CLTI System Response (4B)

Please send corrections (or suggestions) to youngwlim@hotmail.com.

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State Definition

The **state** of a system, $y(t_0)$, at time $t = t_0$ is the information at t_0 that together with the input $x(t)$, $t \geq t_0$, determines uniquely the system's output $y(t)$, for all $t \geq t_0$.

Integrator Example

$$y(t) = \int_{-\infty}^{t} x(t)\,dt$$

$y(t)$ cannot be calculated without knowing the effect of $x(t)$ for all past t

$$y(t) = \int_{-\infty}^{t_0} x(t)\,dt + \int_{t_0}^{t} x(t)\,dt$$

$$= x(t_0) + \int_{t_0}^{t} x(t)\,dt$$

The state summarizes the effect of the past input on the future output

$$y(t) \qquad t \geq t_0$$

Inputs to a system

the input to a system consists of an input & state pair

$[\,y(t_0),\ x(t)\,],\ t \geq t_0 \ \rightarrow y(t),\ t \geq t_0.$

$[\,0+y(t_0),\ x(t)+0\,],\ t \geq t_0 \ \rightarrow y(t),\ t \geq t_0.$

$[\,0,\ x(t)\,],\ t \geq t_0 \ \rightarrow y_{zs}(t),\ t \geq t_0.$ Zero State Response

$[\,y(t_0),\ 0\,],\ t \geq t_0 \ \rightarrow y_{zi}(t),\ t \geq t_0.$ Zero Input Response

$[\,0,\ \delta(t)\,],\ t \geq t_0 \ \rightarrow \mathrm{h}(t),\ t \geq t_0.$ Impulse Response

the output that occurs
when all inputs from t=-∞ to t=0 were held at zero,
ie, the output was at its zero state
and then an impulse function is applied

State and Convolution

$$y(t) = \int_{-\infty}^{t} x(\tau)h(t-\tau)\,d\tau$$

$$y(t) = \int_{-\infty}^{0} x(\tau)h(t-\tau)\,d\tau + \int_{0}^{t} x(\tau)h(t-\tau)\,d\tau$$

ZSR

$-\infty < t < +\infty$

causal h(t)

$\tau \le t$

$y_{x-}(t)$ — response due to x(t)u(-t)

$y_{x+}(t)$ — response due to x(t)u(t)

$$y_{x-}(t) = \int_{-\infty}^{0} x(\tau)h(t-\tau)\,d\tau = y^{-}(t) + y^{+}(t)$$

ZIR

$(t < 0)$ $\quad (t \ge 0)$

response before t=0
due to x(t)u(-t)

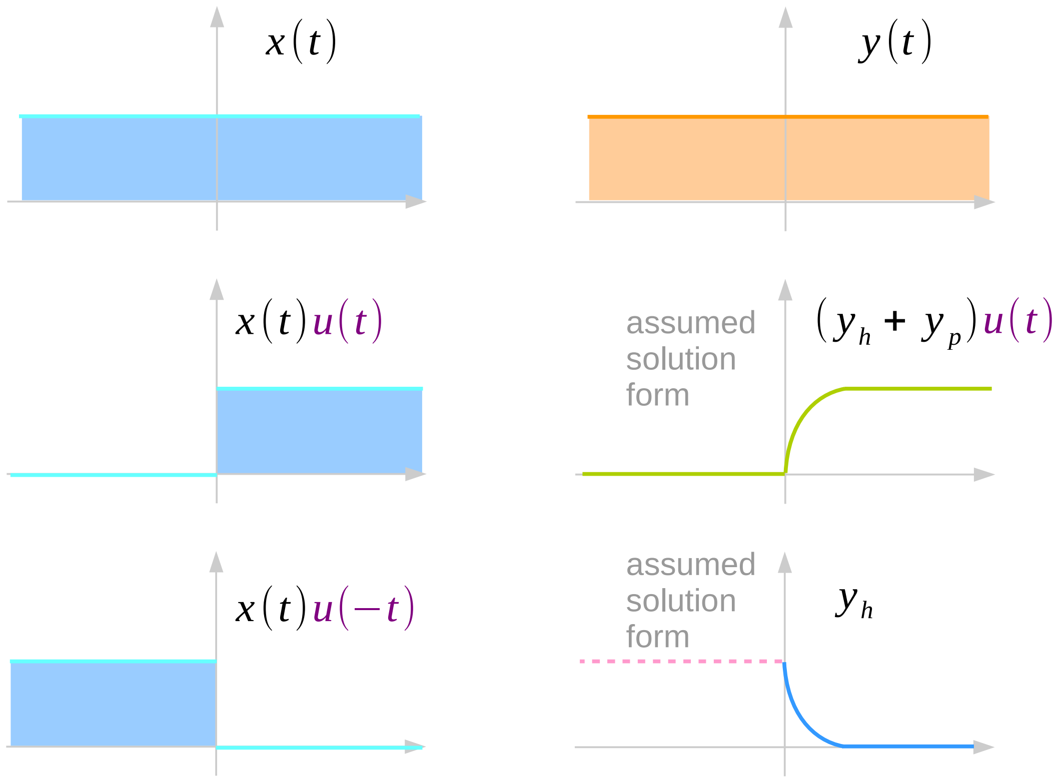
$$y^{-}(t) = y_{x-}(t)\cdot u(-t) = \int_{-\infty}^{0} x(\tau)h(t-\tau)\,d\tau \quad (t < 0)$$

response after t=0
due to x(t)u(-t)

$$y^{+}(t) = y_{x-}(t)u(+t) = \int_{-\infty}^{0} x(\tau)h(t-\tau)\,d\tau \quad (t \ge 0)$$

ZIR

Responses after $t = 0$



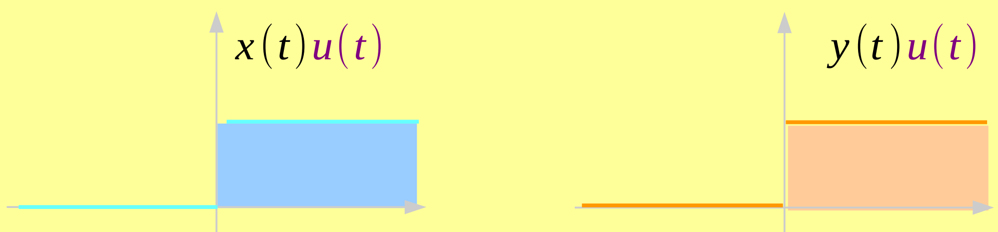
$$y(t) = \int_{-\infty}^{t} x(\tau)h(t-\tau)\,d\tau$$

ZSR

$$y_{x+}(t) = \int_{0}^{t} x(\tau)h(t-\tau)\,d\tau$$

ZIR

$$y^{+}(t) = \left(\int_{-\infty}^{0} x(\tau)h(t-\tau)\,d\tau\right)\cdot u(t)$$



ZSR + ZIR

Natural + Forced

- Initial Conditions and System Responses
  - ZIR & Initial Conditions
  - ZSR & Initial Conditions

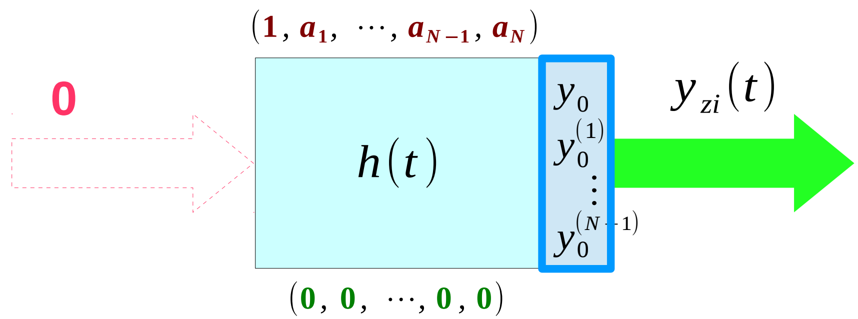
ZIR & Initial Conditions

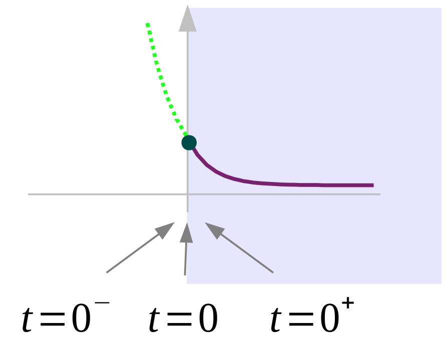
$$(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N)\, y(t) = (b_0 D^N + b_1 D^{N-1} + \cdots + b_{N-1} D + b_N)\, x(t)$$

0

Non-zero initial conditions

$$\{y^{(N-1)}(0^-),\ y^{(N-2)}(0^-),\ \cdots,\ y^{(1)}(0^-),\ y^{(0)}(0^-)\}$$





Only $y_{zi}(t)$ is present at t=0-
and $y_{zi}(t)$ exists for $t \geq 0$

Application of the input
$x(t) = 0$ at $t = 0$
does not affect $y_{zi}(t)$

inductor current
capacitor voltage

continuous (no jumps)

non-zero initial conditions

$\exists\, i,\quad k_i \neq 0$

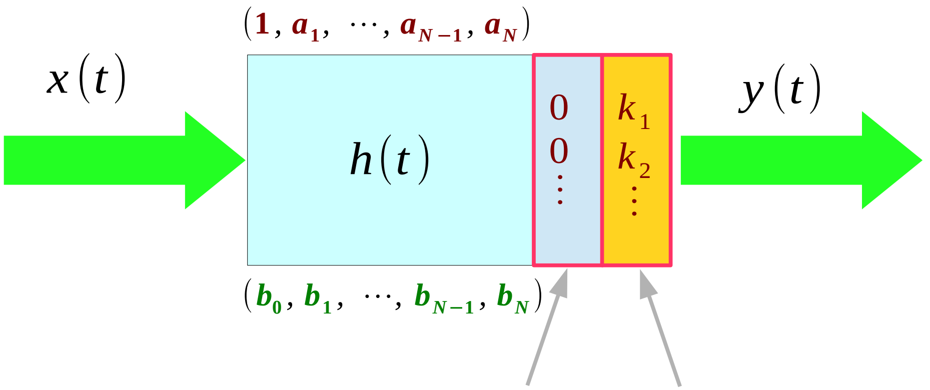
$$\begin{aligned}
y^{(N-1)}(0^-) &= y^{(N-1)}(0) = y^{(N-1)}(0^+) = k_{N-1} \\
y^{(N-2)}(0^-) &= y^{(N-2)}(0) = y^{(N-2)}(0^+) = k_{N-2} \\
&\vdots \\
y^{(1)}(0^-) &= y^{(1)}(0) = y^{(1)}(0^+) = k_1 \\
y(0^-) &= y(0) = y(0^+) = k_0
\end{aligned}$$

ZSR & Initial Conditions

$$(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N)\, y(t) \;=\; (b_0 D^N + b_1 D^{N-1} + \cdots + b_{N-1} D + b_N)\, x(t)$$

