Laurent Series and z-Transform - Geometric Series Time Shift A

20180912 Wed

Causal Signal $a(n) \Rightarrow f(z), \quad X(z)$

$(n \geq 0) \qquad (|z| < p) \quad (|z| > p^{-1})$

$$a_n = \left(\frac{1}{2}\right)^n \quad (n \geq 0)$$

$$f(z) = \left(\frac{1}{2}\right)^0 z^0 + \left(\frac{1}{2}\right)^1 z^1 + \left(\frac{1}{2}\right)^2 z^2 + \cdots = \frac{1}{1 - \left(\frac{z}{2}\right)}$$

$$X(z) = \left(\frac{1}{2}\right)^0 z^0 + \left(\frac{1}{2}\right)^1 z^{-1} + \left(\frac{1}{2}\right)^2 z^{-2} + \cdots = \frac{1}{1 - \left(\frac{1}{2z}\right)}$$

$$a_n = \left(\frac{1}{2}\right)^n \quad (n \geq 0)$$

$$f(z) = \frac{1}{1 - \left(\frac{z}{2}\right)} \Rightarrow \frac{2}{2 - z} \qquad (|z| < 2)$$

$$X(z) = \frac{1}{1 - \left(\frac{1}{2z}\right)} \Rightarrow \frac{z}{z - 0.5} \qquad (|z| > 0.5)$$

Anti-Causal Signal $\quad a(n) \Rightarrow -f(z), -X(z)$

$(n < 0) \qquad (|z| > p) \quad (|z| < p^{-1})$

$$a_n = \left(\frac{1}{2}\right)^n \quad (n < 0)$$

$$f_2(z) = \left(\frac{1}{2}\right)^{-1} z^{-1} + \left(\frac{1}{2}\right)^{-2} z^{-2} + \left(\frac{1}{2}\right)^{-3} z^{-3} + \cdots = \frac{\left(\frac{2}{z}\right)}{1 - \left(\frac{2}{z}\right)}$$

$$X_2(z) = \left(\frac{1}{2}\right)^{-1} z^{1} + \left(\frac{1}{2}\right)^{-2} z^{2} + \left(\frac{1}{2}\right)^{-3} z^{3} + \cdots = \frac{(2z)}{1 - (2z)}$$

$$a_n = \left(\frac{1}{2}\right)^n \quad (n < 0)$$

$$f_2(z) = \frac{\left(\frac{2}{z}\right)}{1 - \left(\frac{2}{z}\right)} \Rightarrow \frac{2}{z - 2} = -f(z) \quad (|z| > 2)$$

$$X_2(z) = \frac{(2z)}{1 - (2z)} \Rightarrow \frac{z}{0.5 - z} = -X(z) \quad (|z| < 0.5)$$

$$a'_n = -\left(\frac{1}{2}\right)^n \quad (n < 0)$$

$$f(z) = \frac{2}{2 - z} \Rightarrow \frac{-\left(\frac{2}{z}\right)}{1 - \left(\frac{2}{z}\right)} \quad (|z| > 2)$$

$$X(z) = \frac{z}{z - 0.5} \Rightarrow \frac{-(2z)}{1 - (2z)} \quad (|z| < 0.5)$$

Inverse z

$z \leftarrow z^{-1}$, $\quad ROC(z) \leftarrow ROC(z^{-1})$

Causal $\quad z^{-1} \quad$ anti-Causal

$$f(z) = \frac{2}{2-z} \quad (|z| < 2) \qquad f(z^{-1}) = \frac{z}{z-0.5} \quad (|z| > 0.5)$$

$$X(z) = \frac{z}{z-0.5} \quad (|z| > 0.5) \qquad X(z^{-1}) = \frac{2}{2-z} \quad (|z| < 2)$$

$$f(z^{-1}) = \frac{2}{2-z^{-1}} \quad (|z^{-1}| < 2) \Rightarrow f(z^{-1}) = X(z) = \frac{z}{z-0.5} \quad (|z| > 0.5)$$

$$X(z^{-1}) = \frac{z^{-1}}{z^{-1}-0.5} \quad (|z^{-1}| > 0.5) \Rightarrow X(z^{-1}) = f(z) = \frac{2}{2-z} \quad (|z| < 2)$$

$f(z^{-1}) = X(z)$ Laurent Series (anti-causal signal) with the same formula as causal X(z)

$X(z^{-1}) = f(z)$ z-Transform (anti-causal signal) with the same formula as causal f(z)

Causal $\qquad$ anti-Causal

$$f(z) = \frac{2}{2-z} \quad (|z| < 2) \qquad X(z^{-1}) = \frac{2}{2-z} \quad (|z| < 2)$$

$$X(z) = \frac{z}{z-0.5} \quad (|z| > 0.5) \qquad f(z^{-1}) = \frac{z}{z-0.5} \quad (|z| > 0.5)$$

Inverse z $\quad f(z^{-1}),\ ROC(z^{-1}) \Rightarrow a_{-n}$

Causal | **anti-Causal**

$$f(z) = \frac{2}{2-z} \quad (|z| < 2) \qquad\qquad f(z^{-1}) = \frac{z}{z-0.5} \quad (|z| > 0.5)$$

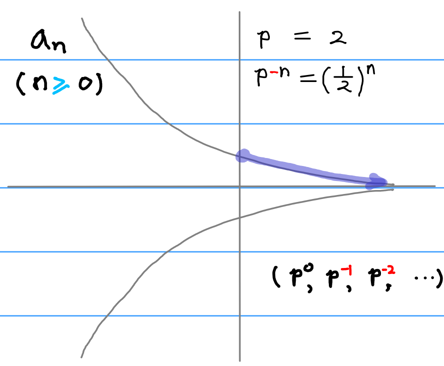
$$X(z) = \frac{z}{z-0.5} \quad (|z| > 0.5) \qquad\qquad X(z^{-1}) = \frac{2}{2-z} \quad (|z| < 2)$$

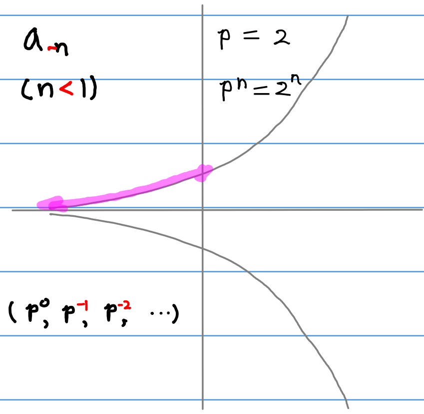
$$f(z) = \frac{2}{2-z} = \frac{1}{1-\left(\frac{z}{2}\right)} = \left(\tfrac{1}{2}\right)^0 z^0 + \left(\tfrac{1}{2}\right)^1 z^1 + \left(\tfrac{1}{2}\right)^2 z^2 + \cdots \qquad a_n = \left(\tfrac{1}{2}\right)^n$$

$$= p^0 z^0 + p^{-1} z^1 + p^{-2} z^2 + \cdots \qquad n = 0, 1, 2, \cdots$$

$$f(z^{-1}) = \frac{2}{2-z^{-1}} = \frac{1}{1-\left(\frac{1}{2z}\right)} = \left(\tfrac{1}{2}\right)^0 z^0 + \left(\tfrac{1}{2}\right)^1 z^{-1} + \left(\tfrac{1}{2}\right)^2 z^{-2} + \cdots \qquad a_{-n} = \left(\tfrac{1}{2}\right)^{-n}$$

$$= p^0 z^0 + p^1 z^{-1} + p^2 z^{-2} + \cdots \qquad n = 0, -1, -2, \cdots$$





Inverse z $\quad X(z^{-1}),\ ROC(z^{-1}) \Rightarrow a_{-n}$

Causal | anti-Causal

$$f(z) = \frac{2}{2-z} \quad (|z| < 2) \qquad f(z^{-1}) = \frac{z}{z-0.5} \quad (|z| > 0.5)$$

$$X(z) = \frac{z}{z-0.5} \quad (|z| > 0.5) \qquad X(z^{-1}) = \frac{2}{2-z} \quad (|z| < 2)$$

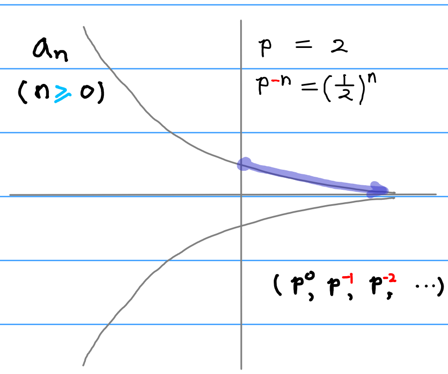
$$X(z) = \frac{z}{z-0.5} = \frac{1}{1-\left(\frac{1}{2z}\right)} = \left(\tfrac{1}{2}\right)^0 z^0 + \left(\tfrac{1}{2}\right)^1 z^{-1} + \left(\tfrac{1}{2}\right)^2 z^{-2} + \cdots \qquad a_n = \left(\tfrac{1}{2}\right)^n$$

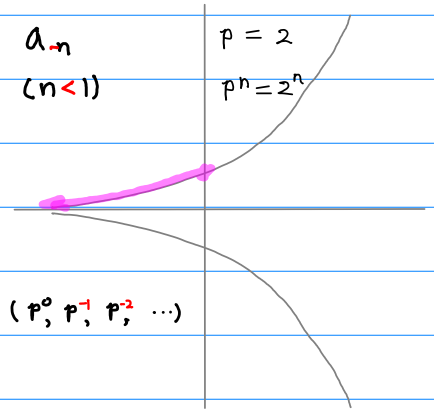
$$= p^0 z^0 + p^{-1} z^{-1} + p^{-2} z^{-2} + \cdots \qquad n = 0, 1, 2, \cdots$$

$$X(z^{-1}) = \frac{z^{-1}}{z^{-1}-0.5} = \frac{1}{1-\left(\frac{z}{2}\right)} = \left(\tfrac{1}{2}\right)^0 z^0 + \left(\tfrac{1}{2}\right)^1 z^1 + \left(\tfrac{1}{2}\right)^2 z^2 + \cdots \qquad a_{-n} = \left(\tfrac{1}{2}\right)^{-n}$$

$$= p^0 z^0 + p^{-1} z^1 + p^{-2} z^2 + \cdots \qquad n = 0, -1, -2, \cdots$$

z





Inverse ROC

$ROC(z) \leftarrow ROC(z^{-1})$

Causal | anti-Causal

$$f(z) = \frac{2}{2-z} \quad (|z| < 2) \quad \Rightarrow \quad -f(z) = -\frac{2}{2-z} \quad (|z| > 2)$$

