

Vector Calculus (H.1) Identities

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https://en.wikipedia.org/wiki/Vector_calculus_identities

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = (\nabla \times \mathbf{A}) \cdot \mathbf{B} - \mathbf{A} \cdot (\nabla \times \mathbf{B})$$

$$\begin{aligned}\nabla \times (\mathbf{A} \times \mathbf{B}) &= \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} \\ &= (\nabla \cdot \mathbf{B} + \mathbf{B} \cdot \nabla)\mathbf{A} - (\nabla \cdot \mathbf{A} + \mathbf{A} \cdot \nabla)\mathbf{B} \\ &= \nabla \cdot (\mathbf{B}\mathbf{A}^T) - \nabla \cdot (\mathbf{A}\mathbf{B}^T) \\ &= \nabla \cdot (\mathbf{B}\mathbf{A}^T - \mathbf{A}\mathbf{B}^T)\end{aligned}$$

- The scalar triple product is invariant under a **circular shift** of its three operands (**a**, **b**, **c**):

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$$

- Swapping the positions of the operators without re-ordering the operands leaves the triple product unchanged. This follows from the preceding property and the commutative property of the dot product.

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$$

- Swapping any two of the three operands **negates** the triple product. This follows from the circular-shift property and the **anticommutativity** of the cross product.

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = -\mathbf{a} \cdot (\mathbf{c} \times \mathbf{b})$$

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = -\mathbf{b} \cdot (\mathbf{a} \times \mathbf{c})$$

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = -\mathbf{c} \cdot (\mathbf{b} \times \mathbf{a})$$

- The scalar triple product can also be understood as the **determinant** of the 3×3 matrix (thus also its **inverse**) having the three vectors either as its rows or its columns (a matrix has the same determinant as its **transpose**):

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \det \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}.$$

- If the scalar triple product is equal to zero, then the three vectors **a**, **b**, and **c** are **coplanar**, since the "parallelepiped" defined by them would be flat and have no volume.
- If any two vectors in the triple scalar product are equal, then its value is zero:

$$\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{a}) = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{b}) = \mathbf{a} \cdot (\mathbf{a} \times \mathbf{a}) = 0$$

- Moreover,

$$[\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})]\mathbf{a} = (\mathbf{a} \times \mathbf{b}) \times (\mathbf{a} \times \mathbf{c})$$

- The **simple product** of two triple products (or the square of a triple product), may be expanded in terms of dot products:^[1]

$$((\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}) ((\mathbf{d} \times \mathbf{e}) \cdot \mathbf{f}) = \det \left[\begin{pmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \end{pmatrix} \cdot (\mathbf{d} \quad \mathbf{e} \quad \mathbf{f}) \right] = \det \left[\begin{matrix} \mathbf{a} & \mathbf{d} & \mathbf{e} & \mathbf{f} \\ \mathbf{b} & \mathbf{d} & \mathbf{e} & \mathbf{f} \\ \mathbf{c} & \mathbf{d} & \mathbf{e} & \mathbf{f} \end{matrix} \right]$$

https://en.wikipedia.org/wiki/Triple_product

The **vector triple product** is defined as the **cross product** of one vector with the cross product of the other two. The following relationship holds:

$$\ast \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}).$$

This is known as **triple product expansion**, or **Lagrange's formula**,^{[2][3]} although the latter name is also used for **several other formulae**. Its right hand side can be remembered by using the **mnemonic** "BAC – CAB", provided one keeps in mind which vectors are dotted together. A proof is provided **below**.

Since the cross product is anticommutative, this formula may also be written (up to permutation of the letters) as:

$$\ast (\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = -\mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = -(\mathbf{c} \cdot \mathbf{b})\mathbf{a} + (\mathbf{c} \cdot \mathbf{a})\mathbf{b}$$

From Lagrange's formula it follows that the vector triple product satisfies:

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = 0$$

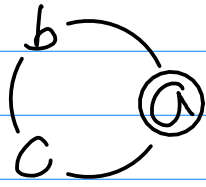
which is the **Jacobi identity** for the cross product. Another useful formula follows:

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) - \mathbf{b} \times (\mathbf{a} \times \mathbf{c})$$

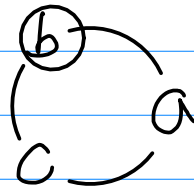
These formulas are very useful in simplifying vector calculations in **physics**. A related identity regarding **gradients** and useful in **vector calculus** is Lagrange's formula of vector cross-product identity:^[4]

$$\nabla \times (\nabla \times \mathbf{f}) = \nabla(\nabla \cdot \mathbf{f}) - (\nabla \cdot \nabla)\mathbf{f}$$