All initial conditions are zero

$$y^{(N-1)}(0^-) \;=\; y^{(N-2)}(0^-) \;=\; \cdots \;=\; y^{(1)}(0^-) \;=\; y^{(0)}(0^-) \;=\; 0$$



zero state
at t=0^-,

non-zero
initial
conditions
at t=0^+,

possible discontinuity
(finite jumps)

Coefficients of a ZSR

$$(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N)\, y(t) = (b_0 D^N + b_1 D^{N-1} + \cdots + b_{N-1} D + b_N)\, x(t)$$

All initial conditions are zero

$$y^{(N-1)}(0^-) = y^{(N-2)}(0^-) = \cdots = y^{(1)}(0^-) = y^{(0)}(0^-) = 0$$

coefficients in h(t)

ZSR in a convolution form

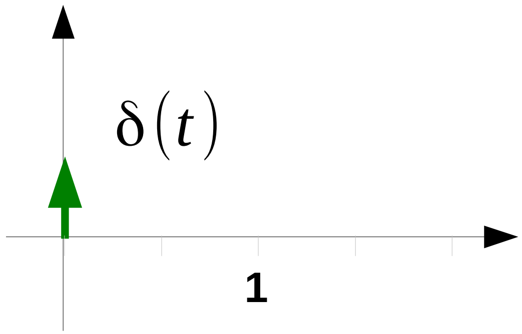
$$y_{zs}(t) = x(t) * h(t) = x(t) * \left(\sum_i d_i e^{\lambda_i t} + b_0 \delta(t) \right)$$

coefficients in $y_h(t)$

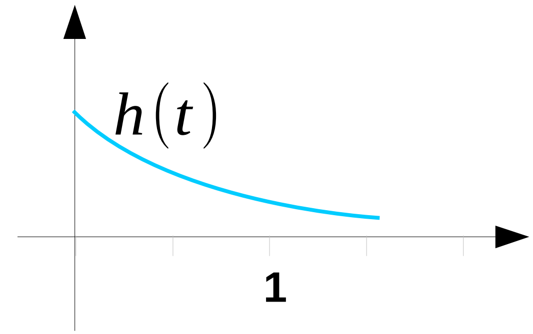
ZSR in a unit step form

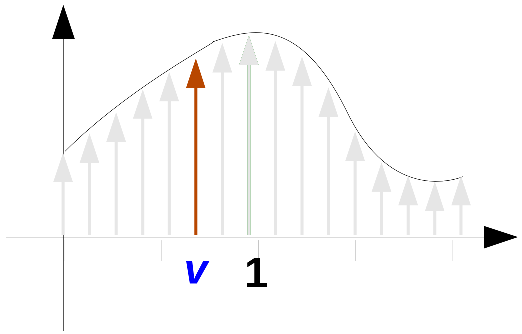
$$y_{zs}(t) = u(t) \cdot \{y_h(t) + y_p(t)\} = u(t) \cdot \left(\sum_i k_i e^{\lambda_i t} + y_p(t) \right)$$

ZSR in a convolution form



causal signal $x(t)$ $(t \geq 0)$ → causal system $h(t)$ $(t \geq 0)$ → causal signal $y(t)$ $(t \geq 0)$

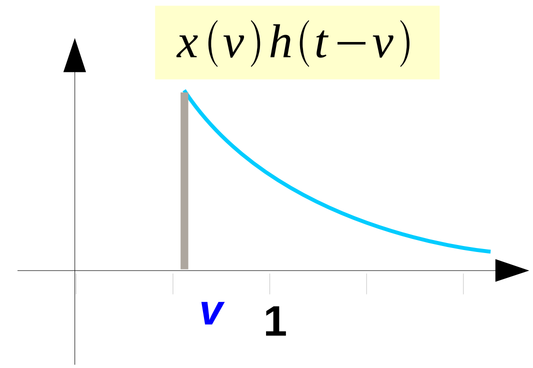


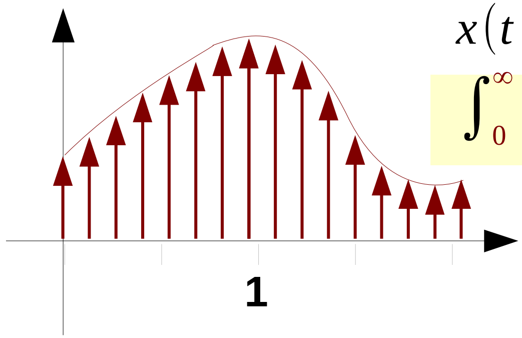


$\delta(t-v)$ → $h(t-v)$

$x(v)\,\delta(t-v)$ → $x(v)\,h(t-v)$

input value at time v
→ x(v)
delayed impulse response
→ x(v) h(t - v)

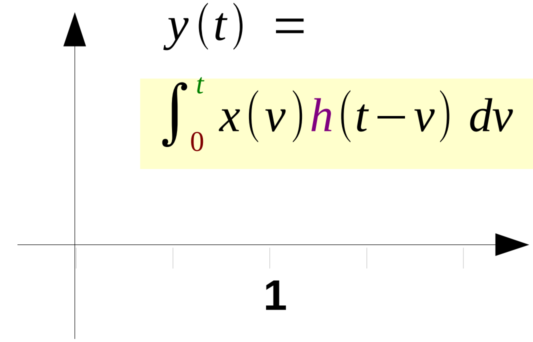




$$x(t) = \int_0^{\infty} x(v)\delta(t-v)\,dv$$

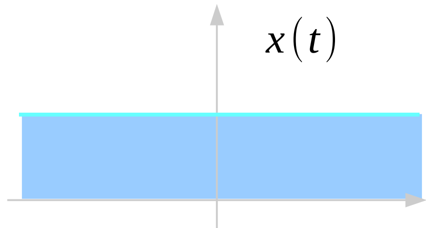
→

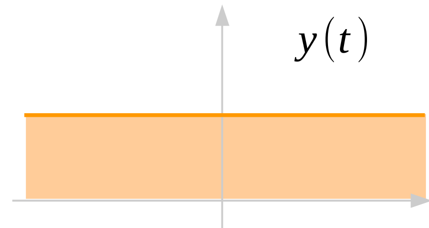
$$y(t) = \int_0^{t} x(v)h(t-v)\,dv$$



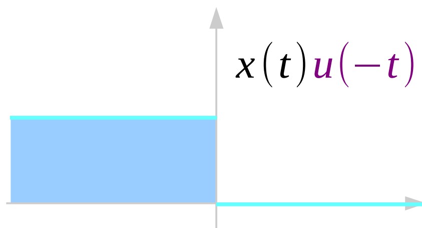
Zero State (initially at rest at $t=0^-$)

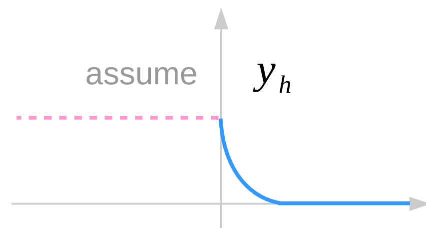
ZSR in a unit step form





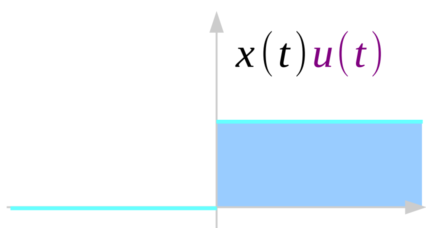
$$y(t) = \int_{-\infty}^{t} x(\tau)h(t-\tau)\,d\tau$$

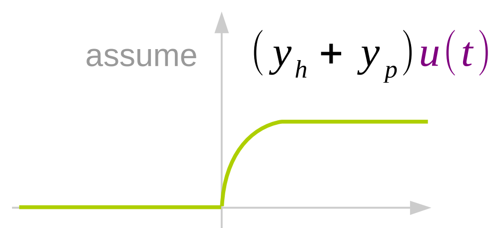




ZIR

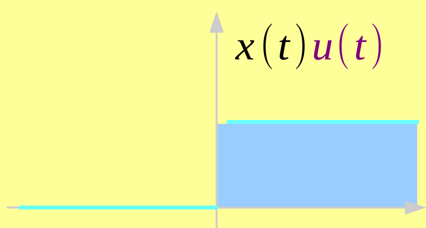
$t > 0$ part of the response to $x(t)u(-t)$

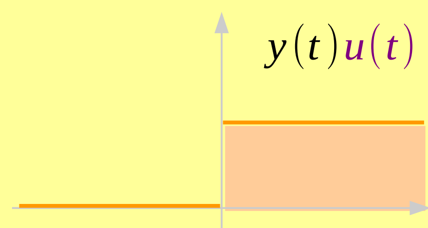




ZSR

$t > 0$ part of the response after $t = 0$ to $x(t)u(t)$





ZSR + ZIR

Natural + Forced

- Finding ZSR in a convolution form
  - Impulse Matching
- Finding ZSR in a unit step form
  - Direct Inspection
  - Balancing Singularities

Finding ZSR in a convolution from

ZSR in a convolution form

$$y_{zs}(t) = x(t) * h(t) = x(t) * \left(\sum_i d_i e^{\lambda_i t} + b_0 \delta(t) \right)$$

$$\begin{aligned} h^{(N-1)}(0^-) &= 0 \\ h^{(N-2)}(0^-) &= 0 \\ \vdots \quad & \quad \vdots \\ h^{(1)}(0^-) &= 0 \\ h(0^-) &= 0 \end{aligned}$$

initially at rest

$$\begin{aligned} h^{(N)}(0) &= f_{N-1}(k_i, \delta^{(i)}) \\ h^{(N-1)}(0) &= f_{N-2}(k_i, \delta^{(i)}) \\ \vdots \quad & \quad \vdots \\ h^{(2)}(0) &= f_1(k_i, \delta^{(i)}) \\ h^{(1)}(0) &= f_0(k_i, \delta^{(i)}) \end{aligned}$$

assumed finite jumps

$$\begin{aligned} h^{(N-1)}(0^+) &= k_{N-1} \\ h^{(N-2)}(0^+) &= k_{N-2} \\ \vdots \quad & \quad \vdots \\ h^{(1)}(0^+) &= k_1 \\ h(0^+) &= k_0 \end{aligned}$$

non-zero initial conditions

$$\exists\, i, \quad k_i \neq 0$$

$$(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N)\, h(t) = (b_0 D^N + b_1 D^{N-1} + \cdots + b_{N-1} D + b_N)\, \delta(t)$$

Impulse Matching

Finding ZSR in a unit step form

ZSR in a unit step form

$$y_{zs}(t) = u(t) \cdot \{y_h(t) + y_p(t)\} = u(t) \cdot \left(\sum_i k_i e^{\lambda_i t} + y_p(t)\right)$$

Homogeneous Equation

$$(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N)\, y_h(t) = 0$$

$$(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N)\, y_p(t) = (b_0 D^N + b_1 D^{N-1} + \cdots + b_{N-1} D + b_N)\, x(t)$$