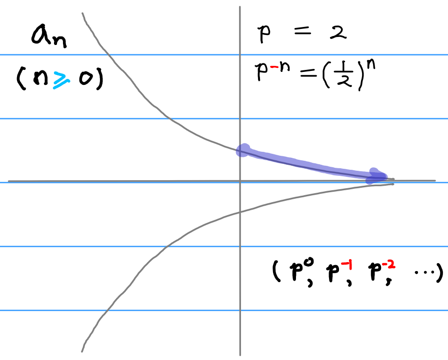
$$X(z) = \frac{z}{z-0.5} \quad (|z| > 0.5) \quad \Rightarrow \quad -X(z) = -\frac{z}{z-0.5} \quad (|z| < 0.5)$$

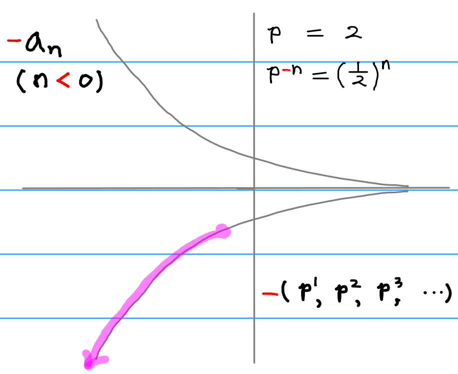
$$f(z) = \frac{2}{2-z} = \frac{1}{1-\left(\frac{z}{2}\right)} = \left(\tfrac{1}{2}\right)^0 z^0 + \left(\tfrac{1}{2}\right)^1 z^1 + \left(\tfrac{1}{2}\right)^2 z^2 + \cdots \qquad a_n = \left(\tfrac{1}{2}\right)^n$$

$$= p^0 z^0 + p^{-1} z^1 + p^{-2} z^2 + \cdots \qquad n = 0, 1, 2, \cdots$$

$$-f(z) = \frac{2}{z-2} = \frac{\left(\frac{2}{z}\right)}{1-\left(\frac{2}{z}\right)} = \left(\tfrac{1}{2}\right)^{-1} z^{-1} + \left(\tfrac{1}{2}\right)^{-2} z^{-2} + \left(\tfrac{1}{2}\right)^{-3} z^{-3} + \cdots \qquad a_n = -\left(\tfrac{1}{2}\right)^n$$

$$= p^1 z^{-1} + p^2 z^{-2} + p^3 z^{-3} + \cdots \qquad n = -1, -2, -3, \cdots$$





Inverse ROC $\quad f(z),\ ROC(z^{-1}) \Rightarrow -a_n$

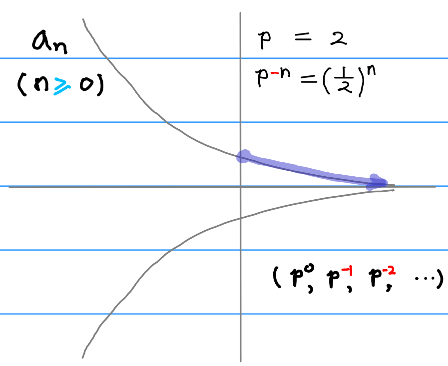
Causal	anti-Causal
$f(z) = \dfrac{2}{2-z} \quad (\lvert z \rvert < 2)$	$f(z) = \dfrac{2}{2-z} \quad (\lvert z \rvert > 0.5)$
$X(z) = \dfrac{z}{z-0.5} \quad (\lvert z \rvert > 0.5)$	$X(z) = \dfrac{z}{z-0.5} \quad (\lvert z \rvert < 2)$

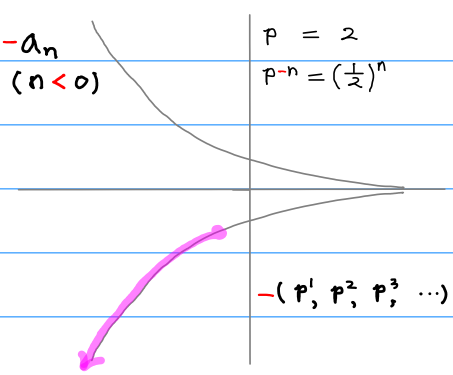
$$f(z) = \frac{2}{2-z} = \frac{1}{1-\left(\frac{z}{2}\right)} = \left(\tfrac{1}{2}\right)^0 z^0 + \left(\tfrac{1}{2}\right)^1 z^1 + \left(\tfrac{1}{2}\right)^2 z^2 + \cdots \qquad \therefore a_n = \left(\tfrac{1}{2}\right)^n$$

$$= p^0 z^0 + p^{-1} z^1 + p^{-2} z^2 + \cdots \qquad n = 0, 1, 2, \cdots$$

$$f(z) = \frac{2}{2-z} = \frac{-\left(\frac{2}{z}\right)}{1-\left(\frac{2}{z}\right)} = -\left[(2)^1 z^{-1} + (2)^2 z^{-2} + \left(\tfrac{1}{2}\right)^3 z^{-3} + \cdots\right] \qquad \therefore -a_n = -\left(\tfrac{1}{2}\right)^n$$

$$= -\left[p^1 z^{-1} + p^2 z^{-2} + p^3 z^{-3} + \cdots\right] \qquad n = -1, -2, -3, \cdots$$





Inverse ROC $X(z)$, $ROC(z^{-1}) \Rightarrow -a_n$

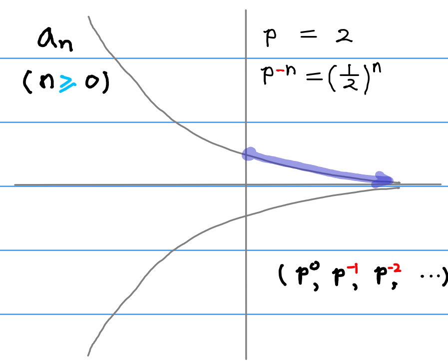
Causal	anti-Causal
$f(z) = \dfrac{2}{2-z} \quad (\lvert z \rvert < 2)$	$f(z) = \dfrac{2}{2-z} \quad (\lvert z \rvert > 0.5)$
$X(z) = \dfrac{z}{z-0.5} \quad (\lvert z \rvert > 0.5)$	$X(z) = \dfrac{z}{z-0.5} \quad (\lvert z \rvert < 2)$

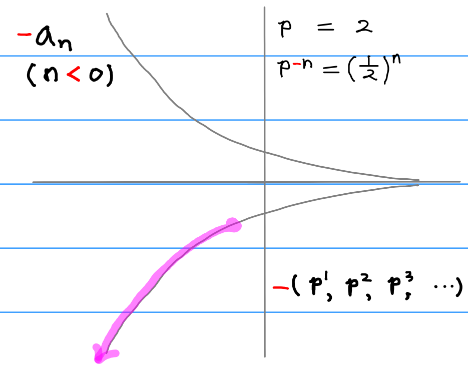
$$X(z) = \frac{z}{z-0.5} = \frac{1}{1-\left(\frac{1}{2z}\right)} = \left(\tfrac{1}{2}\right)^0 z^0 + \left(\tfrac{1}{2}\right)^1 z^{-1} + \left(\tfrac{1}{2}\right)^2 z^{-2} + \cdots \qquad \therefore a_n = \left(\tfrac{1}{2}\right)^n$$

$$= p^0 z^0 + p^{-1} z^{-1} + p^{-2} z^{-2} + \cdots \qquad n = 0, 1, 2, \cdots$$

$$X(z) = \frac{z}{z-0.5} = \frac{-(2z)}{1-(2z)} = -\left[(2)^1 z^1 + (2)^2 z^2 + \left(\tfrac{1}{2}\right)^3 z^3 + \cdots\right] \qquad \therefore -a_n = -\left(\tfrac{1}{2}\right)^n$$

$$= -\left[p^1 z^1 + p^2 z^2 + p^3 z^3 + \cdots\right] \qquad n = -1, -2, -3, \cdots$$





p=2

causal	anti-causal
$a_n = \left(\frac{1}{2}\right)^n \quad (n \geq 0)$	$a_n = \left(\frac{1}{2}\right)^n \quad (n < 0)$
$f(z) = \frac{1}{1-\left(\frac{z}{2}\right)} \quad (\lvert z\rvert < 2)$ $X(z) = \frac{1}{1-\left(\frac{1}{2z}\right)} \quad (\lvert z\rvert > 0.5)$	$f(z) = \frac{\left(\frac{2}{z}\right)}{1-\left(\frac{2}{z}\right)} \quad (\lvert z\rvert > 2)$ $X(z) = \frac{(2z)}{1-(2z)} \quad (\lvert z\rvert < 0.5)$
$f(z) = \frac{2}{2-z} \quad (\lvert z\rvert < 2)$ $X(z) = \frac{z}{z-0.5} \quad (\lvert z\rvert > 0.5)$	$f(z) = \frac{2}{z-2} \quad (\lvert z\rvert > 2)$ $X(z) = \frac{z}{0.5-z} \quad (\lvert z\rvert < 0.5)$

