 0$) $\left(\frac{1}{q}\right)^{n+1}$
 $\left(\frac{1}{p}\right)^{n+1}$ ($n < 0$) ($n \leq 0$) $\left(\frac{1}{p}\right)^{n+1}$
 $p_1 = 1 = p$ $p_2 = 2 = q$ $p_1 = 0.5 = \frac{1}{q}$ $p_2 = 1 = \frac{1}{p}$ 

I **4**  $(p)^{n-1} - (q)^{n-1}$ ($n > 0$) ($n \geq 0$) 0
 0 ($n \leq 0$) ($n < 0$) $(p)^{n-1} - (q)^{n-1}$
 $p_1 = 0.5 = \frac{1}{q}$ $p_2 = 1 = \frac{1}{p}$ $p_1 = 1 = p$ $p_2 = 2 = q$ 

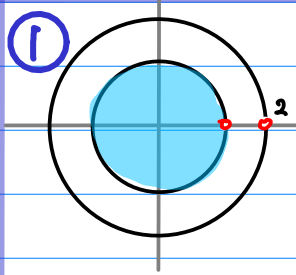
II **5**  0 ($n > 0$) ($n \geq 0$) $(q)^{n-1} - (p)^{n-1}$
 $(q)^{n-1} - (p)^{n-1}$ ($n \leq 0$) ($n < 0$) 0
 $p_1 = 0.5 = \frac{1}{q}$ $p_2 = 1 = \frac{1}{p}$ $p_1 = 1 = p$ $p_2 = 2 = q$ 

III **6**  $(p)^{n-1}$ ($n > 0$) ($n \geq 0$) $(p)^{n-1}$
 $(q)^{n-1}$ ($n \leq 0$) ($n < 0$) $(q)^{n-1}$
 $p_1 = 0.5 = \frac{1}{q}$ $p_2 = 1 = \frac{1}{p}$ $p_1 = 1 = p$ $p_2 = 2 = q$ 

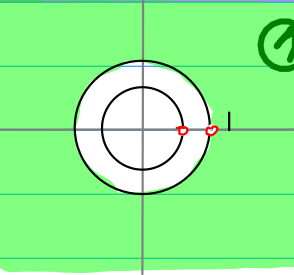
$$f(z) \xleftrightarrow{z^{-1}} X(z)$$

$$a_n = x_n$$

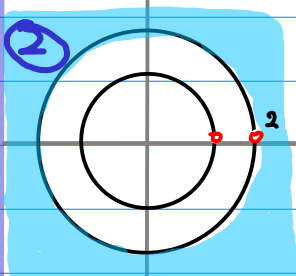
I **1**



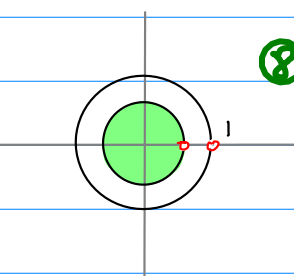
$(\frac{1}{q})^{n+1} - (\frac{1}{p})^{n+1}$	$(n \geq 0) (n > 0)$	0
0	$(n < 0) (n \leq 0)$	$(\frac{1}{q})^{n+1} - (\frac{1}{p})^{n+1}$
$p_1 = 1 = p$		$p_1 = 0.5 = 1/2$
$p_2 = 2 = q$		$p_2 = 1 = 1/p$



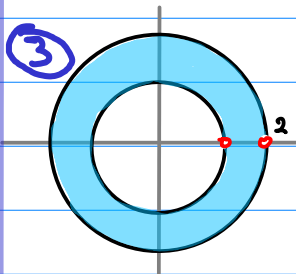
II **2**



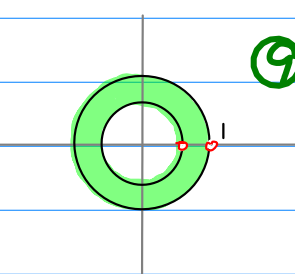
0	$(n \geq 0) (n > 0)$	$(\frac{1}{p})^{n+1} - (\frac{1}{q})^{n+1}$
$(\frac{1}{p})^{n+1} - (\frac{1}{q})^{n+1}$	$(n < 0) (n \leq 0)$	0
$p_1 = 1 = p$		$p_1 = 0.5 = 1/2$
$p_2 = 2 = q$		$p_2 = 1 = 1/p$