Non-homogeneous Equation

Homogeneous Solution y_h

$$y_h(t) = \sum_i k_i e^{\lambda_i t}$$

Particular Solution y_p

$$y_p(t) = 0 \quad \Leftarrow \quad x(t) = \delta(t)$$

$$y_p(t) = \beta \quad \Leftarrow \quad x(t) = k\, u(t)$$

$$y_p(t) = \beta_1 t + \beta_0 \quad \Leftarrow \quad x(t) = t\, u(t)$$

$$y_p(t) = \beta e^{\zeta t} \quad \Leftarrow \quad x(t) = e^{\zeta t} u(t) \quad \zeta \neq \lambda_i$$

Associated Linear ODE's

$$y''(t) + 3y'(t) + 2y(t) = x(t)$$

$$\begin{cases} y'(0^-) = \dfrac{3}{2} \\ y(0^-) = 1 \end{cases}$$

$$x(t) = u(t)$$

$$x(t) = e^{-3t}u(t)$$

Associated Linear ODE when $x(t) = u(t)$

$$y''(t) + 3y'(t) + 2y(t) = 1$$

$$y(t) = y_n(t) + y_p(t)$$

Associated Linear ODE when $x(t) = e^{-3t}u(t)$

$$y''(t) + 3y'(t) + 2y(t) = e^{-3t}$$

$$y(t) = y_n(t) + y_p(t)$$

Find the general solution of the Linear ODE

$$y(t) = y_h(t) + y_p(t)$$

Direct Inspection
Balancing Singularities

Natural Response +
Forced Response

all determined coefficients

$$y_h(t) \Rightarrow y_n(t)$$

Methods of finding coefficients in y_h

$$y''(t) + 3y'(t) + 2y(t) = x(t)$$

$$\begin{cases} y'(0^-) = \dfrac{3}{2} \\ y(0^-) = 1 \end{cases}$$

$x(t) = u(t)$

$x(t) = e^{-3t}u(t)$

$$y''(t) + 3y'(t) + 2y(t) = u(t)$$

$x(t) = u(t)$

$$y''(t) + 3y'(t) + 2y(t) = e^{-3t}u(t)$$

$x(t) = e^{-3t}u(t)$

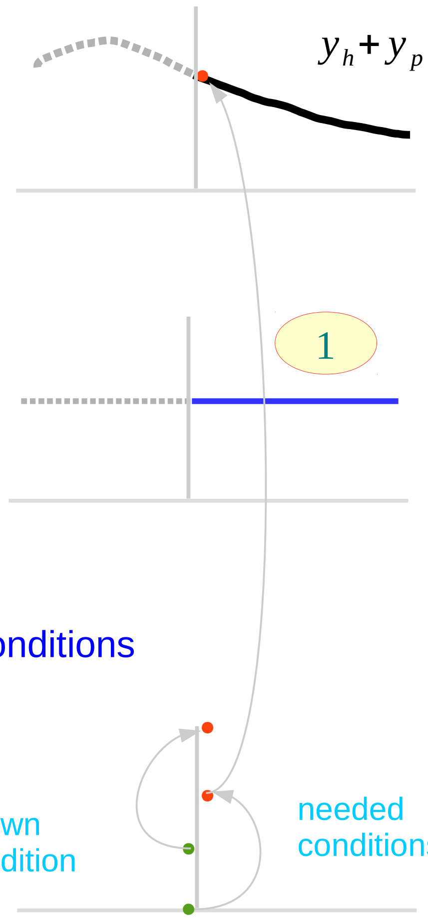
Direct Inspection	finding any jumps in initial conditions
Balancing Singularities	matching derivatives of singularity functions in the both sides

Comparisons of the two solutions

OUTPUT

INPUT

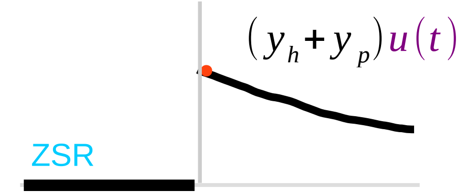
Initial Conditions



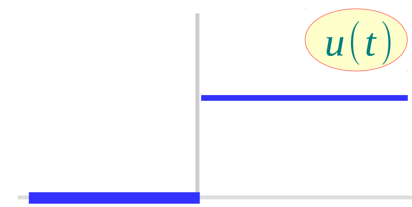
zero state condition

Direct Inspection

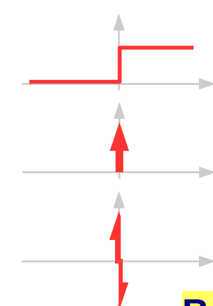
OUTPUT



INPUT



Existence of derivatives of delta functions at t = 0

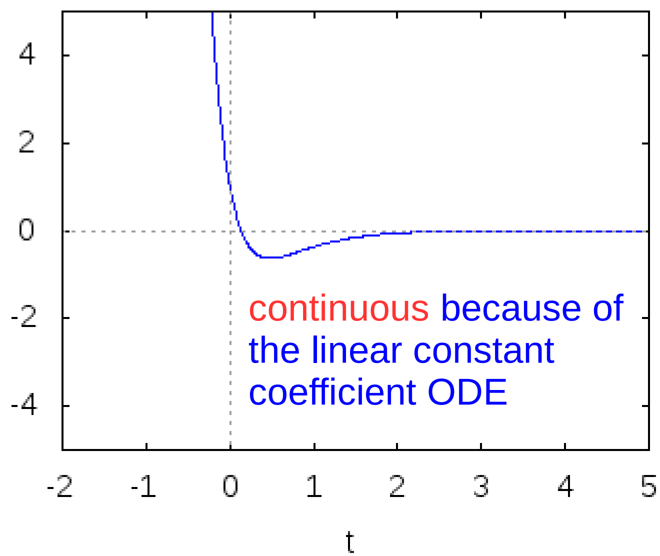


determines the coefficients in y_h

Balancing Singularities

Solution's continuity at t=0

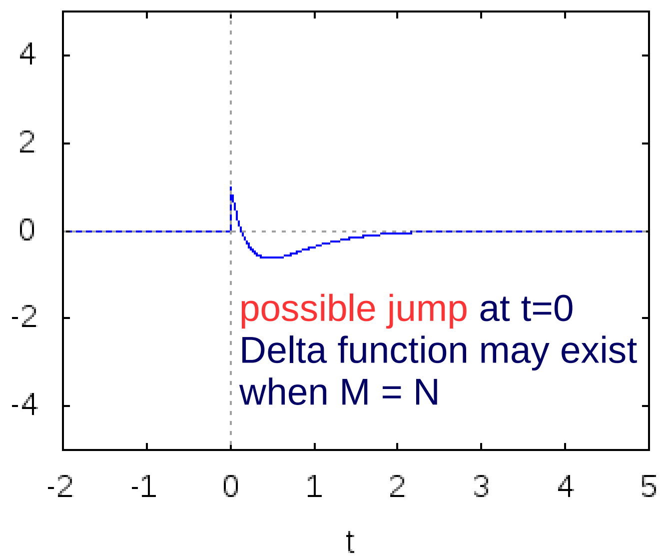
$$y(t) = y_n(t) + y_p(t)$$



- the general solution of the Linear ODE
- take only the portion where t > 0
- ignore the portion where t < 0,

Direct Inspection

$$y(t) = (y_n(t) + y_p(t))u(t)$$



- Explicitly assume that the solution is
- zero for t < 0, and
- (y_h+y_p) for t >0

Balancing Singularities

Direct Inspection

Not limited to finding ZSR only

Can be used in finding Total Response

zero state assumed

~~$y(t) = \{y_h(t) + y_p(t)\}\, u(t)$~~

$$y(t) = y_h(t) + y_p(t)$$

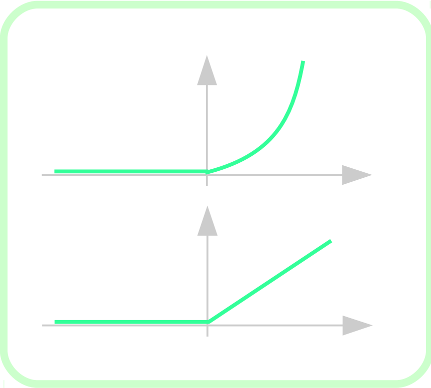
Known I.C.

$$y_n^{(N-1)}(0^-),\; y_n^{(N-2)}(0^-),\; \cdots,\; y_n^{(1)}(0^-),\; y_n(0^-)$$

Needed I.C.

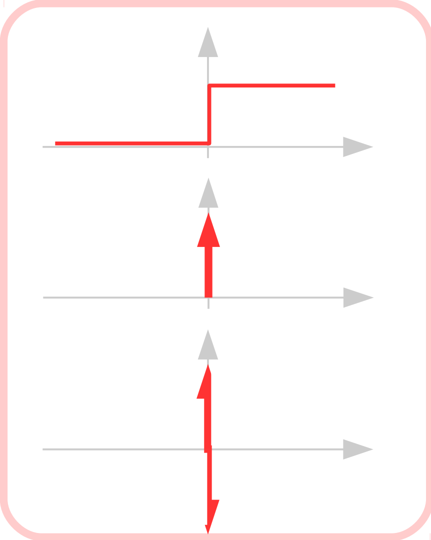
$$y_n^{(N-1)}(0^+),\; y_n^{(N-2)}(0^+),\; \cdots,\; y_n^{(1)}(0^+),\; y_n(0^+)$$

Converting Initial Conditions by Direct Inspection



Continuous Singularities At t=0

$$y^{(n)}(0^+) = y^{(n)}(0^-)$$



finite jumps in the initial conditions

$$y^{(n)}(0^+) = y^{(n)}(0^-) + k$$

Discontinuous Singularities At t=0

Balancing Singularities

$$y_{zs}(t) = u(t) \cdot \{y_h(t) + y_p(t)\} = u(t) \cdot \left(\sum_i k_i e^{\lambda_i t} + y_p(t)\right)$$

Assumed ZSR and its derivatives

$$y_{zs}(t) = \{y_h(t) + y_p(t)\}\, u(t)$$

$$y'_{zs}(t) = \quad \delta(t) + \quad u(t)$$

$$y''_{zs}(t) = \quad \delta'(t) + \quad \delta(t) + \quad u(t)$$

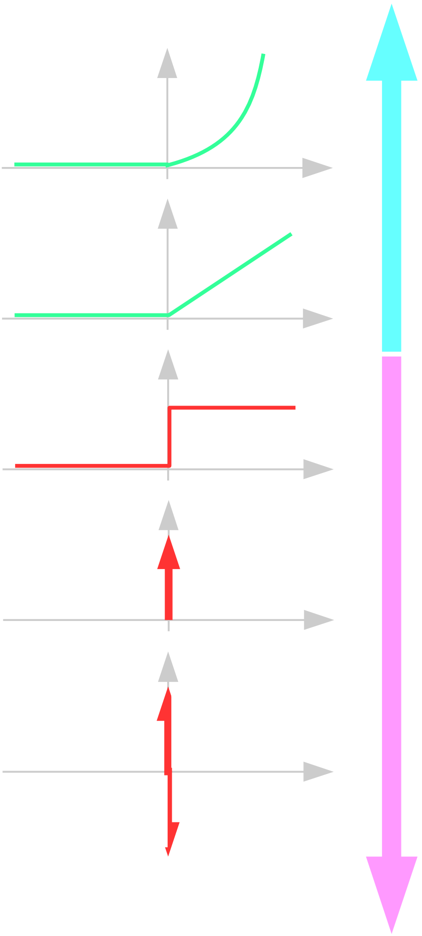
$$f(t)\,\delta(t) = f(0)\,\delta(t)$$

$$(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N)\, y(t) = (b_0 D^N + b_1 D^{N-1} + \cdots + b_{N-1} D + b_N)\, x(t)$$