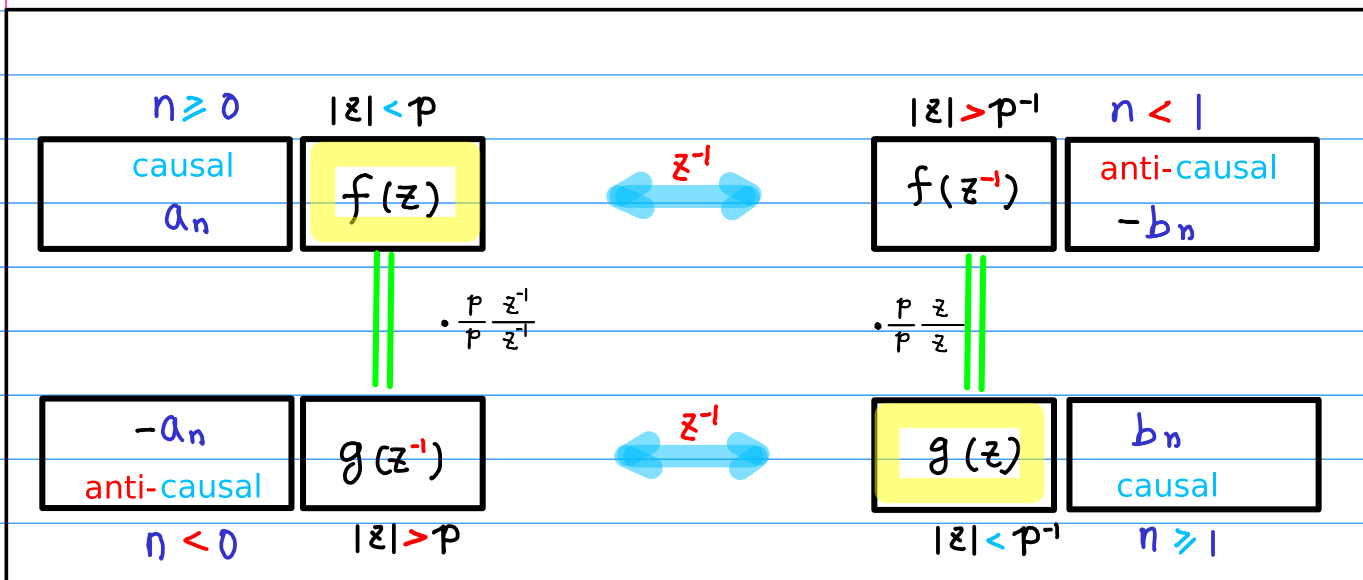
p=1/2

causal	anti-causal
$a_n = (2)^n \quad (n \geq 0)$	$a_n = (2)^n \quad (n < 0)$
$f(z) = \frac{1}{1-(2z)} \quad (\lvert z\rvert < 0.5)$ $X(z) = \frac{1}{1-\left(\frac{2}{z}\right)} \quad (\lvert z\rvert > 2)$	$f(z) = \frac{\left(\frac{1}{2z}\right)}{1-\left(\frac{1}{2z}\right)} \quad (\lvert z\rvert > 0.5)$ $X(z) = \frac{\left(\frac{z}{2}\right)}{1-\left(\frac{z}{2}\right)} \quad (\lvert z\rvert < 2)$
$f(z) = \frac{0.5}{0.5-z} \quad (\lvert z\rvert < 0.5)$ $X(z) = \frac{z}{z-2} \quad (\lvert z\rvert > 2)$	$f(z) = \frac{0.5}{z-0.5} \quad (\lvert z\rvert > 0.5)$ $X(z) = \frac{z}{2-z} \quad (\lvert z\rvert < 2)$

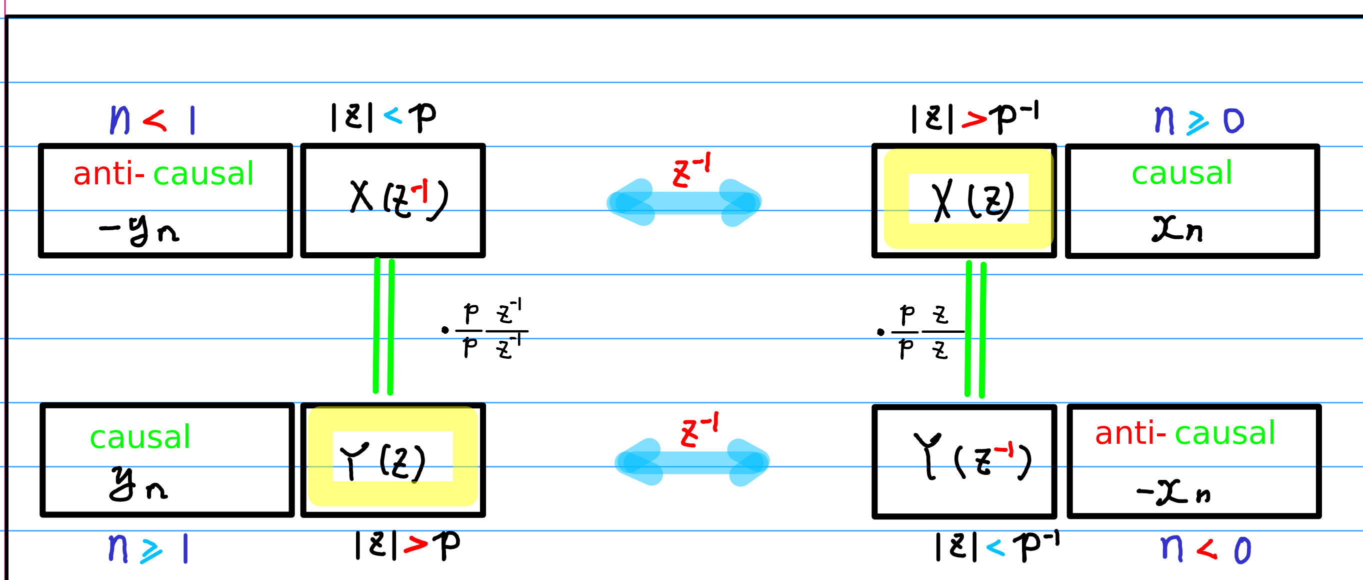
$f(z)$ $g(z^{-1})$	$f(z^{-1})$ $g(z)$

$X(z^{-1})$ $Y(z)$	$X(z)$ $Y(z^{-1})$

Laurent Series $\quad a_n \leftrightarrow f(z) \quad b_n \leftrightarrow g(z)$



Z- Transform $\quad X(z) \leftrightarrow x_n \quad Y(z) \leftrightarrow y_n$



$f(z)$	$f(z^{-1})$	$X(z^{-1})$	$X(z)$
$g(z^{-1})$	$g(z)$	$Y(z)$	$Y(z^{-1})$

Laurent Series $a_n \leftrightarrow f(z)$ $\quad -a_{-n} = b_n \leftrightarrow g(z)$

causal
a_n

$\xleftrightarrow{z^{-1}}$

anti-causal
a_{-n}

$-a_n$
anti-causal

$\xleftrightarrow{z^{-1}}$

$-a_{-n}$
causal

Z-Transform $X(z) \leftrightarrow x_n$ $\quad Y(z) \leftrightarrow y_n = -x_{-n}$

anti-causal
x_{-n}

$\xleftrightarrow{z^{-1}}$

causal
x_n

causal
$-x_{-n}$

$\xleftrightarrow{z^{-1}}$

anti-causal
$-x_n$

$$f(z),\ \ X(z) \ = \ \frac{2}{2-z}\ ,\ \frac{z}{z-0.5}$$

f(z) Laurent Series

$n \in [0,\infty)$ causal $a_n = 0.5^n$	$\lvert z\rvert < 2$ $\frac{2}{2-z}$	$\xleftrightarrow{z^{-1}}$	$\lvert z\rvert > 0.5$ $\frac{z}{z-0.5}$	$n \in (-\infty,0]$ anti-causal $a_{-n} = 0.5^{-n}$
	$\cdot \frac{p}{p}\frac{z^{-1}}{z^{-1}}$		$\cdot \frac{p}{p}\frac{z}{z}$	
$-a_n = -0.5^n$ anti-causal $n \in (-\infty,-1]$	$\frac{2}{2-z}$ $\lvert z\rvert > 2$	$\xleftrightarrow{z^{-1}}$	$\frac{z}{z-0.5}$ $\lvert z\rvert < 0.5$	$-a_{-n} = -0.5^{-n}$ causal $n \in [1,\infty)$

X(z) z-Transform

$n \in (-\infty,0]$ anti-causal X(z) $a_{-n} = 0.5^{-n}$	$\lvert z\rvert < p$ $\frac{2}{2-z}$	$\xleftrightarrow{z^{-1}}$	$\lvert z\rvert > p^{-1}$ $\frac{z}{z-0.5}$	$n \in [0,\infty)$ causal X(z) $a_n = 0.5^n$
	$\cdot \frac{p}{p}\frac{z^{-1}}{z^{-1}}$		$\cdot \frac{p}{p}\frac{z}{z}$	
$-a_{-n} = -0.5^{-n}$ causal X(z) $n \in [1,\infty)$	$\frac{2}{2-z}$ $\lvert z\rvert > p$	$\xleftrightarrow{z^{-1}}$	$\frac{z}{z-0.5}$ $\lvert z\rvert < p^{-1}$	$-a_n = -0.5^n$ anti-causal X(z) $n \in (-\infty,-1]$

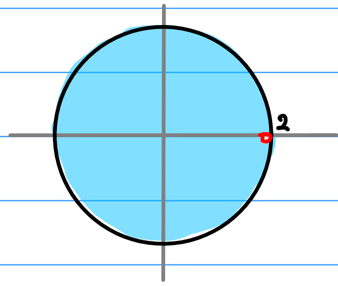
$$f(z) = \frac{z}{2-z}, \quad \frac{z}{z-0.5}$$