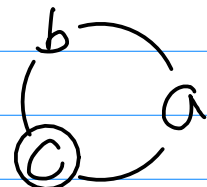
$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) - \mathbf{b} \times (\mathbf{a} \times \mathbf{c})$$



$$\vec{a} \times (\vec{b} \times \vec{c})$$



$$\vec{b} \times (\vec{c} \times \vec{a})$$

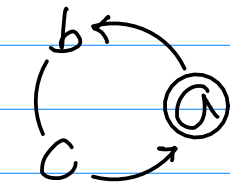
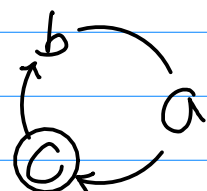
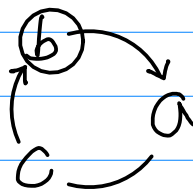
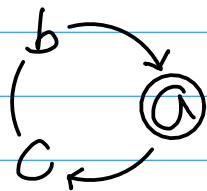


$$\vec{c} \times (\vec{a} \times \vec{b})$$

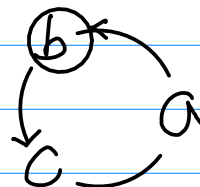
$$b(a \cdot c)$$

$$c(b \cdot a)$$

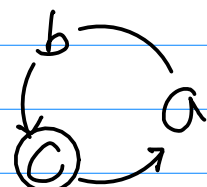
$$a(c \cdot b)$$



$$-c(a \cdot b)$$



$$-a(b \cdot c)$$



$$-b(c \cdot a)$$

$$\vec{a} \times (\vec{b} \times \vec{c}) + \vec{b} \times (\vec{c} \times \vec{a}) + \vec{c} \times (\vec{a} \times \vec{b}) = 0$$

||

$$b(a \cdot c)$$

||

$$c(b \cdot a)$$

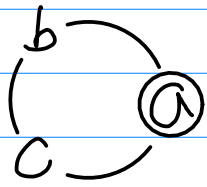
||

$$a(c \cdot b)$$

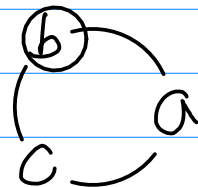
$$-c(a \cdot b)$$

$$-a(b \cdot c)$$

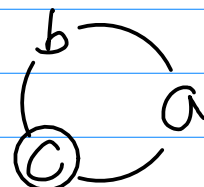
$$-b(c \cdot a)$$



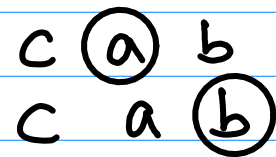
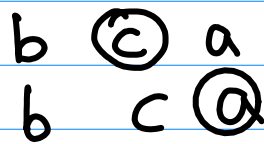
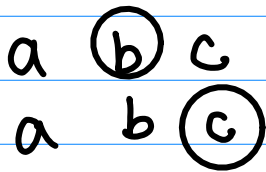
$$\vec{a} \times (\vec{b} \times \vec{c})$$



$$\vec{b} \times (\vec{c} \times \vec{a})$$



$$\vec{c} \times (\vec{a} \times \vec{b})$$



$$b(a \cdot c) - c(a \cdot b)$$

$$c(b \cdot a) - a(b \cdot c)$$

$$a(c \cdot b) - b(c \cdot a)$$

$$\boxed{\vec{a} \times (\vec{b} \times \vec{c})} + \boxed{\vec{b} \times (\vec{c} \times \vec{a})} + \boxed{\vec{c} \times (\vec{a} \times \vec{b})} = 0$$

$$\begin{array}{ccc} \parallel & \parallel & \parallel \\ \boxed{\begin{array}{c} b(a \cdot c) \\ -c(a \cdot b) \end{array}} & \boxed{\begin{array}{c} c(b \cdot a) \\ -a(b \cdot c) \end{array}} & \boxed{\begin{array}{c} a(c \cdot b) \\ -b(c \cdot a) \end{array}} \end{array}$$

$$\boxed{\vec{a} \times (\vec{b} \times \vec{c})} - \boxed{\vec{b} \times (\vec{c} \times \vec{a})} = \boxed{\vec{c} \times (\vec{a} \times \vec{b})}$$

$$\begin{array}{ccc} \parallel & \parallel & \parallel \\ \boxed{\begin{array}{c} b(a \cdot c) \\ -c(a \cdot b) \end{array}} - \boxed{\begin{array}{c} c(b \cdot a) \\ -a(b \cdot c) \end{array}} & = & \boxed{\begin{array}{c} a(c \cdot b) \\ -b(c \cdot a) \end{array}} \end{array}$$

$$\begin{aligned} \nabla \times (\mathbf{A} \times \mathbf{B}) &= \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} \\ &= (\nabla \cdot \mathbf{B} + \mathbf{B} \cdot \nabla)\mathbf{A} - (\nabla \cdot \mathbf{A} + \mathbf{A} \cdot \nabla)\mathbf{B} \\ &= \nabla \cdot (\mathbf{B}\mathbf{A}^T) - \nabla \cdot (\mathbf{A}\mathbf{B}^T) \\ &= \nabla \cdot (\mathbf{B}\mathbf{A}^T - \mathbf{A}\mathbf{B}^T) \end{aligned}$$