Balancing Singularities

Singularity Functions and Generalized Derivatives

A singularity of order **-3**	$\frac{t^2}{2} u(t)$	$\delta^{(-3)}(t)$
A singularity of order **-2**	$t\, u(t)$	$\delta^{(-2)}(t)$
A singularity of order **-1**	$u(t)$	$\delta^{(-1)}(t)$
A singularity of order **zero**	$\delta(t)$	$\delta^{(0)}(t)$
A singularity of order **one**	$\dot{\delta}(t)$	$\delta^{(1)}(t)$
A singularity of order **two**	$\ddot{\delta}(t)$	$\delta^{(2)}(t)$



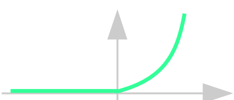
continuous at t=0

discontinuous at t=0

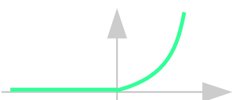
Direct Inspection (1)

$$y^{(N)}(t) + a_1 y^{(N-1)}(t) + a_2 y^{(N-2)}(t) + \cdots + a_{N-1} y^{(1)}(t) + a_N y(t) = x(t)$$

$t^2 u(t)$ $t^3 u(t)$ $t^4 u(t)$ $t^2 u(t)$ $x(t) = t^2$



all continuous at t=0



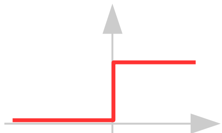
$$\begin{aligned}
y^{(N-1)}(0^+) &= y^{(N-1)}(0^-) \\
y^{(N-2)}(0^+) &= y^{(N-2)}(0^-) \\
&\vdots \\
y^{(1)}(0^+) &= y^{(1)}(0^-) \\
y(0^+) &= y(0^-)
\end{aligned}$$

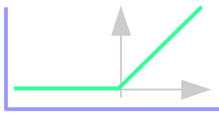
Direct Inspection (2)

$$y^{(N)}(t) + a_1 y^{(N-1)}(t) + a_2 y^{(N-2)}(t) + \cdots + a_{N-1} y^{(1)}(t) + a_N y(t) = x(t)$$

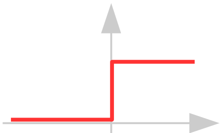
$u(t)$ ⟶ $y^{(N)}(t)$, $t\,u(t)$ ⟶ $y^{(N-1)}(t)$, $t^2 u(t)$ ⟶ $y^{(N-2)}(t)$, $u(t)$ ⟶ $x(t)$

$$x(t) = e^t$$





all continuous at t=0



$$e^t = 1 + \frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots$$

$$\begin{aligned}
y^{(N-1)}(0^+) &= y^{(N-1)}(0^-) \\
y^{(N-2)}(0^+) &= y^{(N-2)}(0^-) \\
&\vdots \\
y^{(1)}(0^+) &= y^{(1)}(0^-) \\
y(0^+) &= y(0^-)
\end{aligned}$$

$$e^t u(t) = \left(1 + \frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\right) u(t)$$

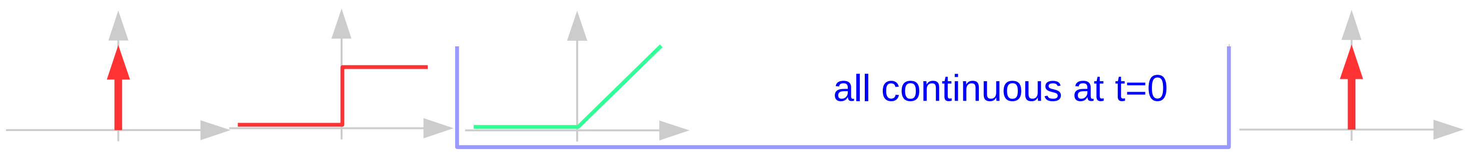
The highest order term = u(t)

Direct Inspection (3)

$$y^{(N)}(t) + a_1 y^{(N-1)}(t) + a_2 y^{(N-2)}(t) + \cdots + a_{N-1} y^{(1)}(t) + a_N h(t) = x^{(1)}(t)$$

$\delta(t)$ $\quad u(t)$ $\quad t\,u(t)$ $\quad \delta(t)$

$x(t) = u(t)$



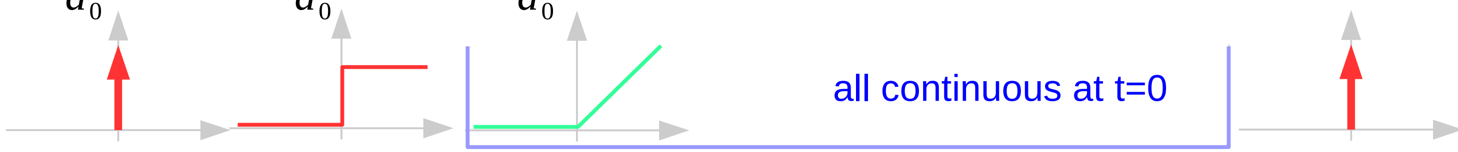
$$\begin{aligned}
y^{(N-1)}(0^+) &= y^{(N-1)}(0^-) + 1 \\
y^{(N-2)}(0^+) &= y^{(N-2)}(0^-) \\
&\vdots \\
y^{(1)}(0^+) &= y^{(1)}(0^-) \\
y(0^+) &= y(0^-)
\end{aligned}$$

Direct Inspection (4)

$$a_0\, y^{(N)}(t) + a_1\, y^{(N-1)}(t) + a_2\, y^{(N-2)}(t) + \cdots + a_{N-1}\, y^{(1)}(t) + a_N\, h(t) = x^{(1)}(t)$$

$\frac{1}{a_0}\delta(t)$ $\quad \frac{1}{a_0}u(t)$ $\quad \frac{1}{a_0}t\,u(t)$ $\quad \delta(t)$

$x(t) = u(t)$



$$\begin{aligned}
y^{(N-1)}(0^+) &= y^{(N-1)}(0^-) + \frac{1}{a_0} \\
y^{(N-2)}(0^+) &= y^{(N-2)}(0^-) \\
&\vdots \\
y^{(1)}(0^+) &= y^{(1)}(0^-) \\
y(0^+) &= y(0^-)
\end{aligned}$$

- System Response Example (I)

$$y''(t) + 3y'(t) + 2y(t) = x(t)$$

$$\begin{cases} y'(0^-) = \frac{3}{2} \\ y(0^-) = 1 \end{cases}$$

- Case I $\quad x(t) = u(t)$
- Case II $\quad x(t) = e^{-3t}u(t)$

Associated Linear ODE's

Example (I-1)

$$y''(t) + 3y'(t) + 2y(t) = x(t)$$

$$\begin{cases} y'(0^-) = \frac{3}{2} \\ y(0^-) = 1 \end{cases}$$

Homogeneous Equation

$$y''(t) + 3y'(t) + 2y(t) = 0$$

$$m^2 + 3m + 2 = 0$$

$$(m+2)(m+1) = 0$$

$$m = -1, \ -2$$

$$y_h = c_1 e^{-t} + c_2 e^{-2t}$$

Homogeneous solution

Non-homogeneous Equation

$x(t) = u(t)$

$$y''(t) + 3y'(t) + 2y(t) = 1$$

$$y_p = A$$

$$0 + 3 \cdot 0 + 2A = 1$$

$$y_p = \frac{1}{2}$$

$(t > 0)$

$x(t) = e^{-3t} u(t)$

$$y''(t) + 3y'(t) + 2y(t) = e^{-3t}$$

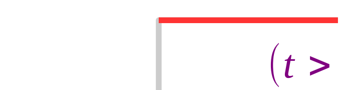
$$y_p = A e^{-3t}$$

$$(9A - 9A + 2A) e^{-3t} = e^{-3t}$$

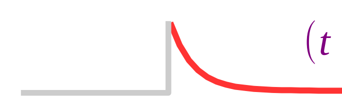
$$y_p = \frac{1}{2} e^{-3t}$$

$(t > 0)$

Particular solutions



$(t > 0)$



$(t > 0)$

Taylor Series Expansion

Example (I-2)

$$y''(t) + 3y'(t) + 2y(t) = x(t)$$

$$x(t) = e^{-3t} \quad (t \geq 0)$$

$$x(t) = e^{-3t} \qquad x(0) = 1$$

$$x^{(1)}(t) = (-3)e^{-3t} \qquad x^{(1)}(0) = (-3)$$

$$x^{(2)}(t) = (-3)^2 e^{-3t} \qquad x^{(2)}(0) = (-3)^2$$

$$x^{(3)}(t) = (-3)^3 e^{-3t} \qquad x^{(3)}(0) = (-3)^3$$

$$x(t) = x(0) + \frac{x^{(1)}(0)}{1!}t + \frac{x^{(2)}(0)}{2!}t^2 + \frac{x^{(3)}(0)}{3!}t^3 + \cdots$$

$$e^{-3t} = 1 + \frac{(-3)}{1!}t + \frac{(-3)^2}{2!}t^2 + \frac{(-3)^3}{3!}t^3 + \cdots$$

$$e^{-3t}u(t) = u(t) + \frac{(-3)}{1!}t\,u(t) + \frac{(-3)^2}{2!}t^2u(t) + \frac{(-3)^3}{3!}t^3u(t) + \cdots$$

highest order singularities

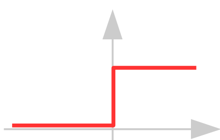
Example (I-3)

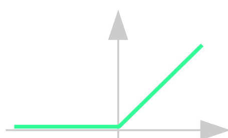
$$y''(t) + 3y'(t) + 2y(t) = x(t)$$

$$y(t) = y_h(t) + y_p(t) \quad (t \geq 0)$$

$$\begin{cases} x(t) = u(t) \\ x(t) = e^{-3t}u(t) \end{cases}$$

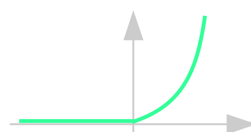
$$y''(t) \qquad + 3y'(t) \qquad + 2y(t) \qquad = x(t)$$





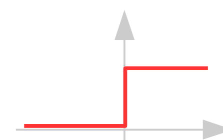
continuous at t=0

$$y^{(1)}(0^+) = y^{(1)}(0^-)$$



continuous at t=0

$$y(0^+) = y(0^-)$$



highest order singularities

For ZSR

$$\begin{cases} y^{(1)}(0^+) = y^{(1)}(0^-) = 0 \\ y(0^+) = y(0^-) = 0 \end{cases} \Rightarrow y_{zs}(t) = y_h(t) + y_p(t) \quad (t \geq 0)$$