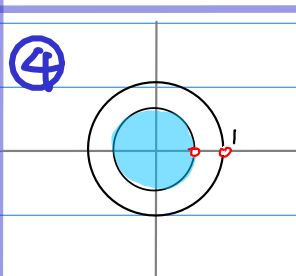
III **3**



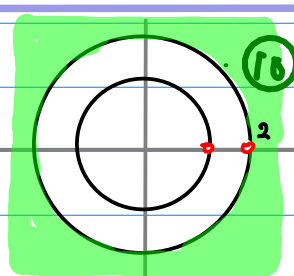
$(\frac{1}{q})^{n+1}$	$(n \geq 0) (n > 0)$	$(\frac{1}{q})^{n+1}$
$(\frac{1}{p})^{n+1}$	$(n < 0) (n \leq 0)$	$(\frac{1}{p})^{n+1}$
$p_1 = 1 = p$		$p_1 = 0.5 = 1/2$
$p_2 = 2 = q$		$p_2 = 1 = 1/p$



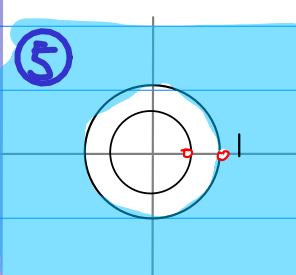
I **4**



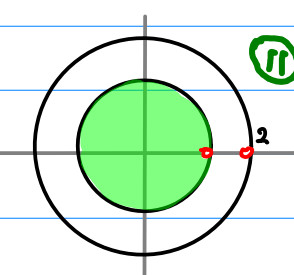
$(p)^{n-1} - (q)^{n-1}$	$(n > 0) (n \geq 0)$	0
0	$(n \leq 0) (n < 0)$	$(p)^{n-1} - (q)^{n-1}$
$p_1 = 0.5 = 1/2$		$p_1 = 1 = p$
$p_2 = 1 = 1/p$		$p_2 = 2 = q$



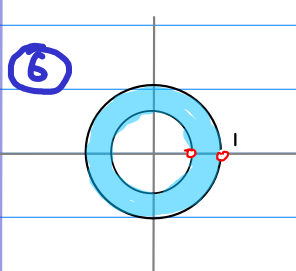
II **5**



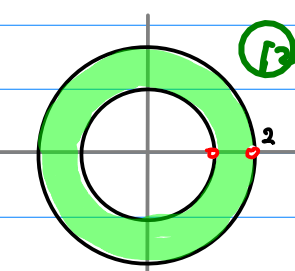
0	$(n > 0) (n \geq 0)$	$(q)^{n-1} - (p)^{n-1}$
$(q)^{n-1} - (p)^{n-1}$	$(n \leq 0) (n < 0)$	0
$p_1 = 0.5 = 1/2$		$p_1 = 1 = p$
$p_2 = 1 = 1/p$		$p_2 = 2 = q$



III **6**



$(p)^{n-1}$	$(n > 0) (n \geq 0)$	$(p)^{n-1}$
$(q)^{n-1}$	$(n \leq 0) (n < 0)$	$(q)^{n-1}$
$p_1 = 0.5 = 1/2$		$p_1 = 1 = p$
$p_2 = 1 = 1/p$		$p_2 = 2 = q$



$$\boxed{n \geq 0}$$

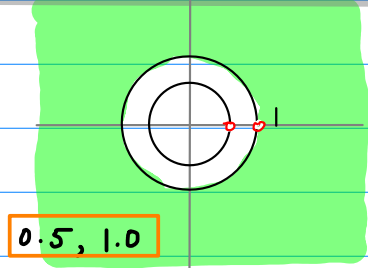
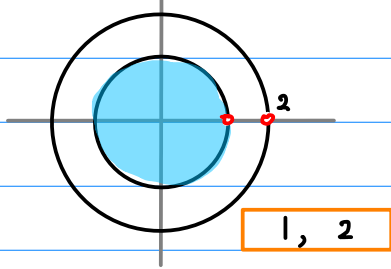
$$\boxed{n < 0}$$

$$\boxed{n > 0}$$

$$\boxed{n \leq 0}$$

$$\left(\frac{1}{q}\right)^{n+1} - \left(\frac{1}{p}\right)^{n+1}$$

$$0$$

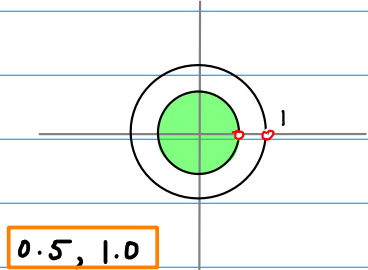
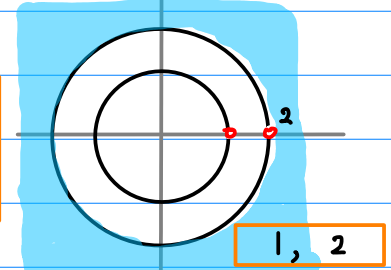


