

# Characteristics of Multiple Random Variables

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Based on  
Probability, Random Variables and Random Signal Principles,  
P.Z. Peebles, Jr. and B. Shi

# Outline

- 1 Expected Value of a Function with Multiple Random Variables

# Expected Value

two random variables

## Definition

the expected value of  $g(x, y)$  is given by

$$\bar{g} = E[g(X, Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) f_{X, Y}(x, y) dx dy$$

where  $g(x, y)$  is some function of two random variables  $X$  and  $Y$

# Expected Value

$N$  random variables

## Definition

for  $N$  random variables  $X_1, X_2, \dots, X_N$ ,  
the expected value of  $g(X_1, X_2, \dots, X_N)$  is given by

$$\bar{g} = E[g(X_1, X_2, \dots, X_N)]$$

$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} g(x_1, \dots, x_N) f_{X_1, \dots, X_N}(x_1, \dots, x_N) dx_1 \cdots dx_N$$

where  $g(X_1, X_2, \dots, X_N)$  is some function of  
 $N$  random variables  $X_1, X_2, \dots, X_N$

# Expected Value

$N$  random variables to a single random variable

If  $g(X_1, X_2, \dots, X_N) = g(X_1)$ , then

$$\bar{g} = E[g(X_1, X_2, \dots, X_N)]$$

$$= \int_{-\infty}^{\infty} g(x_1) f_{X_1}(x_1) dx_1 = E[g(X_1)]$$

$$\bar{g} = E[g(X_1, X_2, \dots, X_N)] = E[g(X_1)]$$

# Joint Moments about the Origin

2 random variables

## Definition

joint moment about the origin  $m_{\{nk\}}$  is defined by

$$m_{\{nk\}} = E[X^n Y^k] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^n y^k f_{X,Y}(x,y) dx dy$$

the second moment  $m_{\{11\}} = E[XY]$  is called the correlation  $R_{\{XY\}}$  of  $X$  and  $Y$

$$R_{\{XY\}} = m_{\{11\}} = E[X^1 Y^1] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^1 y^1 f_{X,Y}(x,y) dx dy$$

# Joint Moments about the Origin

$N$  random variables

## Definition

For  $N$  random variables  $X_1, X_2, \dots, X_N$ , the  $(n_1 + n_2 + \dots + n_N)$ -order joint moment about the origin  $m_{\{n_1, n_2, \dots, n_N\}}$  is defined by

$$\begin{aligned} m_{\{n_1, n_2, \dots, n_N\}} &= E[X_1^{n_1} X_2^{n_2} \dots X_N^{n_N}] \\ &= \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} x_1^{n_1} \dots x_N^{n_N} f_{X_1 \dots X_N}(x_1, \dots, x_N) dx_1 \dots dx_N \end{aligned}$$



# Correlation

## 2 random variables

### Definition

the correlation  $R_{\{XY\}}$  of  $X$  and  $Y$

$$R_{\{XY\}} = m_{\{11\}} = E[X^1 Y^1] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^1 y^1 f_{X,Y}(x,y) dx dy$$

- if the correlation can be written as  $R_{\{XY\}} = E[X]E[Y]$ , then  $X$  and  $Y$  are **uncorrelated**
- **statistical independence** of  $X$  and  $Y$  is sufficient to guarantee they are **uncorrelated**
- the converse of this statement is not generally true
- If  $R_{\{XY\}} = 0$ , then  $X$  and  $Y$  are **orthogonal**

# Joint Central Moments

2 random variables

## Definition

The joint central moments for two random variables  $X$  and  $Y$

$$\begin{aligned}\mu_{nk} &= E[(X - \bar{X})^n (Y - \bar{Y})^k] \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x - \bar{X})^n (y - \bar{Y})^k f_{X,Y}(x,y) dx dy\end{aligned}$$

# 2nd Order Joint Central Moments

2 random variables

## Example

the second order moments  $\mu_{20}$  and  $\mu_{02}$  are just variances of  $X$  and  $Y$

$$\mu_{20} = E[(X - \bar{X})^2]$$

$$\mu_{02} = E[(Y - \bar{Y})^2]$$

the second order moments  $\mu_{11}$  is called the covariance of  $X$  and  $Y$

$$\mu_{11} = E[(X - \bar{X})(Y - \bar{Y})]$$

# Covariance

2 random variables

## Definition

The covariance of two random variables  $X$  and  $Y$

$$\begin{aligned}C_{XY} &= \mu_{11} = E[(X - \bar{X})(Y - \bar{Y})] \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x - \bar{X})(y - \bar{Y}) f_{X,Y}(x,y) dx dy\end{aligned}$$

## Covariance

## 2 random variables

## Theorem

note that  $(x - \bar{X})(y - \bar{Y}) = xy - x\bar{Y} - \bar{X}y + \bar{X}\bar{Y}$

$$C_{XY} = R_{XY} - \bar{X}\bar{Y} = R_{XY} - E[\bar{X}]E[\bar{Y}]$$

if  $X$  and  $Y$  are independent or uncorrelated

$$C_{XY} = 0$$

if  $X$  and  $Y$  are orthogonal

$$C_{XY} = -E[\bar{X}]E[\bar{Y}]$$

if  $X$  or  $Y$  has a zero mean value

$$C_{XY} = 0$$

# Normalized Second Order Moment

2 random variables

## Definition

The normalized second order moment

$$\rho = \mu_{11} / \sqrt{\mu_{20}\mu_{02}} = C_{XY} / \sigma_X \sigma_Y$$

$$\rho = E \left[ \frac{(X - \bar{X})}{\sigma_X} \frac{(Y - \bar{Y})}{\sigma_Y} \right]$$

$\rho$  is also known as the correlation coefficient of  $X$  and  $Y$

$$-1 \leq \rho \leq 1$$

# Joint Central Moments

$N$  random variables

## Definition

The  $(n_1 + n_2 + \dots + n_N)$ -order joint central moments for  $N$  random variables  $X_1, X_2, \dots, X_N$

$$\begin{aligned} \mu_{n_1 n_2 \dots n_N} &= E[(X_1 - \bar{X}_1)^{n_1} (X_2 - \bar{X}_2)^{n_2} \dots (X_N - \bar{X}_N)^{n_N}] \\ &= \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} (x_1 - \bar{X}_1)^{n_1} \dots (x_N - \bar{X}_N)^{n_N} \\ &\quad \cdot f_{X_1, \dots, X_N}(x_1, \dots, x_N) dx_1 \dots dx_N \end{aligned}$$