For Total Response

$$\begin{cases} y^{(1)}(0^+) = y^{(1)}(0^-) = k_1 \\ y(0^+) = y(0^-) = k_2 \end{cases} \Rightarrow y_t(t) = y_h(t) + y_p(t) \quad (t \geq 0)$$

Using Direct Inspection (2)

$x(t) = u(t)$

Example (I-4a)

$$y''(t) + 3y'(t) + 2y(t) = u(t)$$

$$y(t) = (c_1 e^{-t} + c_2 e^{-2t} + \frac{1}{2}) \quad (t \geq 0)$$

For ZSR

$$\begin{cases} y^{(1)}(0^+) = y^{(1)}(0^-) = 0 \\ y(0^+) = y(0^-) = 0 \end{cases}$$

$$\begin{cases} y'(t) = (-c_1 e^{-t} - 2c_2 e^{-2t}) \\ y(t) = (c_1 e^{-t} + c_2 e^{-2t} + \frac{1}{2}) \end{cases}$$

$$\begin{cases} y^{(1)}(0^+) = -c_1 - 2c_2 = 0 \\ y(0^+) = c_1 + c_2 + \frac{1}{2} = 0 \end{cases}$$

$$\begin{cases} c_1 = -1 \\ c_2 = \frac{1}{2} \end{cases}$$

$$y_{zs}(t) = (-e^{-t} + \frac{1}{2} e^{-2t} + \frac{1}{2}) \quad (t \geq 0)$$

For Total Response

$$\begin{cases} y^{(1)}(0^+) = y^{(1)}(0^-) = \frac{3}{2} \\ y(0^+) = y(0^-) = 1 \end{cases}$$

$$\begin{cases} y'(t) = (-c_1 e^{-t} - 2c_2 e^{-2t}) \\ y(t) = (c_1 e^{-t} + c_2 e^{-2t} + \frac{1}{2}) \end{cases}$$

$$\begin{cases} y^{(1)}(0^+) = -c_1 - 2c_2 = \frac{3}{2} \\ y(0^+) = c_1 + c_2 + \frac{1}{2} = 1 \end{cases}$$

$$\begin{cases} c_1 = \frac{5}{2} \\ c_2 = -2 \end{cases}$$

$$y_t(t) = (\frac{5}{2} e^{-t} - 2e^{-2t} + \frac{1}{2}) \quad (t \geq 0)$$

For natural response

$$y_n(t) = y_t(t) - y_p(t)$$

$x(t) = e^{-3t}u(t)$

Example (I-4b)

$$y''(t) + 3y'(t) + 2y(t) = e^{-3t}u(t)$$

$$y(t) = (c_1 e^{-t} + c_2 e^{-2t} + \frac{1}{2}e^{-3t}) \quad (t \geq 0)$$

For ZSR

$$\begin{cases} y^{(1)}(0^+) = y^{(1)}(0^-) = 0 \\ y(0^+) = y(0^-) = 0 \end{cases} \quad \begin{cases} y'(t) = (-c_1 e^{-t} - 2c_2 e^{-2t} - \frac{3}{2}e^{-3t}) \\ y(t) = (c_1 e^{-t} + c_2 e^{-2t} + \frac{1}{2}e^{-3t}) \end{cases} \quad \begin{cases} y^{(1)}(0^+) = -c_1 - 2c_2 - \frac{3}{2} = 0 \\ y(0^+) = c_1 + c_2 + \frac{1}{2} = 0 \end{cases}$$

$$\begin{cases} c_1 = \frac{1}{2} \\ c_2 = -1 \end{cases}$$

$$y_{zs}(t) = (\frac{1}{2}e^{-t} - e^{-2t} + \frac{1}{2}e^{-3t}) \quad (t \geq 0)$$

For Total Response

$$\begin{cases} y^{(1)}(0^+) = y^{(1)}(0^-) = \frac{3}{2} \\ y(0^+) = y(0^-) = 1 \end{cases} \quad \begin{cases} y'(t) = (-c_1 e^{-t} - 2c_2 e^{-2t} - \frac{3}{2}e^{-3t}) \\ y(t) = (c_1 e^{-t} + c_2 e^{-2t} + \frac{1}{2}e^{-3t}) \end{cases} \quad \begin{cases} y^{(1)}(0^+) = -c_1 - 2c_2 - \frac{3}{2} = \frac{3}{2} \\ y(0^+) = c_1 + c_2 + \frac{1}{2} = 1 \end{cases}$$

$$\begin{cases} c_1 = 4 \\ c_2 = -\frac{7}{2} \end{cases}$$

$$y_t(t) = (4e^{-t} - \frac{7}{2}e^{-2t} + \frac{1}{2}e^{-3t}) \quad (t \geq 0)$$

For natural response

$$y_n(t) = y_t(t) - y_p(t)$$

Without Converting Initial Conditions

Example (I-5)

Balancing Singularities

No need the converted initial conditions at t = 0+

Only for zero initial conditions at t = 0- (ZSR)

$$y_{zs}(t) = (y_h + y_p)\cdot u(t)$$

$$\begin{aligned} y^{(N-1)}(0^-) &= 0 \\ y^{(N-2)}(0^-) &= 0 \\ \vdots \quad & \vdots \\ y^{(1)}(0^-) &= 0 \\ y(0^-) &= 0 \end{aligned}$$

Using Balancing Singularities (1)

$x(t) = u(t)$

Example (I-6a)

$$y''(t) + 3y'(t) + 2y(t) = u(t)$$

$$y_{zs}(t) = (c_1 e^{-t} + c_2 e^{-2t} + \frac{1}{2}) u(t)$$

$$y'_{zs}(t) = (-c_1 e^{-t} - 2c_2 e^{-2t}) u(t) + (c_1 e^{-t} + c_2 e^{-2} + \frac{1}{2}) \delta(t)$$

$$= (-c_1 e^{-t} - 2c_2 e^{-2t}) u(t) + (c_1 + c_2 + \frac{1}{2}) \delta(t)$$

$$y''_{zs}(t) = (c_1 e^{-t} + 4c_2 e^{-2t}) u(t) + (-c_1 e^{-t} - 2c_2 e^{-2t}) \delta(t) + (c_1 + c_2 + \frac{1}{2}) \delta'(t)$$

$$= (c_1 e^{-t} + 4c_2 e^{-2t}) u(t) + (-c_1 - 2c_2) \delta(t) + (c_1 + c_2 + \frac{1}{2}) \delta'(t)$$

Using Balancing Singularities (2)

$x(t) = e^{-3t}u(t)$

Example (I-6b)

$$y''(t) + 3y'(t) + 2y(t) = e^{-3t}u(t)$$

$$y_{zs}(t) = (c_1 e^{-t} + c_2 e^{-2t} + \frac{1}{2}e^{-3t})u(t)$$

$$y'_{zs}(t) = (-c_1 e^{-t} - 2c_2 e^{-2t} - \frac{3}{2}e^{-3t})u(t) + (c_1 e^{-t} + c_2 e^{-2} + \frac{1}{2}e^{-3t})\delta(t)$$

$$= (-c_1 e^{-t} - 2c_2 e^{-2t} - \frac{3}{2}e^{-3t})u(t) + (c_1 + c_2 + \frac{1}{2})\delta(t)$$

$$y''_{zs}(t) = (c_1 e^{-t} + 4c_2 e^{-2t} + \frac{9}{2}e^{-3t})u(t) + (-c_1 e^{-t} - 2c_2 e^{-2t} - \frac{3}{2}e^{-3t})\delta(t) + (c_1 + c_2 + \frac{1}{2})\delta'(t)$$

$$= (c_1 e^{-t} + 4c_2 e^{-2t} + \frac{9}{2}e^{-3t})u(t) + (-c_1 - 2c_2 - \frac{3}{2})\delta(t) + (c_1 + c_2 + \frac{1}{2})\delta'(t)$$

Example (I-7a)

$$y''(t) + 3y'(t) + 2y(t) = u(t)$$

$$y''_{zs}(t) = (c_1 e^{-t} + 4c_2 e^{-2t})u(t) + (-c_1 - 2c_2)\delta(t) + (c_1 + c_2 + \frac{1}{2})\delta'(t)$$

$$3y'_{zs}(t) = (-3c_1 e^{-t} - 6c_2 e^{-2t})u(t) + (3c_1 + 3c_2 + \frac{3}{2})\delta(t)$$

$$2y_{zs}(t) = (2c_1 e^{-t} + 2c_2 e^{-2t} + 1)u(t)$$

$$u(t) = u(t) + (2c_1 + c_2 + \frac{3}{2})\delta(t) + (c_1 + c_2 + \frac{1}{2})\delta'(t)$$

$$\begin{cases} (2c_1 + c_2 + \frac{3}{2}) = 0 \\ (c_1 + c_2 + \frac{1}{2}) = 0 \end{cases} \qquad \begin{cases} c_1 = -1 \\ c_2 = \frac{1}{2} \end{cases}$$

$$y_{zs}(t) = (-e^{-t} + \frac{1}{2}e^{-2t} + \frac{1}{2})u(t)$$

Example (I-7b)

$$y''(t) + 3y'(t) + 2y(t) = e^{-3t}u(t)$$

$$y''_{zs}(t) = (c_1 e^{-t} + 4c_2 e^{-2t} + \frac{9}{2}e^{-3t})u(t) + (-c_1 - 2c_2 - \frac{3}{2})\delta(t) + (c_1 + c_2 + \frac{1}{2})\delta'(t)$$

$$3y'_{zs}(t) = 3(-c_1 e^{-t} - 2c_2 e^{-2t} - \frac{3}{2}e^{-3t})u(t) + 3(c_1 + c_2 + \frac{1}{2})\delta(t)$$

$$2y_{zs}(t) = (2c_1 e^{-t} + 2c_2 e^{-2t} + e^{-3t})u(t)$$

$$e^{-3t}u(t) = e^{-3t}u(t) + (2c_1 + c_2)\delta(t) + (c_1 + c_2 + \frac{1}{2})\delta'(t)$$

$$\begin{cases} (2c_1 + c_2) = 0 \\ (c_1 + c_2 + \frac{1}{2}) = 0 \end{cases} \qquad \begin{cases} c_1 = \frac{1}{2} \\ c_2 = -1 \end{cases}$$

$$\Rightarrow \quad y_{zs}(t) = (\frac{1}{2}e^{-t} - e^{-2t} + \frac{1}{2}e^{-3t})u(t)$$