$$0$$

$$\left(\frac{1}{q}\right)^{n+1} - \left(\frac{1}{p}\right)^{n+1}$$

$$0$$

$$\left(\frac{1}{p}\right)^{n+1} - \left(\frac{1}{q}\right)^{n+1}$$

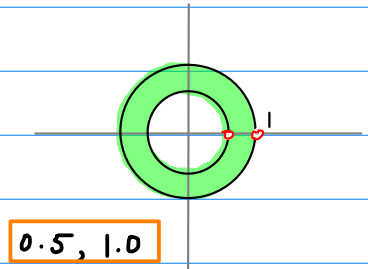
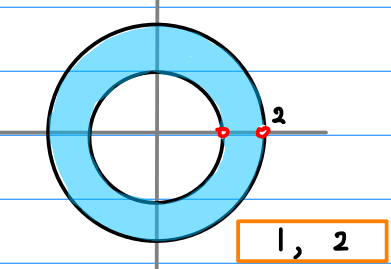


$$\left(\frac{1}{p}\right)^{n+1} - \left(\frac{1}{q}\right)^{n+1}$$

$$0$$

$$\left(\frac{1}{q}\right)^{n+1}$$

$$\left(\frac{1}{p}\right)^{n+1}$$

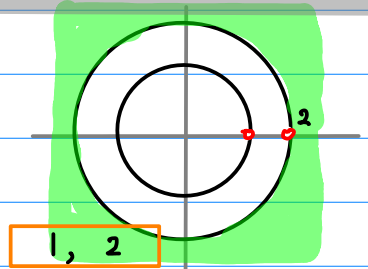
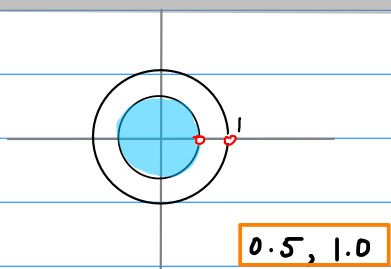


$$\left(\frac{1}{p}\right)^{n+1}$$

$$\left(\frac{1}{q}\right)^{n+1}$$

$$(p)^{n-1} - (q)^{n-1}$$

$$0$$

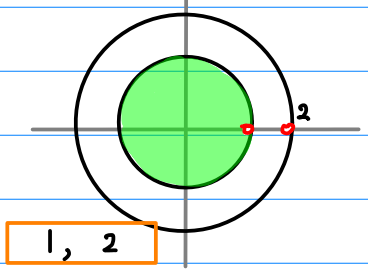
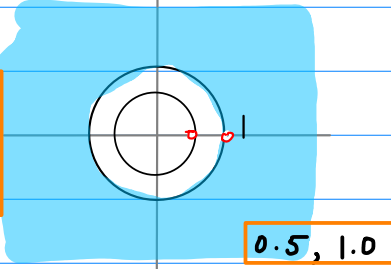


$$0$$

$$(p)^{n-1} - (q)^{n-1}$$

$$0$$

$$(q)^{n-1} - (p)^{n-1}$$

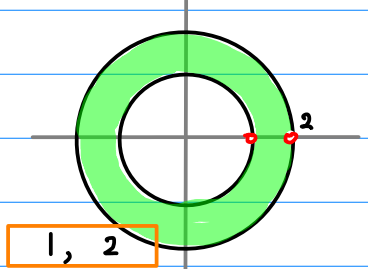
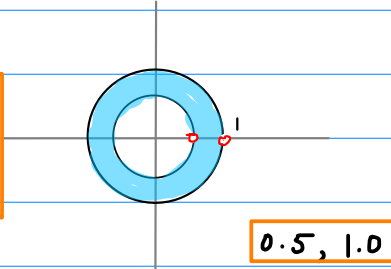


$$(q)^{n-1} - (p)^{n-1}$$

$$0$$

$$(p)^{n-1}$$

$$(q)^{n-1}$$



$$(q)^{n-1}$$

$$(p)^{n-1}$$