0.5^{n}	0.5^{-n}
-0.5^{n}	-0.5^{-n}

$n \in [0, \infty)$ $\quad |z| < 2$

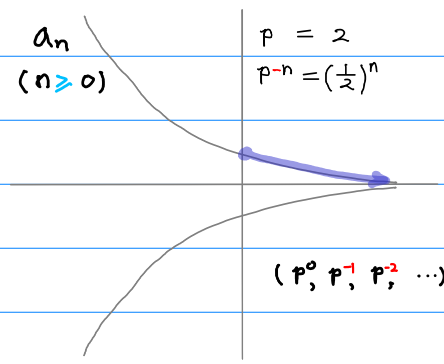
causal

$a_n = 0.5^{n}$ $\quad \boxed{\dfrac{z}{2-z}}$



a_n

$(n \geq 0)$

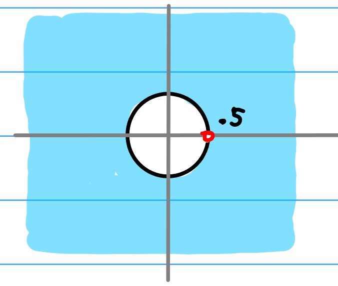


$|z| > 0.5$ $\quad n \in (-\infty, 0]$

$\boxed{\dfrac{z}{z-0.5}}$

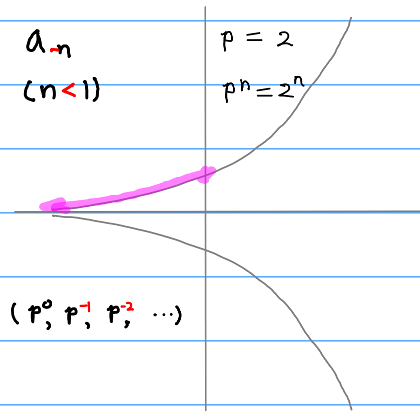
anti-causal

$a_{-n} = 0.5^{-n}$



a_{-n}

$(n < 1)$

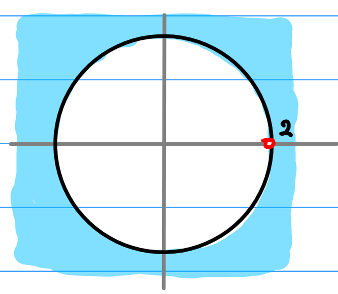


$-a_n = -0.5^{n}$

anti-causal

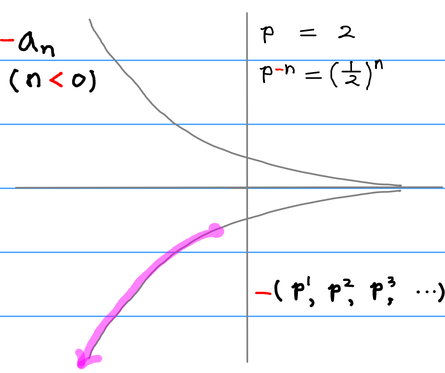
$\boxed{\dfrac{z}{2-z}}$

$n \in (-\infty, -1]$ $\quad |z| > 2$



$-a_n$

$(n < 0)$

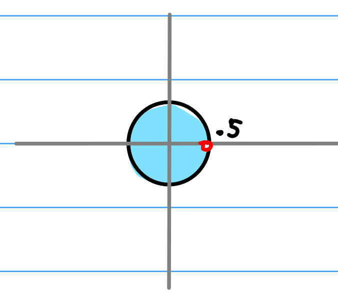


$\boxed{\dfrac{z}{z-0.5}}$

$-a_{-n} = -0.5^{-n}$

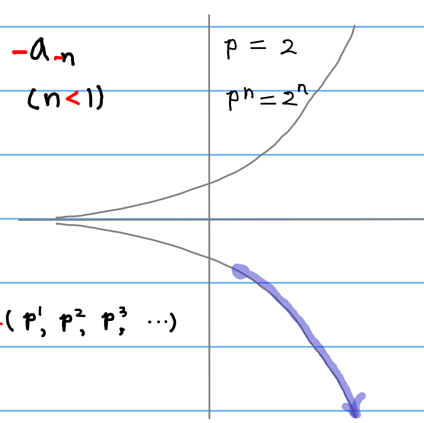
causal

$|z| < 0.5$ $\quad n \in [1, \infty)$



$-a_{-n}$

$(n < 1)$



$$X(z) = \frac{2}{2-z}\,,\quad \frac{z}{z-0.5}$$

0.5^{-n}	0.5^{n}
-0.5^{-n}	-0.5^{n}

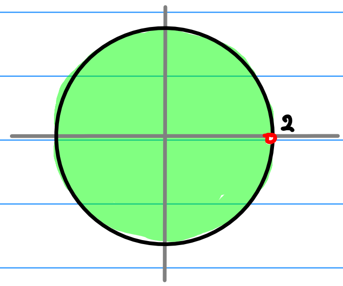
$n < 1$ $\quad |z| < 2$

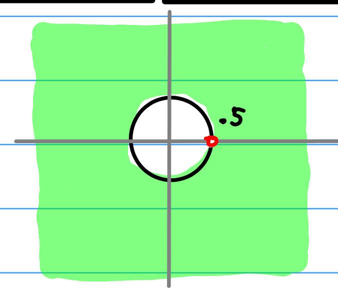
x_{-n} — anti-causal, $x_{-n} = 0.5^{-n}$ — $\boxed{-\dfrac{0.5}{1-0.5z}}$

$\overset{z^{-1}}{\longleftrightarrow}$

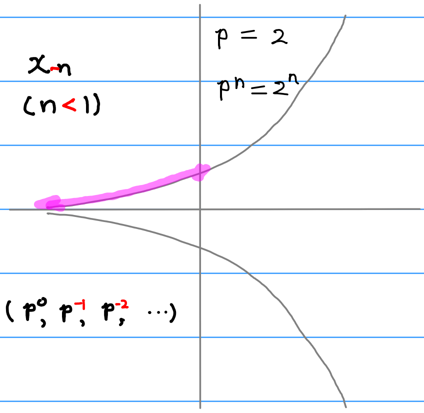
$|z| > 0.5$ $\quad n \geq 0$

$\boxed{-\dfrac{0.5}{1-0.5z^{-1}}}$ — causal, $x_n = 0.5^{n}$ — x_n



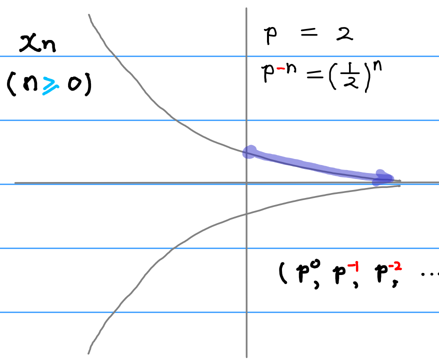


x_{-n}
$(n<1)$



$(p^0, p^{-1}, p^{-2} \cdots)$

x_n
$(n \geq 0)$



$(p^0, p^{-1}, p^{-2} \cdots)$

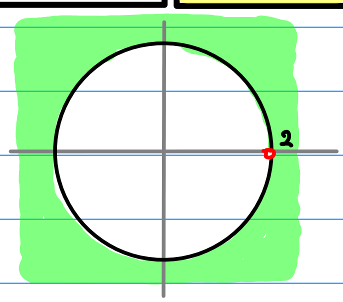
$n \geq 1$ $\quad |z| > 2$

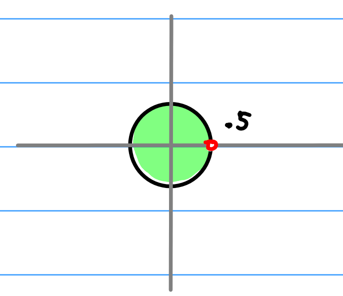
y_n — $-x_{-n} = -0.5^{-n}$, causal — $\boxed{\dfrac{z^{-1}}{1-2z^{-1}}}$

$\overset{z^{-1}}{\longleftrightarrow}$

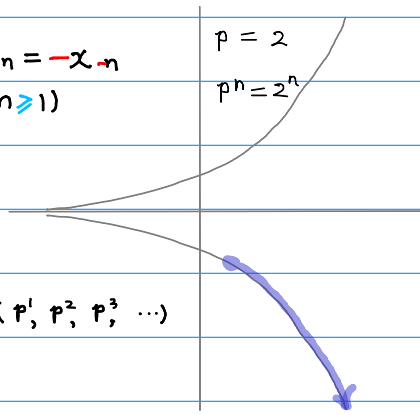
$|z| < 0.5$ $\quad n < 0$

$\boxed{\dfrac{z}{1-2z}}$ — $-x_n = -0.5^{n}$, anti-causal — y_{-n}



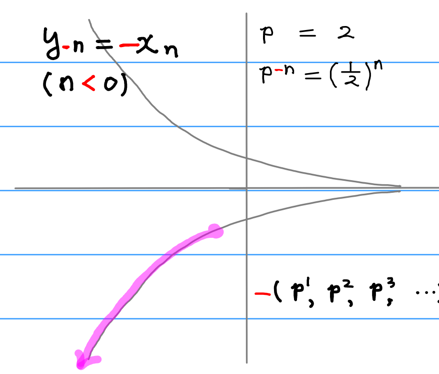


$y_n = -x_{-n}$
$(n \geq 1)$



$-(p^1, p^2, p^3 \cdots)$

$y_{-n} = -x_n$
$(n < 0)$



$-(p^1, p^2, p^3 \cdots)$

2 formulas Ⓐ

$$\frac{1}{z-p} \qquad \frac{1}{z^{-1}-p}$$

2 representations each

$$\frac{1}{z-p} \quad \begin{cases} -\dfrac{p^{-1}}{1-p^{-1}z} \\[2ex] \dfrac{z^{-1}}{1-pz^{-1}} \end{cases}$$

$$\frac{1}{z^{-1}-p} \quad \begin{cases} -\dfrac{p^{-1}}{1-p^{-1}z^{-1}} \\[2ex] \dfrac{z}{1-pz} \end{cases}$$