Finding ZIR

Example (I-8)

$$y''(t) + 3y'(t) + 2y(t) = u(t)$$

$$\begin{cases} y'(0^+) = y'(0^-) = \frac{3}{2} \\ y(0^+) = y(0^-) = 1 \end{cases}$$

$$\begin{cases} y'_{zi}(t) = (-c_1 e^{-t} - 2c_2 e^{-2t}) \\ y_{zi}(t) = (c_1 e^{-t} + c_2 e^{-2t}) \end{cases}$$

$$\begin{cases} y'_{zi}(0^+) = (-c_1 - 2c_2) = \frac{3}{2} \\ y_{zi}(0^+) = (c_1 + c_2) = 1 \end{cases}$$

$$\begin{cases} c_1 = +\frac{7}{2} \\ c_2 = -\frac{5}{2} \end{cases}$$

$$y_{zi}(t) = (\frac{7}{2}e^{-t} - \frac{5}{2}e^{-2t}) \qquad (t \geq 0)$$

$$y''(t) + 3y'(t) + 2y(t) = e^{-3t}u(t)$$

$$\begin{cases} y'(0^+) = y'(0^-) = \frac{3}{2} \\ y(0^+) = y(0^-) = 1 \end{cases}$$

The same ZIR

$$y_{zi}(t) = (\frac{7}{2}e^{-t} - \frac{5}{2}e^{-2t}) \qquad (t \geq 0)$$

System Response Plots

$x(t) = u(t)$

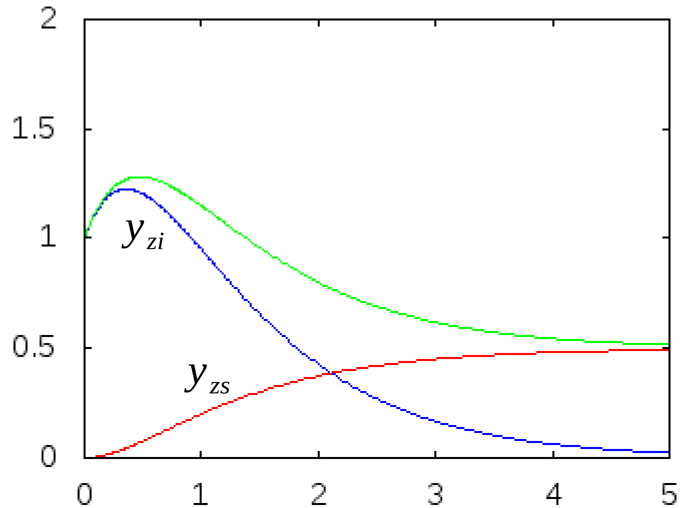
Example (I-9)

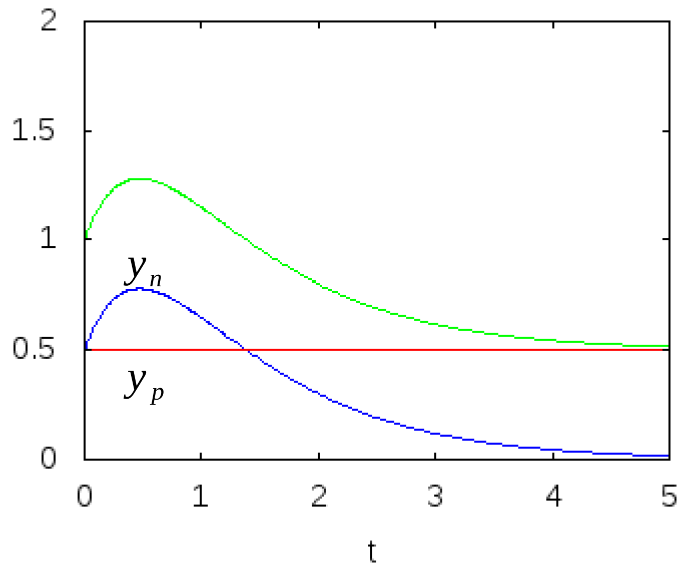
$$y_{zi}(t) = (\frac{7}{2}e^{-t} - \frac{5}{2}e^{-2t}) \qquad (t \geq 0)$$

$$y_{zs}(t) = (-e^{-t} + \frac{1}{2}e^{-2t} + \frac{1}{2}) \quad (t \geq 0)$$

$$y_{n}(t) = (\frac{5}{2}e^{-t} - 2e^{-2t}) \quad (t \geq 0)$$

$$y_{p}(t) = (\frac{1}{2}) \quad (t \geq 0)$$





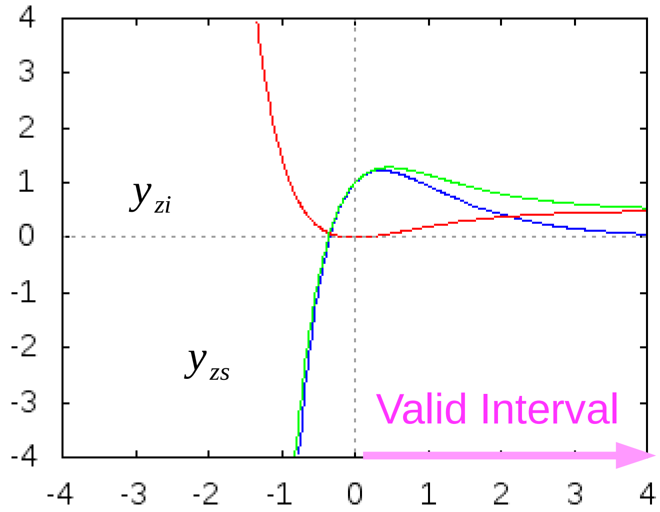
System Response Plots with Valid Intervals

$x(t) = u(t)$

Example (I-10)

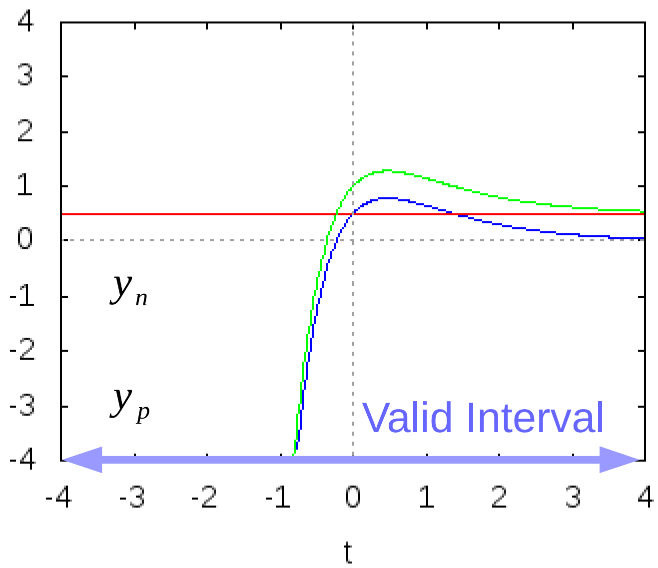
$$y_{zi}(t) = (\frac{7}{2}e^{-t} - \frac{5}{2}e^{-2t}) \quad (t \geq 0)$$

$$y_{zs}(t) = (-e^{-t} + \frac{1}{2}e^{-2t} + \frac{1}{2}) \quad (t \geq 0)$$



$$y_n(t) = (\frac{5}{2}e^{-t} - 2e^{-2t}) \quad (t \geq 0)$$

$$y_p(t) = (\frac{1}{2}) \quad (t \geq 0)$$



System Response Plots

$x(t) = e^{-3t}u(t)$

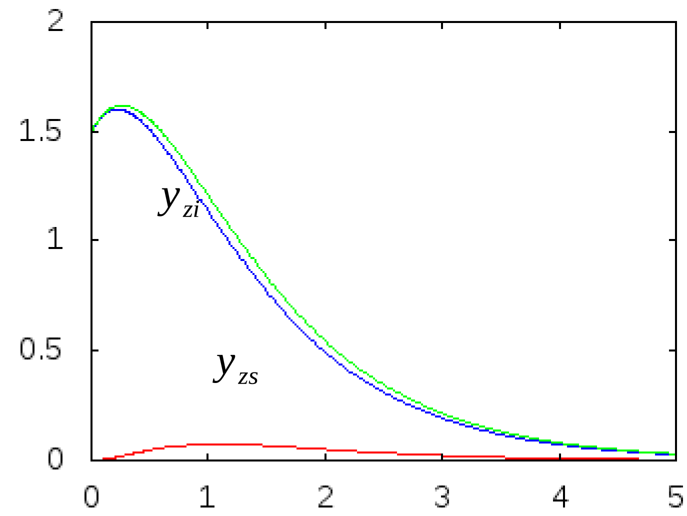
Example (I-11)

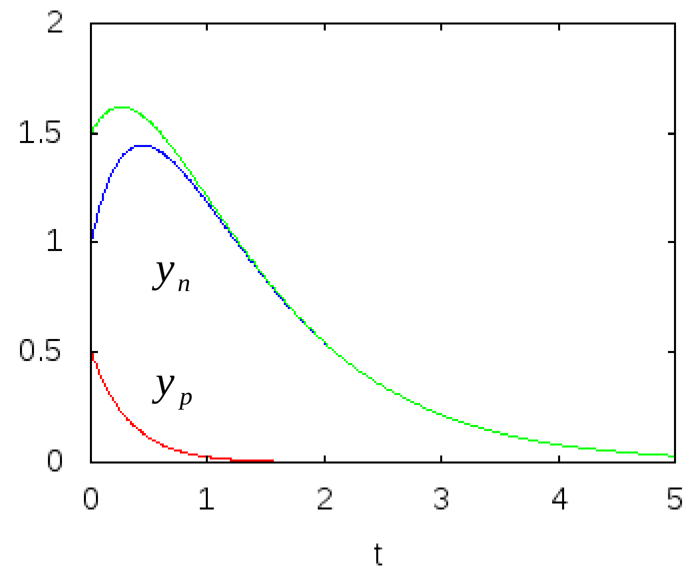
$$y_{zi}(t) = \left(\frac{7}{2}e^{-t} - \frac{5}{2}e^{-2t}\right) \qquad (t \geq 0)$$

$$y_{zs}(t) = \left(\frac{1}{2}e^{-t} - e^{-2t} + \frac{1}{2}e^{-3t}\right) \qquad (t \geq 0)$$

$$y_n(t) = \left(4e^{-t} - \frac{7}{2}e^{-2t}\right) \qquad (t \geq 0)$$

$$y_p(t) = \left(\frac{1}{2}e^{-3t}\right) \qquad (t \geq 0)$$





Young Won Lim
5/16/15

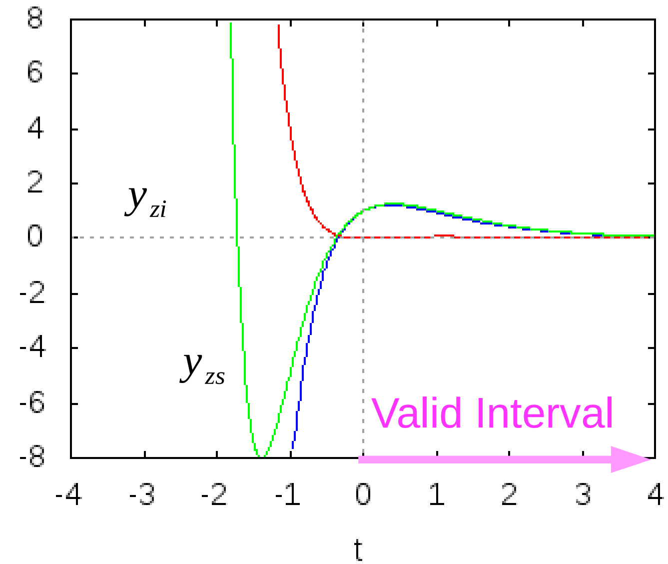
System Response Plots with Valid Intervals

