Characteristics of Multiple Random Variables

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June 13, 2020

Based on
Probability, Random Variables and Random Signal Principles,
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Outline

Expected Value
two random variables

Definition

the expected value of $g(x, y)$ is given by

$$\overline{g} = E[g(X, Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x, y) f_{X,Y}(x, y) dxdy$$

where$g(x, y)$is some function of two random variables X and Y

Expected Value

N random variables

Definition

for N random variables $X_1, X_2, ..., X_N$,
the expected value of $g(X_1, X_2, ..., X_N)$ is given by

$$\overline{g} = E[g(X_1, X_2, ..., X_N)]$$

$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} g(x_1, ..., x_N) f_{x_1,...,x_N}(x_1, ..., x_N) dx_1 \cdots dx_N$$

where $g(X_1, X_2, ..., X_N)$ is some function of
N random variables $X_1, X_2, ..., X_N$

Expected Value
N random variables to a single random variable

If $g(X_1, X_2, ..., X_N) = g(X_1)$, then

$$\overline{g} = E[g(X_1, X_2, ..., X_N)]$$

$$= \int_{-\infty}^{\infty} g(x_1) f_{x_1}(x_1) dx_1 = E[g(X_1)]$$

$$\overline{g} = E[g(X_1, X_2, ..., X_N)] = E[g(X_1)]$$

Joint Moments about the Origin

2 random variables

Definition

joint moment about the origin $m_{\{nk\}}$ is defined by

$$m_{\{nk\}} = E[X^n Y^k] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^n y^k f_{X,Y}(x,y)dxdy$$

the second moment $m_{\{11\}} = E[XY]$ is called the correlation $R_{\{XY\}}$ of X and Y

$$R_{\{XY\}} = m_{\{11\}} = E[X^1 Y^1] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^1 y^1 f_{X,Y}(x,y)dxdy$$

Joint Moments about the Origin
N random variables

Definition

For N random variables $X_1, X_2, ..., X_N$, the $(n_1 + n_2 + ... + n_N)$-order joint moment about the origin $m_{\{n_1, n_2, \cdots, n_N\}}$ is defined by

$$m_{\{n_1, n_2, \cdots, n_N\}} = E[X_1^{n_1} X_2^{n_2} \cdots X_N^{n_N}]$$

$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} x_1^{n_1} \cdots x_N^{n_N} f_{X_1 \cdots X_N}(x_1, \cdots, x_N) dx_1 \cdots dx_N$$

Correlation

2 random variables

Definition

the correlation $R_{\{XY\}}$ of X and Y

$$R_{\{XY\}} = m_{\{11\}} = E[X^1 Y^1] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^1 y^1 f_{X,Y}(x,y) dxdy$$

- if the correlation can be written as $R_{\{XY\}} = E[X]E[Y]$, then X and Y are **uncorrelated**
- **statistical independence** of X and Y is sufficient to guaranttee they are **uncorrelated**
- the converse of this statement is not generally true
- If $R_{\{XY\}} = 0$, then X and Y are **orthogonal**

Joint Central Moments
2 random variables

Definition

The joint central moments for two random variables X and Y

$$\mu_{nk} = E[(X - \overline{X})^n (Y - \overline{Y})^k]$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x - \overline{X})^n (y - \overline{Y})^k f_{X,Y}(x, y) dxdy$$

2nd Order Joint Central Moments

2 random variables

Example

the second order moments μ_{20} and μ_{02} are just variances of X and Y

$$\mu_{20} = E[(X - \overline{X})^2]$$

$$\mu_{02} = E[(Y - \overline{Y})^2]$$

the second order moments μ_{11} is called the covariance of X and Y

$$\mu_{11} = E[(X - \overline{X})(Y - \overline{Y})]$$

Covariance
2 random variables

Definition

The covariance of two random variables X and Y

$$C_{XY} = \mu_{11} = E[(X - \overline{X})(Y - \overline{Y})]$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x - \overline{X})(y - \overline{Y}) f_{X,Y}(x, y) dxdy$$

Covariance
2 random variables

Theorem

note that $(x-\overline{X})(y-\overline{Y}) = xy - x\overline{Y} - \overline{X}y + \overline{X}\,\overline{Y}$

$$C_{XY} = R_{XY} - \overline{X}\,\overline{Y} = R_{XY} - E[\overline{X}]E[\overline{Y}]$$

if X and Y are independent or uncorrelated

$$C_{XY} = 0$$

if X and Y are orthogonal

$$C_{XY} = -E[\overline{X}]E[\overline{Y}]$$

if X or Y has a zero mean value

$$C_{XY} = 0$$

Normalized Second Order Moment

2 random variables

Definition

The normalized second order moment

$$\rho = \mu_{11}/\sqrt{\mu_{20}\mu_{02}} = C_{XY}/\sigma_X\sigma_Y$$

$$\rho = E\left[\frac{(X-\overline{X})}{\sigma_X}\frac{(Y-\overline{Y})}{\sigma_Y}\right]$$

ρ is also known as the correlation coefficient of X and Y

$$-1 \leq \rho \leq 1$$

Joint Central Moments

N random variables

Definition

The $(n_1 + n_2 + ... + n_N)$-order joint central moments for N random variables $X_1, X_2, \cdots, X_N$

$$\mu_{n_1 n_2 \cdots n_N} = E[(X_1 - \overline{X_1})^{n_1}(X_2 - \overline{X_2})^{n_2} \cdots (X_N - \overline{X_N})^{n_N}]$$

$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} (x_1 - \overline{X_1})^{n_1} \cdots (x_N - \overline{X_N})^{n_N}$$

$$\cdot f_{X_1,\cdots,X_N}(x_1, \cdots, x_N) dx_1 \cdots dx_N$$