2 formulas Ⓑ

$$\frac{p}{z-p} \qquad \frac{p}{z^{-1}-p}$$

2 representations each

$$\frac{p}{z-p} \quad \begin{cases} -\dfrac{1}{1-p^{-1}z} \\[2ex] \dfrac{pz^{-1}}{1-pz^{-1}} \end{cases}$$

$$\frac{p}{z^{-1}-p} \quad \begin{cases} -\dfrac{1}{1-p^{-1}z^{-1}} \\[2ex] \dfrac{pz}{1-pz} \end{cases}$$

$$\xrightarrow{\;p\;}$$
$$\xleftarrow[\;1/p\;]{}$$

Ⓐ $f(z) \longrightarrow a_n$

2 formulas Ⓐ

$$\boxed{\frac{1}{z-p}} \qquad \boxed{\frac{1}{z^{-1}-p}}$$

2 representations each

$$\boxed{\frac{1}{z-p}}$$

causal

$$\boxed{-\frac{p^{-1}}{1-p^{-1}z}} \triangleq f(z) \qquad -(p^{-1})^{n+1} \quad (n \geq 0)$$

$$\|$$

$$\boxed{\frac{z^{-1}}{1-pz^{-1}}} \triangleq g(z^{-1}) \qquad (p^{-1})^{n+1} \quad (n<0)$$

anti- causal

$$\boxed{\frac{1}{z^{-1}-p}}$$

anti- causal

$$\boxed{-\frac{p^{-1}}{1-p^{-1}z^{-1}}} \triangleq f(z^{-1}) \qquad -(p^{-1})^{-n+1} \quad (n<1)$$

$$\|$$

$$\boxed{\frac{z}{1-pz}} \triangleq g(z) \qquad (p^{-1})^{-n+1} \quad (n \geq 1)$$

causal

$$\boxed{-\frac{1}{2-z}}$$

$$\boxed{-\frac{0.5}{1-0.5z}}$$ causal -0.5^{n+1} $(n \geq 0)$

$$\boxed{\frac{z^{-1}}{1-2z^{-1}}}$$ anti 0.5^{n+1} $(n<0)$

$$\boxed{-\frac{0.5z}{z-0.5}}$$

$$\boxed{-\frac{0.5}{1-0.5z^{-1}}}$$ anti -0.5^{-n+1} $(n<1)$

$$\boxed{\frac{z}{1-2z}}$$ causal 0.5^{-n+1} $(n \geq 1)$

Ⓐ $f(z) \longrightarrow a_n$ method 1

p^{-1} $\boxed{-\frac{p^{-1}}{1-p^{-1}z}}$ $-(p^{-1}) \rightarrow -(p^{-1})^{n+1} \rightarrow -(p^{-1})^{n+1}$

$n = 0, 1, 2, \cdots$

$-(p^{-1}z^0 + p^{-2}z^1 + p^{-3}z^2 + \cdots)$

p $\boxed{\frac{z^{-1}}{1-pz^{-1}}}$ $(p) \rightarrow (p)^{-n-1} \rightarrow (p^{-1})^{n+1}$

$n = -1, -2, -3, \cdots$

$(p^0z^{-1} + p^1z^{-2} + p^2z^{-3} + \cdots)$

p^{-1} $\boxed{-\frac{p^{-1}}{1-p^{-1}z^{-1}}}$ $-(p^{-1}) \rightarrow -(p^{-1})^{-n+1} \rightarrow -(p^{-1})^{-n+1}$

$n = 0, -1, -2, \cdots$

$-(p^{-1}z^0 + p^{-2}z^{-1} + p^{-3}z^{-2} + \cdots)$

p $\boxed{\frac{z}{1-pz}}$ $(p) \rightarrow (p)^{n-1} \rightarrow (p^{-1})^{-n+1}$

$n = 1, 2, 3, \cdots$

$(p^0z^1 + p^1z^2 + p^2z^3 + \cdots)$

Ⓐ $f(z) \longrightarrow a_n$ method 2

p^{-1} $\boxed{-\dfrac{p^{-1}}{1-p^{-1}z}}$ $\dfrac{-(p^{-1})}{z} \rightarrow n=0,1,2,\cdots \rightarrow \dfrac{-(p^{-1})^{n+1}}{(p)^{-n-1}}$

p $\boxed{\dfrac{z^{-1}}{1-pz^{-1}}}$ $\dfrac{(p)}{z^{-1}/z^{-1}} \rightarrow n=-1,-2,-3,\cdots \rightarrow \dfrac{(p^{-1})^{n+1}}{(p)^{-n-1}}$

p $\boxed{-\dfrac{p^{-1}}{1-p^{-1}z^{-1}}}$ $\dfrac{-(p^{-1})}{z^{-1}} \rightarrow n=0,-1,-2,\cdots \rightarrow \dfrac{-(p^{-1})^{-n+1}}{(p)^{n-1}}$

p^{-1} $\boxed{\dfrac{z}{1-pz}}$ $\dfrac{(p)}{z/z} \rightarrow n=1,2,3,\cdots \rightarrow \dfrac{(p^{-1})^{-n+1}}{(p)^{n-1}}$

Ⓑ $f(z) \longrightarrow a_n$

2 formulas

$$\boxed{\frac{p}{z-p}} \qquad \boxed{\frac{p}{z^{-1}-p}}$$

2 representations each

$$\boxed{\frac{p}{z-p}}$$

$$\boxed{-\frac{1}{1-p^{-1}z}} \triangleq f'(z) \qquad -(p^{-1})^n \qquad (n \geq 0)$$

causal

$$\|$$

$$\boxed{\frac{pz^{-1}}{1-pz^{-1}}} \triangleq g'(z^{-1}) \qquad (p^{-1})^n \qquad (n < 0)$$

anti- causal

$$\boxed{\frac{p}{z^{-1}-p}}$$

anti- causal

$$\boxed{-\frac{1}{1-p^{-1}z^{-1}}} \triangleq f'(z^{-1}) \qquad -(p^{-1})^{-n} \qquad (n < 1)$$

$$\|$$

$$\boxed{\frac{pz}{1-pz}} \triangleq g'(z) \qquad (p^{-1})^{-n} \qquad (n \geq 1)$$

causal

$$\boxed{\frac{2}{2-z}}$$

$$\boxed{\frac{1}{1-0.5z}}$$ causal $0.5^n \ (n \geq 0)$

$$\boxed{\frac{-2z^{-1}}{1-2z^{-1}}}$$ anti $-0.5^n \ (n<0)$

$$\boxed{\frac{z}{z-0.5}}$$

$$\boxed{\frac{1}{1-0.5z^{-1}}}$$ anti $0.5^{-n} \ (n<1)$

$$\boxed{\frac{-2z}{1-2z}}$$ causal $-0.5^{-n} \ (n \geq 1)$

Ⓑ $f(z) \longrightarrow a_n$ method 1

p^{-1} $\boxed{-\dfrac{1}{1-p^{-1}z}}$ $-(p^{-1}) \rightarrow -(p^{-1})^n \rightarrow -(p^{-1})^n$

$n = 0, 1, 2, \cdots$

$-(p^0 z^0 + p^{-1} z^1 + p^{-2} z^2 + \cdots)$

p $\boxed{\dfrac{p z^{-1}}{1-p z^{-1}}}$ $(p) \rightarrow (p)^{-n} \rightarrow (p^{-1})^n$

$n = -1, -2, -3, \cdots$

$(p^1 z^{-1} + p^2 z^{-2} + p^3 z^{-3} + \cdots)$

p $\boxed{-\dfrac{1}{1-p^{-1}z^{-1}}}$ $-(p^{-1}) \rightarrow -(p^{-1})^{-n} \rightarrow -(p^{-1})^{-n}$

$n = 0, -1, -2, \cdots$

$-(p^0 z^0 + p^{-1} z^{-1} + p^{-2} z^{-2} + \cdots)$

p^{-1} $\boxed{\dfrac{pz}{1-pz}}$ $(p) \rightarrow (p)^n \rightarrow (p^{-1})^{-n}$

$n = 1, 2, 3, \cdots$

$(p^1 z^1 + p^2 z^2 + p^3 z^3 + \cdots)$

Ⓑ $f(z) \longrightarrow a_n$ method 2

p^{-1} $\boxed{-\dfrac{1}{1-p^{-1}z}}$ $\quad -(p^{-1})$ z $\rightarrow$ $n=0,1,2,\cdots$ $\rightarrow$ $-p^{-n}$ $(p)^{-n}$

p $\boxed{\dfrac{pz^{-1}}{1-pz^{-1}}}$ $\quad (p)$ z^{-1}/z^{-1} $\rightarrow$ $n=-1,-2,-3,\cdots$ $\rightarrow$ p^{-n} $(p)^{-n}$

p $\boxed{-\dfrac{1}{1-p^{-1}z^{-1}}}$ $\quad -(p^{-1})$ z^{-1} $\rightarrow$ $n=0,-1,-2,\cdots$ $\rightarrow$ $-(p^{-1})^{-n}$ $(p)^{n}$

p^{-1} $\boxed{\dfrac{pz}{1-pz}}$ $\quad (p)$ z/z $\rightarrow$ $n=1,2,3,\cdots$ $\rightarrow$ $(p^{-1})^{-n}$ $(p)^{n}$

2 formulas

$$\boxed{\frac{p}{z-p}} \qquad \boxed{\frac{p}{z^{-1}-p}}$$

2 representations each

$$\boxed{\frac{p}{z-p}}$$

causal, anti- causal

$$\boxed{-\frac{1}{1-p^{-1}z}} \triangleq f'(z) = X'(z^{-1})$$

$$\Vert \qquad \Vert$$

$$\boxed{\frac{pz^{-1}}{1-pz^{-1}}} \triangleq Y'(z) = g'(z^{-1})$$

causal, anti- causal

$$\boxed{\frac{p}{z^{-1}-p}}$$

causal, anti- causal

$$\boxed{-\frac{1}{1-p^{-1}z^{-1}}} \triangleq X'(z) = f'(z^{-1})$$

$$\Vert \qquad \Vert$$

$$\boxed{\frac{pz}{1-pz}} \triangleq g'(z) = Y'(z^{-1})$$

causal, anti- causal

$n \in [0, \infty)$ causal $a_n = 0.5^n$ $\quad$ $|z| < 2$ $\quad$ $\boxed{\dfrac{2}{2-z}}$ $\xleftrightarrow{z^{-1}}$ $\boxed{\dfrac{z}{z-0.5}}$ $\quad$ $|z| > 0.5$ $\quad$ $n \in (-\infty, 0]$ anti-causal $a_{-n} = 0.5^{-n}$

$$\cdot \frac{p}{p}\,\frac{z^{-1}}{z^{-1}} \qquad\qquad \cdot \frac{p}{p}\,\frac{z}{z}$$