$x(t) = e^{-3t}u(t)$

Example (I-12)

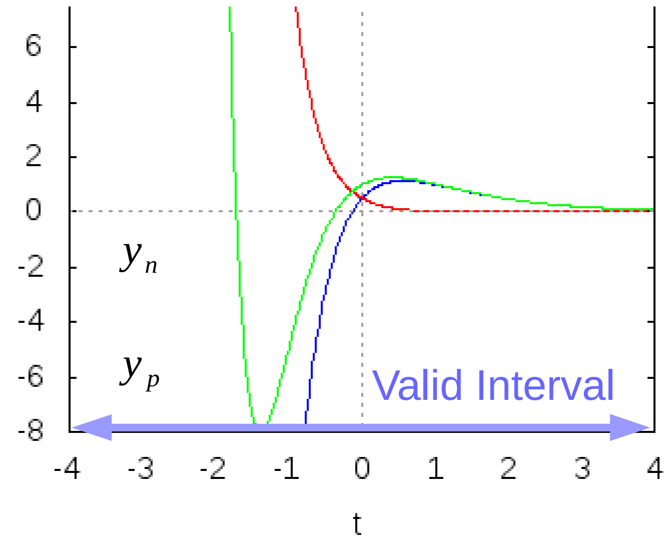
$$y_{zi}(t) = \left(\frac{7}{2}e^{-t} - \frac{5}{2}e^{-2t}\right) \qquad (t \geq 0)$$

$$y_{zs}(t) = \left(\frac{1}{2}e^{-t} - e^{-2t} + \frac{1}{2}e^{-3t}\right) \qquad (t \geq 0)$$



$$y_n(t) = \left(4e^{-t} - \frac{7}{2}e^{-2t}\right) \qquad (t \geq 0)$$

$$y_p(t) = \left(\frac{1}{2}e^{-3t}\right) \qquad (t \geq 0)$$



- System Response Example (II)

$$y''(t) + 3y'(t) + 2y(t) = x'(t)$$

$$\begin{cases} y'(0^-) = \frac{3}{2} \\ y(0^-) = 1 \end{cases}$$

- Case I $\quad x(t) = u(t)$
- Case II $\quad x(t) = e^{-3t}u(t)$

Solutions of Linear ODEs

Example (II-1)

$$y''(t) + 3y'(t) + 2y(t) = x'(t)$$

$$\begin{cases} y'(0^-) = \frac{3}{2} \\ y(0^-) = 1 \end{cases}$$

Homogeneous Equation

$x(t) = u(t)$

Non-homogeneous Equation

$x(t) = e^{-3t}u(t)$

$$y''(t) + 3y'(t) + 2y(t) = 0$$

$$m^2 + 3m + 2 = 0$$

$$(m+2)(m+1) = 0$$

$$m = -1, \ -2$$

$$y_h = c_1 e^{-t} + c_2 e^{-2t}$$

Homogeneous solution

$$y''(t) + 3y'(t) + 2y(t) = 0$$

$$y_p = A$$

$$0 + 3\cdot 0 + 2A = 1$$

$$y_p = 0$$

$$y''(t) + 3y'(t) + 2y(t) = e^{-3t}$$

$$y_p = A e^{-3t}$$

$$(9A - 9A + 2A)e^{-3t} = e^{-3t}$$

$$y_p = \frac{1}{2}e^{-3t}$$

Particular solutions

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5/16/15

Using Direct Inspection (1)

Example (II-2)

$$y''(t) + 3y'(t) + 2y(t) = x'(t)$$

$$y(t) = y_h(t) + y_p(t) \quad (t \geq 0)$$

$$y''(t) \qquad + 3y'(t) \qquad + 2y(t) \qquad = x(t)$$

$$\begin{cases} x(t) = u(t) \\ x(t) = e^{-3t}u(t) \end{cases}$$

jump at t=0

$$y^{(1)}(0^+) = y^{(1)}(0^-) + 1$$

continuous at t=0

$$y(0^+) = y(0^-)$$

highest order singularities

For ZSR

$$\begin{cases} y^{(1)}(0^+) = y^{(1)}(0^-) + 1 = 1 \\ y(0^+) = y(0^-) = 0 \end{cases} \Rightarrow y_{zs}(t) = y_h(t) + y_p(t) \quad (t \geq 0)$$

For Total Response

$$\begin{cases} y^{(1)}(0^+) = y^{(1)}(0^-) + 1 = k_1 \\ y(0^+) = y(0^-) = k_2 \end{cases} \Rightarrow y_t(t) = y_h(t) + y_p(t) \quad (t \geq 0)$$

Using Direct Inspection (2)

Example (II-3)

$$y''(t) + 3y'(t) + 2y(t) = u'(t)$$

$$y(t) = (c_1 e^{-t} + c_2 e^{-2t} + 0) \quad (t \geq 0)$$

For ZSR

$$\begin{cases} y^{(1)}(0^+) = y^{(1)}(0^-) + 1 = 1 \\ y(0^+) = y(0^-) = 0 \end{cases} \quad \begin{cases} y'(t) = (-c_1 e^{-t} - 2c_2 e^{-2t}) \\ y(t) = (c_1 e^{-t} + c_2 e^{-2t}) \end{cases} \quad \begin{cases} y^{(1)}(0^+) = -c_1 - 2c_2 = 1 \\ y(0^+) = c_1 + c_2 = 0 \end{cases}$$

$$\begin{cases} c_1 = +1 \\ c_2 = -1 \end{cases}$$

$$y_{zs}(t) = (e^{-t} - e^{-2t} + 0) \quad (t \geq 0)$$

For Total Response

$$\begin{cases} y^{(1)}(0^+) = y^{(1)}(0^-) + 1 = \frac{5}{2} \\ y(0^+) = y(0^-) = 1 \end{cases} \quad \begin{cases} y'(t) = (-c_1 e^{-t} - 2c_2 e^{-2t}) \\ y(t) = (c_1 e^{-t} + c_2 e^{-2t}) \end{cases} \quad \begin{cases} y^{(1)}(0^+) = -c_1 - 2c_2 = \frac{5}{2} \\ y(0^+) = c_1 + c_2 = 1 \end{cases}$$

$$\begin{cases} c_1 = +\frac{9}{2} \\ c_2 = -\frac{7}{2} \end{cases}$$

$$y_t(t) = (\frac{9}{2} e^{-t} - \frac{7}{2} e^{-2t} + 0) \quad (t \geq 0)$$

For natural response

$$y_n(t) = y_t(t) - y_p(t)$$

Using Balancing Singularities (1)

Example (II-4)

$$y''(t) + 3y'(t) + 2y(t) = u'(t)$$

$$y_{zs}(t) = (c_1 e^{-t} + c_2 e^{-2t} + 0)u(t)$$

$$y'_{zs}(t) = (-c_1 e^{-t} - 2c_2 e^{-2t})u(t) + (c_1 e^{-t} + c_2 e^{-2})\delta(t)$$

$$= (-c_1 e^{-t} - 2c_2 e^{-2t})u(t) + (c_1 + c_2)\delta(t)$$

$$y''_{zs}(t) = (c_1 e^{-t} + 4c_2 e^{-2t})u(t) + (-c_1 e^{-t} - 2c_2 e^{-2t})\delta(t) + (c_1 + c_2)\delta'(t)$$

$$= (c_1 e^{-t} + 4c_2 e^{-2t})u(t) + (-c_1 - 2c_2)\delta(t) + (c_1 + c_2)\delta'(t)$$

Using Balancing Singularities (2)

Example (II-5)

$$y''(t) + 3y'(t) + 2y(t) = u'(t)$$

$$y''_{zs}(t) = (c_1 e^{-t} + 4c_2 e^{-2t})u(t) + (-c_1 - 2c_2)\delta(t) + (c_1 + c_2)\delta'(t)$$

$$3y'_{zs}(t) = (-3c_1 e^{-t} - 6c_2 e^{-2t})u(t) + (3c_1 + 3c_2)\delta(t)$$

$$2y_{zs}(t) = (2c_1 e^{-t} + 2c_2 e^{-2t})u(t)$$

$$\delta(t) = (2c_1 + c_2)\delta(t) + (c_1 + c_2)\delta'(t)$$

$$\begin{cases} (2c_1 + c_2) = 1 \\ (c_1 + c_2) = 0 \end{cases} \qquad \begin{cases} c_1 = +1 \\ c_2 = -1 \end{cases} \Rightarrow \quad y_{zs}(t) = (e^{-t} - e^{-2t})u(t)$$

Finding ZIR

Example (II-6)

$$y''(t) + 3y'(t) + 2y(t) = u'(t)$$

$$\begin{cases} y'(0^+) = y'(0^-) = \frac{3}{2} \\ y(0^+) = y(0^-) = 1 \end{cases}$$

$$\begin{cases} y'_{zi}(t) = (-c_1 e^{-t} - 2c_2 e^{-2t}) \\ y_{zi}(t) = (c_1 e^{-t} + c_2 e^{-2t}) \end{cases}$$

$$\begin{cases} y'_{zi}(0^+) = (-c_1 - 2c_2) = \frac{3}{2} \\ y_{zi}(0^+) = (c_1 + c_2) = 1 \end{cases}$$

$$\begin{cases} c_1 = \frac{5}{2} \\ c_2 = -\frac{3}{2} \end{cases}$$

$$y_{zi}(t) = (\frac{5}{2} e^{-t} - \frac{3}{2} e^{-2t}) \qquad (t \geq 0)$$

$$y''(t) + 3y'(t) + 2y(t) = -3e^{-3t}$$

$$\begin{cases} y'(0^+) = y'(0^-) = \frac{3}{2} \\ y(0^+) = y(0^-) = 1 \end{cases}$$

The same ZIR

$$y_{zi}(t) = (\frac{5}{2} e^{-t} - \frac{3}{2} e^{-2t}) \qquad (t \geq 0)$$

System Response Plots

$x(t) = u(t)$

Example (II-7)