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) - \mathbf{b} \times (\mathbf{a} \times \mathbf{c})$$

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = -\mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = -(\mathbf{c} \cdot \mathbf{b})\mathbf{a} + (\mathbf{c} \cdot \mathbf{a})\mathbf{b}$$

$$\begin{aligned}\nabla \times (\mathbf{A} \times \mathbf{B}) &= \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} \\ &= (\nabla \cdot \mathbf{B} + \mathbf{B} \cdot \nabla)\mathbf{A} - (\nabla \cdot \mathbf{A} + \mathbf{A} \cdot \nabla)\mathbf{B} \\ &= \nabla \cdot (\mathbf{B}\mathbf{A}^T) - \nabla \cdot (\mathbf{A}\mathbf{B}^T) \\ &= \nabla \cdot (\mathbf{B}\mathbf{A}^T - \mathbf{A}\mathbf{B}^T)\end{aligned}$$

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = \nabla \times (\mathbf{A} \times \mathbf{B}) + \nabla \times (\mathbf{A} \times \mathbf{B})$$

$$\begin{aligned}\nabla \times (\mathbf{A} \times \mathbf{B}) &= \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) \\ &= (\mathbf{B} \cdot \nabla)\mathbf{A} - \mathbf{B}(\nabla \cdot \mathbf{A})\end{aligned}$$

$$\begin{aligned}\nabla \times (\mathbf{A} \times \mathbf{B}) &= \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) \\ &= (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B}\end{aligned}$$

$$\begin{aligned}\nabla \times (\mathbf{A} \times \mathbf{B}) &= \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) \\ &+ (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B}\end{aligned}$$

Directional Derivatives

$$\begin{aligned} \mathbf{B} \cdot \nabla &= \langle B_x, B_y, B_z \rangle \cdot \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \\ &= B_x \frac{\partial}{\partial x} + B_y \frac{\partial}{\partial y} + B_z \frac{\partial}{\partial z} \end{aligned}$$

$$\times \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z}$$

$$\begin{aligned} (\mathbf{B} \cdot \nabla) f &= \langle B_x, B_y, B_z \rangle \cdot \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle f \\ &= B_x \frac{\partial f}{\partial x} + B_y \frac{\partial f}{\partial y} + B_z \frac{\partial f}{\partial z} \\ &= \mathbf{B} \cdot (\nabla f) \quad f_{\mathbf{B}} \\ &\quad \text{grad} \end{aligned}$$

$$\vec{a} = \langle a, b, c \rangle$$

$$\begin{aligned} D_{\vec{a}} f(x, y, z) &= a \cdot f_x(x, y, z) + b \cdot f_y(x, y, z) + c \cdot f_z(x, y, z) \\ &= \langle f_x, f_y, f_z \rangle \cdot \langle a, b, c \rangle \end{aligned}$$

$$\begin{aligned} \mathbf{B} \cdot \nabla &= B_x, B_y, B_z \\ &= B_x \frac{\partial}{\partial x} + B_y \frac{\partial}{\partial y} + B_z \frac{\partial}{\partial z} \end{aligned}$$

$$\cancel{\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z}}$$

$$\nabla \square = \frac{\partial \square}{\partial x} \vec{i} + \frac{\partial \square}{\partial y} \vec{j} + \frac{\partial \square}{\partial z} \vec{k}$$

Op

$$\begin{aligned} \mathbf{B} \cdot \nabla &= (B_x \vec{i} + B_y \vec{j} + B_z \vec{k}) \cdot \left(\frac{\partial \square}{\partial x} \vec{i} + \frac{\partial \square}{\partial y} \vec{j} + \frac{\partial \square}{\partial z} \vec{k} \right) \\ &= B_x \frac{\partial \square}{\partial x} + B_y \frac{\partial \square}{\partial y} + B_z \frac{\partial \square}{\partial z} \end{aligned}$$

Scalar

$$\begin{aligned} \nabla \cdot \mathbf{B} &= \left(\frac{\partial \square}{\partial x} \vec{i} + \frac{\partial \square}{\partial y} \vec{j} + \frac{\partial \square}{\partial z} \vec{k} \right) \cdot (B_x \vec{i} + B_y \vec{j} + B_z \vec{k}) \\ &= \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} \end{aligned}$$

Dot Del Operator ∇

Op

$$\mathbf{B} \cdot \nabla = B_x \frac{\partial}{\partial x} + B_y \frac{\partial}{\partial y} + B_z \frac{\partial}{\partial z}$$

Scalar

$$\nabla \cdot \mathbf{B} = \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z}$$

$$\mathbf{B} \cdot \nabla \neq \nabla \cdot \mathbf{B}$$

$$\mathbf{B} \cdot \nabla = \nabla \cdot \mathbf{B}$$

dot del