$-a_n = -0.5^n$ anti-causal $\quad$ $n \in (-\infty, -1]$ $\quad$ $\boxed{\dfrac{2}{2-z}}$ $\quad$ $|z| > 2$ $\xleftrightarrow{z^{-1}}$ $\boxed{\dfrac{z}{z-0.5}}$ $\quad$ $|z| < 0.5$ $\quad$ $-a_{-n} = -0.5^{-n}$ causal $\quad$ $n \in [1, \infty)$

$$\begin{tikzcd}
\dfrac{2}{2-z} \arrow[r, leftrightarrow, "z^{-1}"] \arrow[d, "-\frac{1}{2}"] & \dfrac{z}{z-0.5} \arrow[d, "-\frac{1}{2}"] \\
-\dfrac{1}{2-z} \arrow[r, leftrightarrow, "z^{-1}"] & -\dfrac{0.5z}{z-0.5}
\end{tikzcd}$$

$$0.5^{-n+1} = \left(\frac{1}{2}\right)^{-n+1}$$

$$= 2^{+n-1}$$

- $\dfrac{2}{2-z}$
  - $\dfrac{1}{1-0.5z}$ causal: $0.5^{n}\ (n\geq 0)$
  - $\dfrac{-2z^{-1}}{1-2z^{-1}}$ anti: $-0.5^{n}\ (n<0)$

- $\dfrac{z}{z-0.5}$
  - $\dfrac{1}{1-0.5z^{-1}}$ anti: $0.5^{-n}\ (n<1)$
  - $\dfrac{-2z}{1-2z}$ causal: $-0.5^{-n}\ (n\geq 1)$

- $-\dfrac{1}{2-z}$
  - $\dfrac{-0.5}{1-0.5z}$ causal: $-0.5^{n+1}\ (n\geq 0)$
  - $\dfrac{z^{-1}}{1-2z^{-1}}$ anti: $0.5^{n+1}\ (n<0)$

- $-\dfrac{0.5z}{z-0.5}$
  - $\dfrac{-0.5}{1-0.5z^{-1}}$ anti: $-0.5^{-n+1}\ (n<1)$
  - $\dfrac{z}{1-2z}$ causal: $0.5^{-n+1}\ (n\geq 1)$

$$\frac{2}{2-z},\ \frac{z}{z-0.5} \quad \text{vs.} \quad -\frac{0.5}{1-0.5z},\ \frac{z}{1-2z}$$

$n \in [0,\infty)$ $\quad |z| < 2$ $\quad\quad |z| > 0.5$ $\quad n \in (-\infty, 0]$

causal $a_n = 0.5^n$	$\frac{2}{2-z}$	$\xleftrightarrow{z^{-1}}$	$\frac{z}{z-0.5}$	anti-causal $a_{-n} = 0.5^{-n}$

$\cdot \frac{p}{p}\ \frac{z^{-1}}{z^{-1}}$ $\quad\quad \cdot \frac{p}{p}\ \frac{z}{z}$

$-a_n = -0.5^n$ anti-causal	$\frac{2}{2-z}$	$\xleftrightarrow{z^{-1}}$	$\frac{z}{z-0.5}$	$-a_{-n} = -0.5^{-n}$ causal

$n \in (-\infty, -1]$ $\quad |z| > 2$ $\quad\quad |z| < 0.5$ $\quad n \in [1,\infty)$

$n \in [0,\infty)$ $\quad |z| < 2$ $\quad\quad |z| > 0.5$ $\quad n \in (-\infty, 0]$

a_n	causal -0.5^{n+1}	$-\frac{0.5}{1-0.5z}$	$\xleftrightarrow{z^{-1}}$	$-\frac{0.5}{1-0.5z^{-1}}$	anti-causal -0.5^{-n+1}

-2^{+n-1}

2^{+n-1}

b.	0.5^{n+1} anti-causal	$\frac{z^{-1}}{1-2z^{-1}}$	$\xleftrightarrow{z^{-1}}$	$\frac{z}{1-2z}$	0.5^{-n+1} causal

$n \in (-\infty, -1]$ $\quad |z| > 2$ $\quad\quad |z| < 0.5$ $\quad n \in [1,\infty)$

$$\frac{2}{2-z}$$

$$\frac{1}{1-0.5z}$$ causal 0.5^{n} $(n \geq 0)$

$$\frac{-2z^{-1}}{1-2z^{-1}}$$ anti -2^{-n} $(n<0)$

$$\frac{z}{z-0.5}$$

$$\frac{1}{1-0.5z^{-1}}$$ anti 0.5^{-n} $(n<1)$

$$\frac{-2z}{1-2z}$$ causal -0.5^{-n} $(n \geq 1)$

$n \in [0,\infty)$ $|z|<2$

causal $a_n = 0.5^{n}$ $\boxed{\frac{2}{2-z}}$ $\xleftrightarrow{z^{-1}}$ $\boxed{\frac{z}{z-0.5}}$ anti-causal $a_{-n} = 0.5^{-n}$

$|z|>0.5$ $n \in (-\infty, 0]$

$\cdot \frac{p}{p} \frac{z^{-1}}{z^{-1}}$ $\cdot \frac{p}{p} \frac{z}{z}$

$-a_n = -0.5^{n}$ anti-causal $\boxed{\frac{2}{2-z}}$ $\xleftrightarrow{z^{-1}}$ $\boxed{\frac{z}{z-0.5}}$ $-a_{-n} = -0.5^{-n}$ causal

$n \in (-\infty, -1]$ $|z|>2$ $|z|<0.5$ $n \in [1,\infty)$

$$-\frac{1}{2-z}$$

- $\dfrac{-0.5}{1-0.5z}$ causal: -0.5^{n+1} $(n \geq 0)$
- $\dfrac{z^{-1}}{1-2z^{-1}}$ anti: 0.5^{n+1} $(n < 0)$

$$-\frac{0.5z}{z-0.5}$$

- $\dfrac{-0.5}{1-0.5z^{-1}}$ anti: -0.5^{-n+1} $(n < 1)$
- $\dfrac{z}{1-2z}$ causal: 0.5^{-n+1} $(n \geq 1)$

a_n: $n \in [0, \infty)$, causal, -0.5^{n+1} | $|z| < 2$, $-\dfrac{0.5}{1-0.5z}$ $\xleftrightarrow{z^{-1}}$ $|z| > 0.5$, $-\dfrac{0.5}{1-0.5z^{-1}}$ | $n \in (-\infty, 0]$, anti-causal, -0.5^{-n+1}

b_n: 0.5^{n+1}, anti-causal, $n \in (-\infty, -1]$ | $\dfrac{z^{-1}}{1-2z^{-1}}$, $|z| > 2$ $\xleftrightarrow{z^{-1}}$ $\dfrac{z}{1-2z}$, $|z| < 0.5$ | 0.5^{-n+1}, causal, $n \in [1, \infty)$

a_n: $n \in [0, \infty)$, causal, -0.5^{n+1} | $|z| < 2$, $-\dfrac{0.5}{1-0.5z}$ $\xleftrightarrow{z^{-1}}$ $|z| > 0.5$, $-\dfrac{0.5}{1-0.5z^{-1}}$ | $n \in (-\infty, 0]$, anti-causal, -2^{+n-1}

b_n: 0.5^{n+1}, anti-causal, $n \in (-\infty, -1]$ | $\dfrac{z^{-1}}{1-2z^{-1}}$, $|z| > 2$ $\xleftrightarrow{z^{-1}}$ $\dfrac{z}{1-2z}$, $|z| < 0.5$ | 2^{+n-1}, causal, $n \in [1, \infty)$

Time Shift

$p = 2$

1 $(n \geq 0)$ $a_n = \left(\frac{1}{2}\right)^n$ $f(z) = \frac{2}{2-z}$ $X(z) = \frac{z}{z-0.5}$

3 $(n < 0)$ $a_n = \left(\frac{1}{2}\right)^n$ $f(z) = -\frac{2}{2-z}$ $X(z) = -\frac{z}{z-0.5}$

5 $(n \geq 1)$ $a_{n-1} = \left(\frac{1}{2}\right)^{n-1}$ $f(z) = \frac{2z}{2-z}$ $X(z) = \frac{1}{z-0.5}$

7 $(n < 1)$ $a_{n-1} = \left(\frac{1}{2}\right)^{n-1}$ $f(z) = -\frac{2z}{2-z}$ $X(z) = -\frac{1}{z-0.5}$

9 $(n \geq -1)$ $a_{n+1} = \left(\frac{1}{2}\right)^{n+1}$ $f(z) = \frac{2}{(2-z)z}$ $X(z) = \frac{z^2}{z-0.5}$

11 $(n < -1)$ $a_{n+1} = \left(\frac{1}{2}\right)^{n+1}$ $f(z) = -\frac{2}{(2-z)z}$ $X(z) = -\frac{z^2}{z-0.5}$

Time Shift

$p = \frac{1}{2}$

② $(n \geq 0)$ $a_n = (2)^n$ $f(z) = \frac{0.5}{0.5 - z}$ $X(z) = \frac{z}{z-2}$

④ $(n < 0)$ $a_n = (2)^n$ $f(z) = -\frac{0.5}{0.5 - z}$ $X(z) = -\frac{z}{z-2}$

⑥ $(n \geq 1)$ $a_{n-1} = (2)^{n-1}$ $f(z) = \frac{0.5z}{0.5 - z}$ $X(z) = \frac{1}{z-2}$