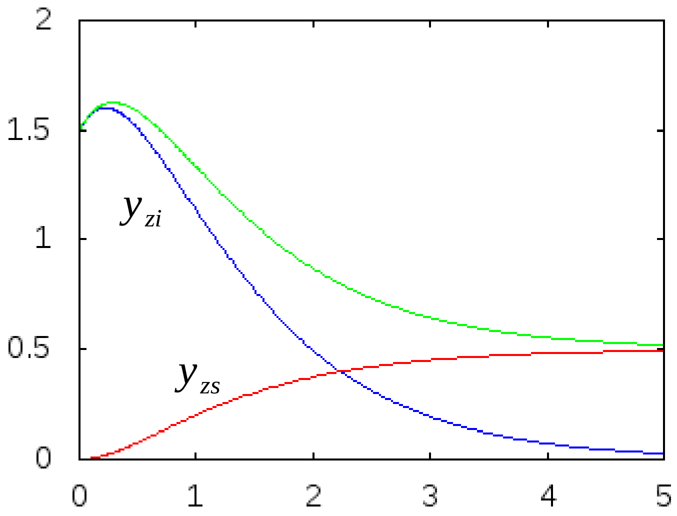
$$y_{zi}(t) = (4e^{-t} - \frac{5}{2}e^{-2t}) \qquad (t \geq 0)$$

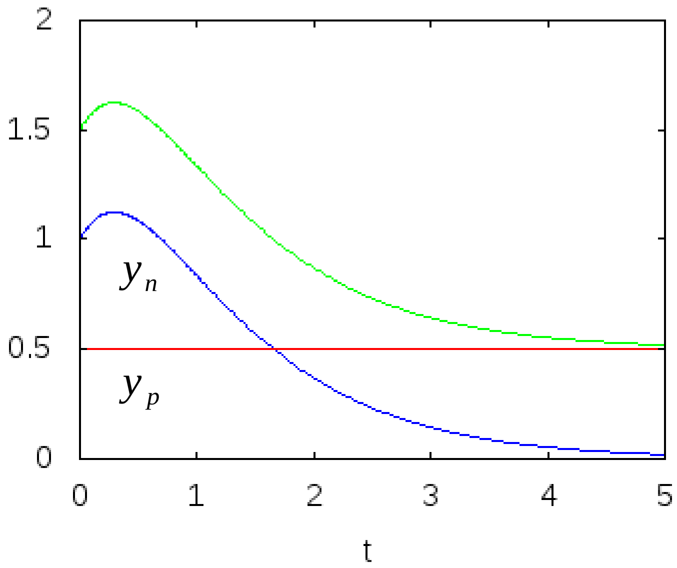
$$y_{zs}(t) = (e^{-t} - e^{-2t})u(t)$$

$$y_n(t) = (3e^{-t} - 2e^{-2t}) \quad (t \geq 0)$$

$$y_t(t) = (\frac{9}{2}e^{-t} - \frac{7}{2}e^{-2t} + 0) \quad (t \geq 0)$$

$$y_p(t) = 0 \quad (t \geq 0)$$





ZIR

$$x(t) = 10e^{-3t}u(t)$$

$$h(t) = (-e^{-t} + 2e^{-2t})u(t)$$

$$y(t) = \int_0^t h(\tau)x(t-\tau)\,d\tau$$

$$= \int_0^t (-e^{-\tau}+2e^{-2\tau})10e^{-3(t-\tau)}\,d\tau$$

$$= -5e^{-t}+20e^{-2t}-15e^{-3t}$$

$$x(t) = 10e^{-3t}u(t+c)$$

$$h(t) = (-e^{-t} + 2e^{-2t})u(t)$$

$$y(t) = \int_{-c}^t h(\tau)x(t-\tau)\,d\tau$$

$$= \int_{-c}^t (-e^{-\tau}+2e^{-2\tau})10e^{-3(t-\tau)}\,d\tau$$

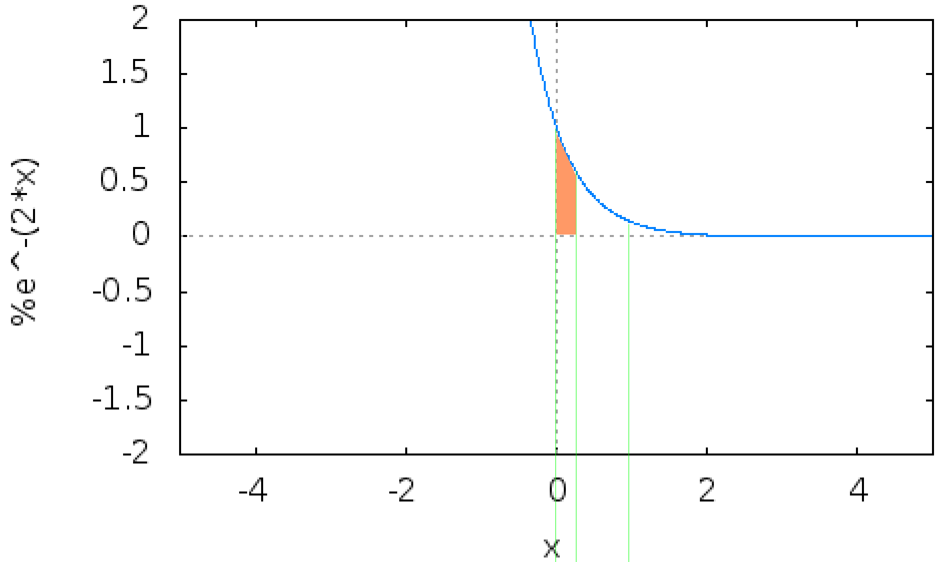
$$= -5e^{-t+2c}+20e^{-2t+c}-15e^{-3t}$$

$$x(t) = 10e^{-3t}(u(t+c)-u(t))$$

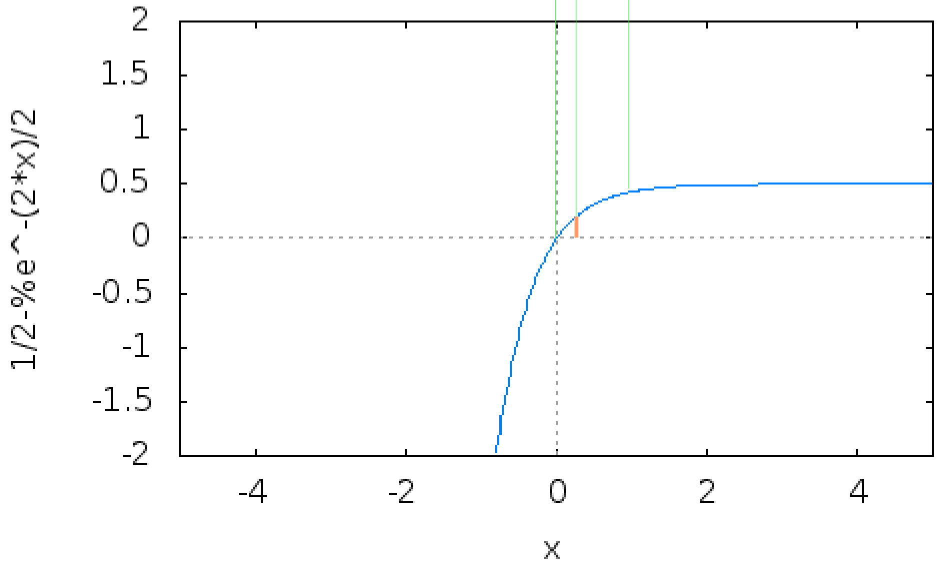
$$h(t) = (-e^{-t} + 2e^{-2t})u(t)$$

$$y(t) = \int_{-c}^t h(\tau)x(t-\tau)\,d\tau$$

$$= -5(e^{-t+2c}-e^{-t})+20(e^{-2t+c}-e^{-2t})$$

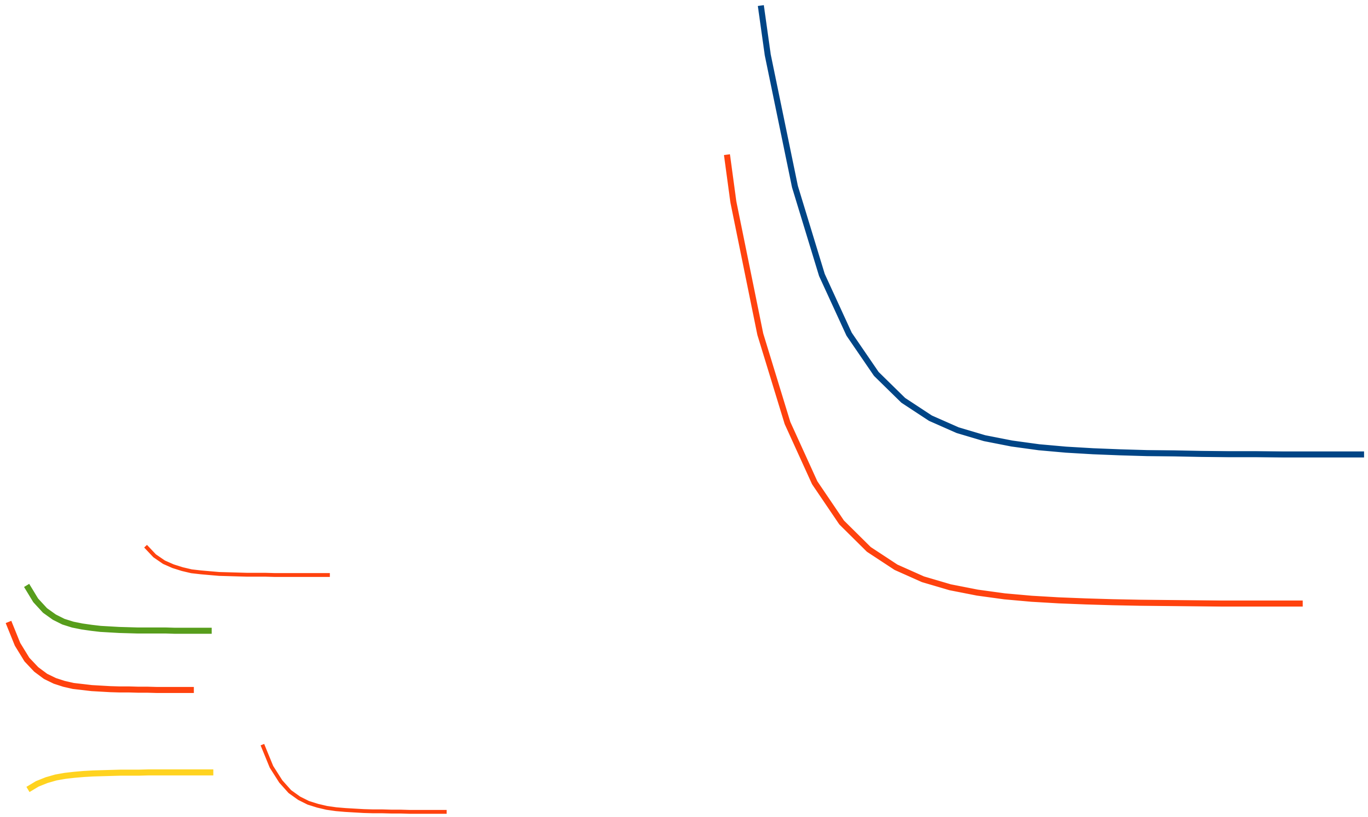


$$y_1(t) = e^{-2t}$$



$$y_2(x) = \int_0^x e^{-2t}\,dt \quad = 1 - \frac{1}{2}e^{-2x}$$

Impulse Response h(t)



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