⑧ $(n < 1)$ $a_{n-1} = (2)^{n-1}$ $f(z) = -\frac{0.5z}{0.5 - z}$ $X(z) = -\frac{1}{z-2}$

⑩ $(n \geq -1)$ $a_{n+1} = (2)^{n+1}$ $f(z) = \frac{0.5}{(0.5 - z)z}$ $X(z) = \frac{z^2}{z-2}$

⑫ $(n < -1)$ $a_{n+1} = (2)^{n+1}$ $f(z) = -\frac{0.5}{(0.5 - z)z}$ $X(z) = -\frac{z^2}{z-2}$

Time Shift

$2 \longleftrightarrow \frac{1}{2}$

1	$(n \geq 0)$	$a_n = \left(\frac{1}{2}\right)^n$	$f(z) = \frac{2}{2-z}$	$X(z) = \frac{z}{z-0.5}$
2	$(n \geq 0)$	$a_n = (2)^n$	$f(z) = \frac{0.5}{0.5-z}$	$X(z) = \frac{z}{z-2}$
3	$(n < 0)$	$a_n = \left(\frac{1}{2}\right)^n$	$f(z) = -\frac{2}{2-z}$	$X(z) = -\frac{z}{z-0.5}$
4	$(n < 0)$	$a_n = (2)^n$	$f(z) = -\frac{0.5}{0.5-z}$	$X(z) = -\frac{z}{z-2}$
5	$(n \geq 1)$	$a_{n-1} = \left(\frac{1}{2}\right)^{n-1}$	$f(z) = \frac{2z}{2-z}$	$X(z) = \frac{1}{z-0.5}$
6	$(n \geq 1)$	$a_{n-1} = (2)^{n-1}$	$f(z) = \frac{0.5z}{0.5-z}$	$X(z) = \frac{1}{z-2}$
7	$(n < 1)$	$a_{n-1} = \left(\frac{1}{2}\right)^{n-1}$	$f(z) = -\frac{2z}{2-z}$	$X(z) = -\frac{1}{z-0.5}$
8	$(n < 1)$	$a_{n-1} = (2)^{n-1}$	$f(z) = -\frac{0.5z}{0.5-z}$	$X(z) = -\frac{1}{z-2}$
9	$(n \geq -1)$	$a_{n+1} = \left(\frac{1}{2}\right)^{n+1}$	$f(z) = \frac{2}{(2-z)z}$	$X(z) = \frac{z^2}{z-0.5}$
10	$(n \geq -1)$	$a_{n+1} = (2)^{n+1}$	$f(z) = \frac{0.5}{(0.5-z)z}$	$X(z) = \frac{z^2}{z-2}$
11	$(n < -1)$	$a_{n+1} = \left(\frac{1}{2}\right)^{n+1}$	$f(z) = -\frac{2}{(2-z)z}$	$X(z) = -\frac{z^2}{z-0.5}$
12	$(n < -1)$	$a_{n+1} = (2)^{n+1}$	$f(z) = -\frac{0.5}{(0.5-z)z}$	$X(z) = -\frac{z^2}{z-2}$

Shift to the right

shift to the right →	$\times z$	$\times z^{-1}$
delete a_0		
① $(n \geq 0)$ $a_n = \left(\frac{1}{2}\right)^n$	$f(z) = \frac{2}{2-z}$	$X(z) = \frac{z}{z-0.5}$
⑤ $(n \geq 1)$ $a_{n-1} = \left(\frac{1}{2}\right)^{n-1}$	$f(z) = \frac{2z}{2-z}$	$X(z) = \frac{1}{z-0.5}$
② $(n \geq 0)$ $a_n = (2)^n$	$f(z) = \frac{0.5}{0.5-z}$	$X(z) = \frac{z}{z-2}$
⑥ $(n \geq 1)$ $a_{n-1} = (2)^{n-1}$	$f(z) = \frac{0.5z}{0.5-z}$	$X(z) = \frac{1}{z-2}$

shift to the right →	$\times z$	$\times z^{-1}$
insert a_0		
③ $(n < 0)$ $a_n = \left(\frac{1}{2}\right)^n$	$f(z) = -\frac{2}{2-z}$	$X(z) = -\frac{z}{z-0.5}$
⑦ $(n < 1)$ $a_{n-1} = \left(\frac{1}{2}\right)^{n-1}$	$f(z) = -\frac{2z}{2-z}$	$X(z) = -\frac{1}{z-0.5}$
④ $(n < 0)$ $a_n = (2)^n$	$f(z) = -\frac{0.5}{0.5-z}$	$X(z) = -\frac{z}{z-2}$
⑧ $(n < 1)$ $a_{n-1} = (2)^{n-1}$	$f(z) = -\frac{0.5z}{0.5-z}$	$X(z) = -\frac{1}{z-2}$

Shift to the left

Shift to the left $\leftarrow$ $\times z^{-1}$ $\times z$

delete a_0

1 $(n \geq 0)$ $a_n = \left(\frac{1}{2}\right)^n$ $f(z) = \frac{2}{2-z}$ $X(z) = \frac{z}{z-0.5}$

9 $(n \geq -1)$ $a_{n+1} = \left(\frac{1}{2}\right)^{n+1}$ $f(z) = \frac{2}{(2-z)z}$ $X(z) = \frac{z}{z-0.5}$

2 $(n \geq 0)$ $a_n = (2)^n$ $f(z) = \frac{0.5}{0.5-z}$ $X(z) = \frac{z}{z-2}$

10 $(n \geq -1)$ $a_{n+1} = (2)^{n+1}$ $f(z) = \frac{0.5}{(0.5-z)z}$ $X(z) = \frac{z}{z-2}$

Shift to the left $\leftarrow$ $\times z^{-1}$ $\times z$

insert a_0

3 $(n < 0)$ $a_n = \left(\frac{1}{2}\right)^n$ $f(z) = -\frac{2}{2-z}$ $X(z) = -\frac{z}{z-0.5}$

11 $(n < -1)$ $a_{n+1} = \left(\frac{1}{2}\right)^{n+1}$ $f(z) = -\frac{2}{(2-z)z}$ $X(z) = -\frac{z}{z-0.5}$

4 $(n < 0)$ $a_n = (2)^n$ $f(z) = -\frac{0.5}{0.5-z}$ $X(z) = -\frac{z}{z-2}$

12 $(n < -1)$ $a_{n+1} = (2)^{n+1}$ $f(z) = -\frac{0.5}{(0.5-z)z}$ $X(z) = -\frac{z}{z-2}$

$n=-4$	$n=-3$	$n=-2$	$n=-1$	$n=0$	$n=1$	$n=2$	
b^{-3}	b^{-2}	b^{-1}	b^{0}	b^{1}	b^{2}	b^{3}	

b^{n+1} $n = -2, -3, -4, \cdots$ b^{n+1} $n = -1, 0, 1, \cdots$

$n=-3$	$n=-2$	$n=-1$	$n=0$	$n=1$	$n=2$	$n=3$	
b^{-3}	b^{-2}	b^{-1}	b^{0}	b^{1}	b^{2}	b^{3}	

b^{n} $n = -1, -2, -3, \cdots$ b^{n} $n = 0, 1, 2, \cdots$

$n=-3$

$n=-2$	$n=-1$	$n=0$	$n=1$	$n=2$	$n=3$		
b^{-3}	b^{-2}	b^{-1}	b^{0}	b^{1}	b^{2}	b^{3}	

b^{n-1} $n = 0, -1, -2, \cdots$ b^{n-1} $n = 1, 2, 3, \cdots$

$1 \longleftrightarrow \frac{1}{1}$

1. $(n \geq 0)$ $a_n = (1)^n$ $\quad f(z) = \frac{1}{1-z}$ $\quad X(z) = \frac{z}{z-1}$
2. $(n \geq 0)$ $a_n = (1^{-1})^n$ $\quad f(z) = \frac{1}{1-z}$ $\quad X(z) = \frac{z}{z-1}$
3. $(n < 0)$ $a_n = (1)^n$ $\quad f(z) = -\frac{1}{1-z}$ $\quad X(z) = -\frac{z}{z-1}$
4. $(n < 0)$ $a_n = (1^{-1})^n$ $\quad f(z) = -\frac{1}{1-z}$ $\quad X(z) = -\frac{z}{z-1}$

5. $(n \geq 1)$ $a_{n-1} = (1)^{n-1}$ $\quad f(z) = \frac{z}{1-z}$ $\quad X(z) = \frac{1}{z-1}$
6. $(n \geq 1)$ $a_{n-1} = (1^{-1})^{n-1}$ $\quad f(z) = \frac{z}{1-z}$ $\quad X(z) = \frac{1}{z-1}$
7. $(n < 1)$ $a_{n-1} = (1)^{n-1}$ $\quad f(z) = -\frac{z}{1-z}$ $\quad X(z) = -\frac{1}{z-1}$
8. $(n < 1)$ $a_{n-1} = (1^{-1})^{n-1}$ $\quad f(z) = -\frac{z}{1-z}$ $\quad X(z) = -\frac{1}{z-1}$

9. $(n \geq -1)$ $a_{n+1} = (1)^{n+1}$ $\quad f(z) = \frac{1}{(1-z)z}$ $\quad X(z) = \frac{z}{z-1}$
10. $(n \geq -1)$ $a_{n+1} = (1^{-1})^{n+1}$ $\quad f(z) = \frac{1}{(1-z)z}$ $\quad X(z) = \frac{z}{z-1}$
11. $(n < -1)$ $a_{n+1} = (1)^{n+1}$ $\quad f(z) = -\frac{1}{(1-z)z}$ $\quad X(z) = -\frac{z}{z-1}$
12. $(n < -1)$ $a_{n+1} = (1^{-1})^{n+1}$ $\quad f(z) = -\frac{1}{(1-z)z}$ $\quad X(z) = -\frac{z}{z-1}$

1 $(n \geq 0)$ $a_n = (1)^n$ $f(z) = \frac{1}{1-z}$ $X(z) = \frac{z}{z-1}$

2 $(n \geq 0)$ $a_n = (1^{-1})^n$ $f(z) = \frac{1}{1-z}$ $X(z) = \frac{z}{z-1}$

shift to the right $\rightarrow$ $\times z$ $\times z^{-1}$

delete a_0

5 $(n \geq 1)$ $a_{n-1} = (1)^{n-1}$ $f(z) = \frac{z}{1-z}$ $X(z) = \frac{1}{z-1}$

6 $(n \geq 1)$ $a_{n-1} = (1^{-1})^{n-1}$ $f(z) = \frac{z}{1-z}$ $X(z) = \frac{1}{z-1}$

3 $(n < 0)$ $a_n = (1)^n$ $f(z) = -\frac{1}{1-z}$ $X(z) = -\frac{z}{z-1}$

4 $(n < 0)$ $a_n = (1^{-1})^n$ $f(z) = -\frac{1}{1-z}$ $X(z) = -\frac{z}{z-1}$

shift to the right $\rightarrow$ $\times z$ $\times z^{-1}$

insert a_0

7 $(n < 1)$ $a_{n-1} = (1)^{n-1}$ $f(z) = -\frac{z}{1-z}$ $X(z) = -\frac{1}{z-1}$

8 $(n < 1)$ $a_{n-1} = (1^{-1})^{n-1}$ $f(z) = -\frac{z}{1-z}$ $X(z) = -\frac{1}{z-1}$

① $(n \geq 0)$ $a_n = (1)^n$ $f(z) = \frac{1}{1-z}$ $X(z) = \frac{z}{z-1}$

② $(n \geq 0)$ $a_n = (1^{-1})^n$ $f(z) = \frac{1}{1-z}$ $X(z) = \frac{z}{z-1}$

shift to the left $\leftarrow$ $\times z^{-1}$ $\times z$

delete a_0

⑨ $(n \geq -1)$ $a_{n+1} = (1)^{n+1}$ $f(z) = \frac{1}{(1-z)z}$ $X(z) = \frac{z}{z-1}$

⑩ $(n \geq -1)$ $a_{n+1} = (1^{-1})^{n+1}$ $f(z) = \frac{1}{(1-z)z}$ $X(z) = \frac{z}{z-1}$

③ $(n < 0)$ $a_n = (1)^n$ $f(z) = -\frac{1}{1-z}$ $X(z) = -\frac{z}{z-1}$

④ $(n < 0)$ $a_n = (1^{-1})^n$ $f(z) = -\frac{1}{1-z}$ $X(z) = -\frac{z}{z-1}$

shift to the left $\leftarrow$ $\times z^{-1}$ $\times z$

insert a_0

⑪ $(n < -1)$ $a_{n+1} = (1)^{n+1}$ $f(z) = -\frac{1}{(1-z)z}$ $X(z) = -\frac{z}{z-1}$

⑫ $(n < -1)$ $a_{n+1} = (1^{-1})^{n+1}$ $f(z) = -\frac{1}{(1-z)z}$ $X(z) = -\frac{z}{z-1}$

Causality

$f(z) \quad (|z| < p) \quad \longleftrightarrow \quad a_n \quad (n \geq 0) \qquad -(p^{-1}, p^{-2}, p^{-3}, \cdots)$

$X(z^{-1}) \quad (|z| < p) \quad \longleftrightarrow \quad x_{-n} \quad (n < 1) \qquad -(p^{-1}, p^{-2}, p^{-3}, \cdots)$

$f(z^{-1}) \quad (|z| > p^{-1}) \quad \longleftrightarrow \quad a_{-n} \quad (n < 1) \qquad -(p^{-1}, p^{-2}, p^{-3}, \cdots)$

$X(z) \quad (|z| > p^{-1}) \quad \longleftrightarrow \quad x_n \quad (n \geq 0) \qquad -(p^{-1}, p^{-2}, p^{-3}, \cdots)$

$f(z) \quad (|z| > p) \quad \longleftrightarrow \quad -a_n \quad (n < 0) \qquad (p^0, p^1, p^2, \cdots)$

$X(z^{-1}) \quad (|z| > p) \quad \longleftrightarrow \quad -x_{-n} \quad (n \geq 1) \qquad (p^0, p^1, p^2, \cdots)$

$f(z^{-1}) \quad (|z| < p^{-1}) \quad \longleftrightarrow \quad -a_{-n} \quad (n \geq 1) \qquad (p^0, p^1, p^2, \cdots)$

$X(z) \quad (|z| < p^{-1}) \quad \longleftrightarrow \quad -x_n \quad (n < 0) \qquad (p^0, p^1, p^2, \cdots)$

$f(z)$	$f(z^{-1})$
$g(z^{-1})$	$g(z)$

$X(z^{-1})$	$X(z)$
$Y(z)$	$Y(z^{-1})$

a_n	a_{-n}
b_{-n}	b_n

x_{-n}	x_n
y_n	y_{-n}

$f(z)$	$f(z^{-1})$
$f(z)$	$f(z^{-1})$

$X(z^{-1})$	$X(z)$
$X(z^{-1})$	$X(z)$

$-(p^{-1}, p^{-2}, p^{-3}, \dots)$	$-(p^{-1}, p^{-2}, p^{-3}, \dots)$
$(p^{0}, p^{1}, p^{2}, \dots)$	$(p^{0}, p^{1}, p^{2}, \dots)$

$-(p^{-1}, p^{-2}, p^{-3}, \dots)$	$-(p^{-1}, p^{-2}, p^{-3}, \dots)$
$(p^{0}, p^{1}, p^{2}, \dots)$	$(p^{0}, p^{1}, p^{2}, \dots)$

$-\dfrac{p^{-1}}{1-p^{-1}z}$	$-\dfrac{p^{-1}}{1-p^{-1}z^{-1}}$
$\dfrac{z^{-1}}{1-pz^{-1}}$	$\dfrac{z}{1-pz}$

$-\dfrac{p^{-1}}{1-p^{-1}z}$	$-\dfrac{p^{-1}}{1-p^{-1}z^{-1}}$
$\dfrac{z^{-1}}{1-pz^{-1}}$	$\dfrac{z}{1-pz}$

$f(z)$ $g(z)$ $f(z)$ $g(z)$	$Y(z)$ $X(z)$ $Y(z)$ $X(z)$
$\|z\| < p$ $\|z\| > p^{-1}$ $\|z\| > p$ $\|z\| < p^{-1}$	$\|z\| < p$ $\|z\| > p^{-1}$ $\|z\| > p$ $\|z\| < p^{-1}$
$[0, \infty)$ $(-\infty, 0]$ $(-\infty, -1]$ $[1, \infty)$	$(-\infty, 0]$ $[0, \infty)$ $[1, \infty)$ $(-\infty, -1]$
$-(p^{-1}, p^{-2}, p^{-3}, \cdots)$ $(p^{0}, p^{1}, p^{2}, \cdots)$	$-(p^{-1}, p^{-2}, p^{-3}, \cdots)$ $(p^{0}, p^{1}, p^{2}, \cdots)$

a_n a_{-n}
$-a_n$ $-a_{-n}$

x_{-n} x_n
$-x_{-n}$ $-x_n$

$$\boxed{\begin{array}{ll} a_n & a_{-n} \\ -a_n & -a_{-n} \end{array}} \qquad \boxed{\begin{array}{ll} 2^{-n} & 2^{n} \\ -2^{-n} & -2^{n} \end{array}} \qquad a_n = -2^{-n}$$

$$\boxed{\begin{array}{ll} x_{-n} & x_n \\ -x_{-n} & -x_n \end{array}} \qquad \boxed{\begin{array}{ll} 2^{-n} & 2^{n} \\ -2^{-n} & -2^{n} \end{array}} \qquad x_n = -2^{n}$$

$$\boxed{\begin{array}{ll} -(p^{-1}, p^{-2}, p^{-3}, \cdots) & -(p^{-1}, p^{-2}, p^{-3}, \cdots) \\ (p^{0}, p^{1}, p^{2}, \cdots) & (p^{0}, p^{1}, p^{2}, \cdots) \end{array}} \qquad \boxed{\begin{array}{ll} -(-2^{-1}, -2^{-2}, -2^{-3}, \cdots) & -(-2^{-1}, -2^{-2}, -2^{-3}, \cdots) \\ (-2^{0}, -2^{1}, -2^{2}, \cdots) & (-2^{0}, -2^{1}, -2^{2}, \cdots) \end{array}}$$

$$\boxed{\begin{array}{ll} -\dfrac{p^{-1}}{1-p^{-1}z} & -\dfrac{p^{-1}}{1-p^{-1}z^{-1}} \\ \dfrac{z^{-1}}{1-pz^{-1}} & \dfrac{z}{1-pz} \end{array}} \qquad \boxed{\begin{array}{ll} \dfrac{2^{-1}}{1-2^{-1}z} & \dfrac{2^{-1}}{1-2^{-1}z^{-1}} \\ -\dfrac{z^{-1}}{1-2z^{-1}} & -\dfrac{z}{1-2z} \end{array}} \qquad \begin{array}{ll} \dfrac{\left(\frac{1}{2}\right)}{1-\left(\frac{z}{2}\right)} & \dfrac{\left(\frac{1}{2}\right)}{1-\left(\frac{1}{2z}\right)} \\ -\dfrac{\left(\frac{1}{z}\right)}{1-\left(\frac{2}{z}\right)} & -\dfrac{z}{1-2z} \end{array}$$

$$\boxed{\begin{array}{ll} |z| < p & |z| > p^{-1} \\ |z| > p & |z| < p^{-1} \end{array}} \qquad \boxed{\begin{array}{ll} |z| < 2 & |z| > 2^{-1} \\ |z| > 2 & |z| < 2^{-1} \end{array}}$$

$$\boxed{\begin{array}{ll} [0, \infty) & (-\infty, 0] \\ (-\infty, -1] & [1, \infty) \end{array}} \qquad \boxed{\begin{array}{ll} [0, \infty) & (-\infty, 0] \\ (-\infty, -1] & [1, \infty) \end{array}}$$